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# "Compressed Sensing and Quantization: a Compander Approach."

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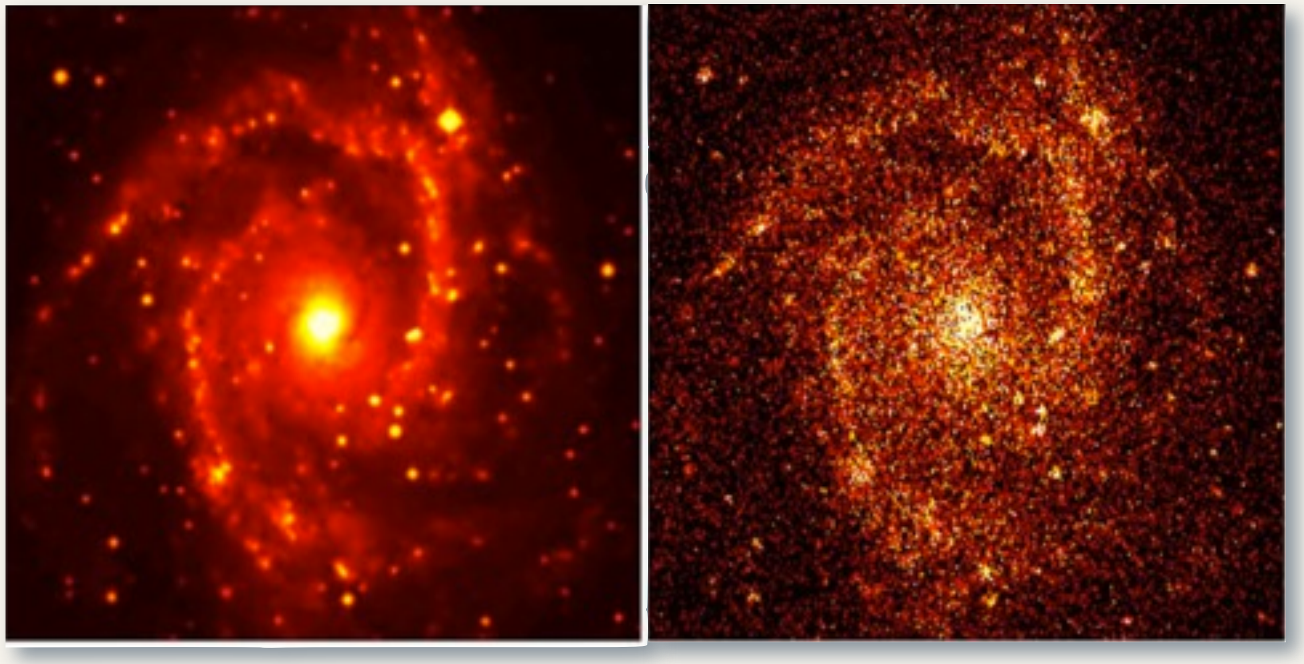
# 1. Context

# A Noisy Affair...

Actual Sensors suffer from more general disturbances than simple AWG noises.

(*e.g.*, Poisson, Impulsive, Gamma, Quantization, GGD, ...)



Court. [B. Zhang et al.]

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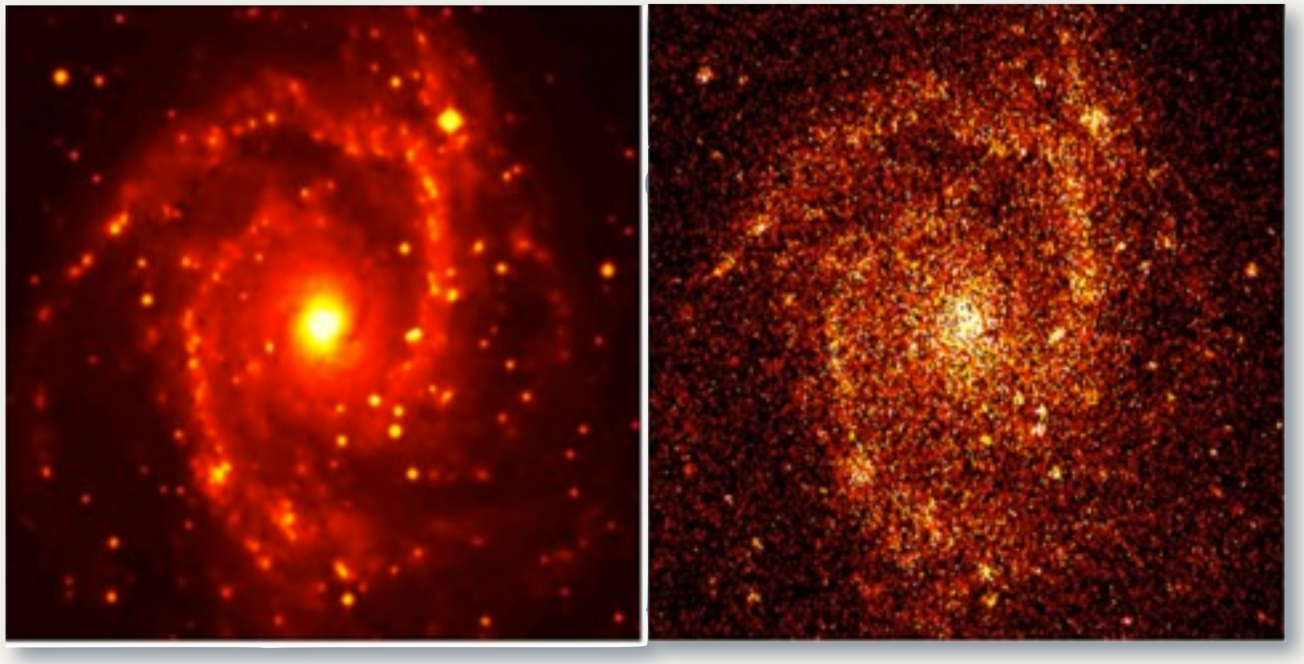

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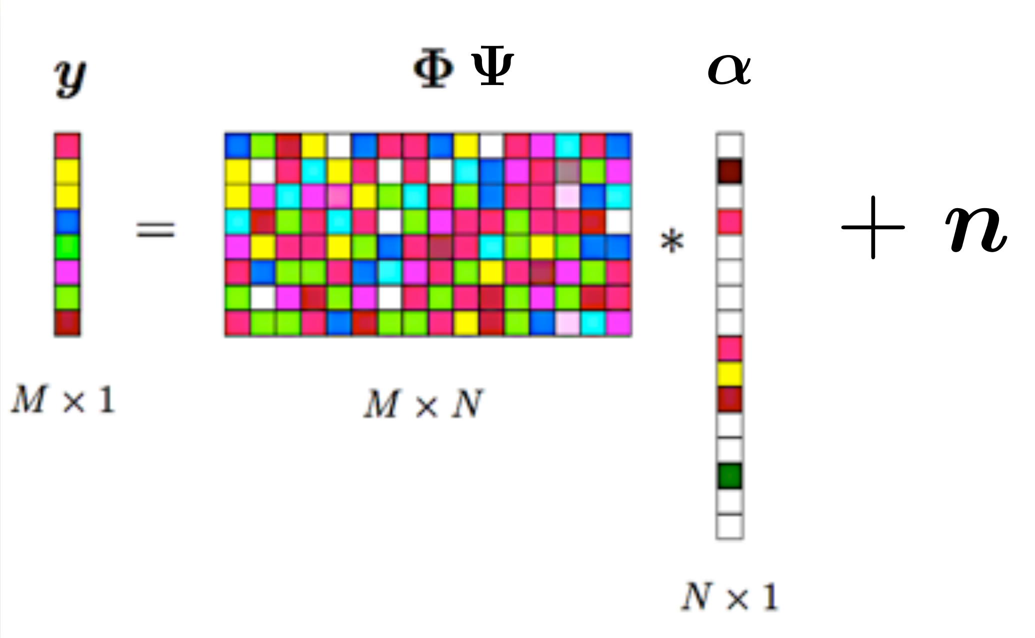
# Noise and Compressed Sensing

Initially, CS tested for AWGN noise

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{\Psi}\boldsymbol{\alpha} + \boldsymbol{n}$$




with:

$n_i \sim \mathcal{N}(0, \sigma^2)$

$\boldsymbol{\alpha}$ sparse or compressible

$\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ the *sensing* and the *sparsity* bases

# Reconstruction with AWGN model

Basis Pursuit DeNoise (BPDN)

If $\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{n}$ (with $\boldsymbol{\Psi} = \mathrm{Id}$ and $\|\boldsymbol{n}\|_2 \leqslant \epsilon$),

$$\boxed{\boldsymbol{x}^* = \operatorname*{argmin}_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\Phi u}\|_2 \leqslant \epsilon}$$

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If

$\exists\ \mu > 0,\ \delta \in (0,1)$ — Restricted Isometry Property

$$\sqrt{1-\delta}\,\|\boldsymbol{v}\|_2 \ \leqslant\ \tfrac{1}{\mu}\|\boldsymbol{\Phi v}\|_2 \ \leqslant\ \sqrt{1+\delta}\,\|\boldsymbol{v}\|_2$$

for all $2K$ sparse signals $\boldsymbol{v}$.

AND $\quad \delta < \delta_0 \quad$ (*e.g.*, $\delta_0 = \sqrt{2}-1$)

Then,

$$\boxed{\|\boldsymbol{x} - \boldsymbol{x}^*\|_2 \ \leqslant\ A\,\epsilon/\mu \ +\ B\,\|\boldsymbol{x} - \boldsymbol{x}_K\|_1/\sqrt{K}}$$

# BDPN Drawbacks (wrt noise sources)

$$\boldsymbol{x}^* = \underset{\boldsymbol{u}}{\operatorname{argmin}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \underline{\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{u}\|_2} \leqslant \epsilon$$

MAP standpoint: adapted to AWGN only

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Can we use other fidelity terms/norms for other noises?

If *heteroscedastic* noise, *e.g.*,

$$n_i \sim \mathcal{N}(0, \sigma_i^2)$$

**Wish to "stabilize" it**, *e.g.*

$$\boldsymbol{n} = \boldsymbol{y} - \boldsymbol{\Phi x} \longrightarrow \boldsymbol{W}(\boldsymbol{y} - \boldsymbol{\Phi x}),$$

with $\boldsymbol{W} = \operatorname{diag}(\sigma_1, \cdots, \sigma_M)^{-1}$

but is it allowed?

What about **non-linear** operations, e.g., scalar **Quantization** ?

$$\boldsymbol{y} = \mathcal{Q}[\boldsymbol{\Phi u}] \ ?$$

and $\boldsymbol{y} \neq \mathcal{Q}[\boldsymbol{\Phi x}^*]$ !

# 2. Quantized CS

# The Quantized CS Problem

Compressed Sensing theory says:

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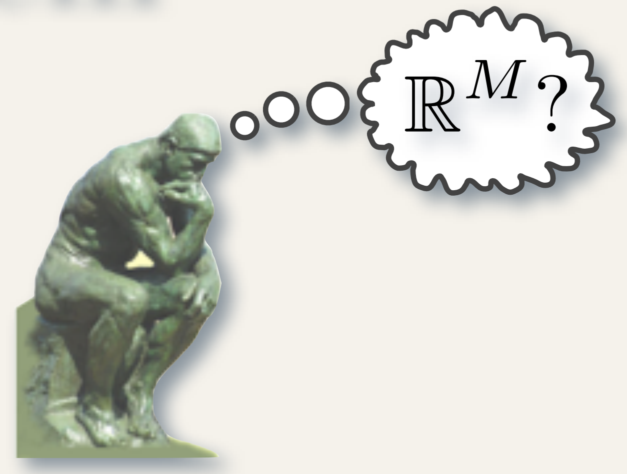
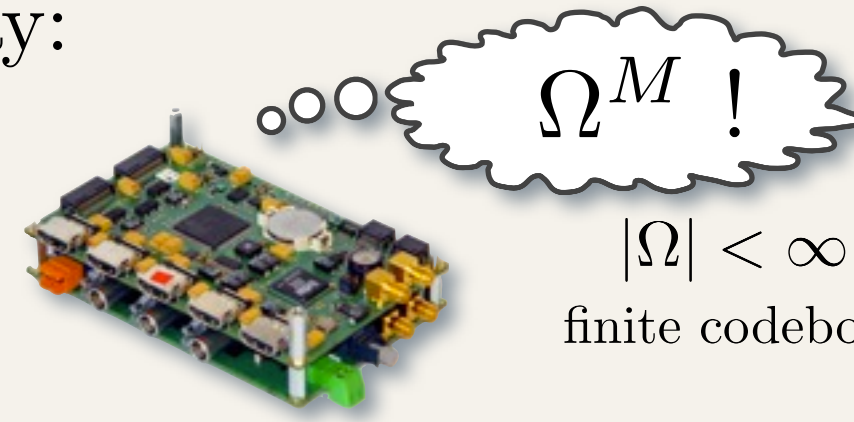



Information Theory and Sensor designer say:

*"Okay, but I need to*
*quantize/digitize (ADC) my measurements!"*




$|\Omega| < \infty$
finite codebook

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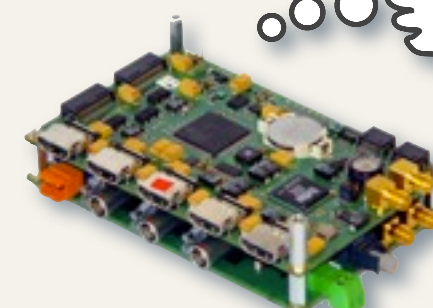



$|\Omega| < \infty$
finite codebook

Question:

"**Given** quantized signal measurements (*quantization is untouched*),
**how to minimize quantization effects in the reconstruction**?"

Our answer:

"Oversample and reconstruct with *compander theory* and
non-gaussian constraints, i.e. using (weighted) $\ell_p$ - norm "
(not the sparsity prior) (for $p > 2$)

# Quantization of CS Measurements

Turning Measurements into bits $\rightarrow$ scalar quantization

$$\boxed{\boldsymbol{y} \;=\; \mathcal{Q}\big[\boldsymbol{\Phi x}\big] \;\in \Omega^M,}$$

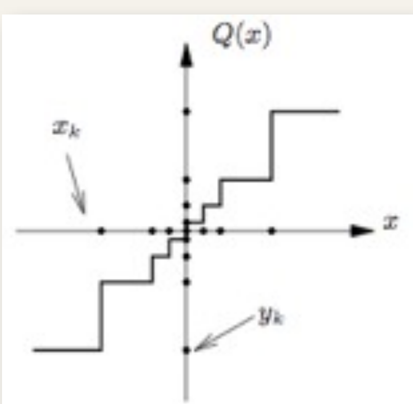
with:

$$\Omega = \{\omega_i \in \mathbb{R} : 1 \leqslant i \leqslant 2^B\}, \qquad \text{(levels)}$$

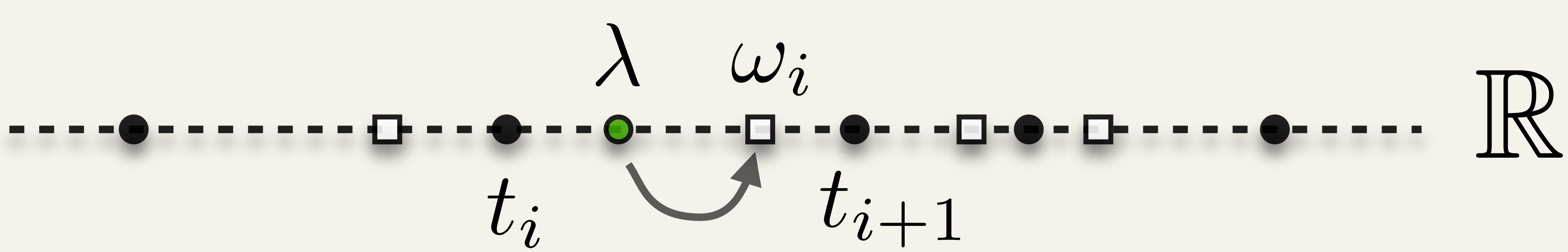
$$T = \{t_i \in \overline{\mathbb{R}} : 1 \leqslant i \leqslant 2^B + 1, t_i \leqslant t_{i+1}\} \qquad \text{(thresholds)}$$

$$\forall \lambda \in \mathbb{R}, \qquad \mathcal{Q}[\lambda] = \omega_i \;\Leftrightarrow\; \lambda \in \mathcal{R}_i \triangleq [t_i, t_{i+1}),$$

$$\forall u \in \mathbb{R}^M, \quad (\mathcal{Q}[u])_j = \mathcal{Q}[u_j]$$



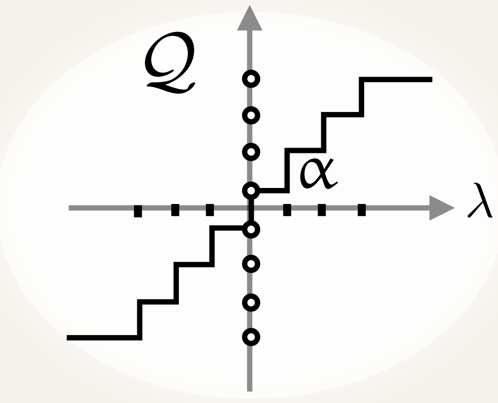

Court. [Connexions]




(e.g. Lloyd-Max)

# Previous work: uniform quantization

(J, Hammond, Fadili, 2009, 2011)




- Distortion model:

$$\boldsymbol{y} = \mathcal{Q}[\boldsymbol{\Phi x}] = \boldsymbol{\Phi x} + \boldsymbol{\varepsilon}, \quad \varepsilon_i \sim U(-\tfrac{\alpha}{2}, \tfrac{\alpha}{2})$$

- Observation: $\|\boldsymbol{y} - \boldsymbol{\Phi x}\|_\infty \leqslant \frac{\alpha}{2}$
- Reconstruction: Generalizing BPDN with BPDQ

$$\boxed{\boldsymbol{x}^* \;=\; \operatorname*{arg\,min}_{\boldsymbol{v}\in\mathbb{R}^N} \|\boldsymbol{v}\|_1 \text{ s.t. } \|\boldsymbol{y} - \boldsymbol{\Phi v}\|_p \;\leqslant \epsilon_p}$$

Towards $p = \infty$

Related to GGD MAP

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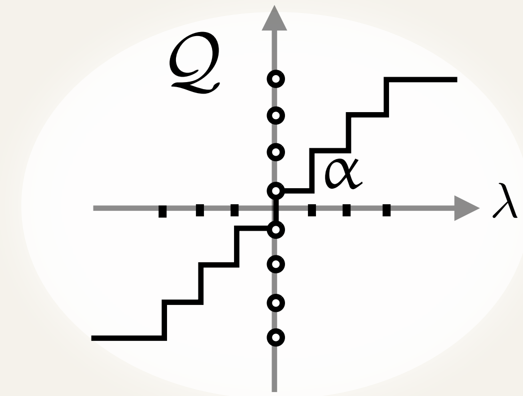


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Towards $p = \infty$

Related to GGD MAP

If $\boldsymbol{\Phi}$ is RIP$_p$ of order $K$, *i.e.*,

$\exists\ \mu_p > 0,\ \delta \in (0,1),$

$$\sqrt{1-\delta}\,\|\boldsymbol{v}\|_2 \;\leqslant\; \tfrac{1}{\mu_p}\,\|\boldsymbol{\Phi v}\|_p \;\leqslant\; \sqrt{1+\delta}\,\|\boldsymbol{v}\|_2,$$

for all $K$ sparse signals $\boldsymbol{v}$.

Gain over BPDN (for tight $\epsilon_p(\alpha, M)$)

$$\boxed{\Rightarrow \|\boldsymbol{x} - \boldsymbol{x}^*\| = O(\alpha/\sqrt{p+1})}$$

But no free lunch:

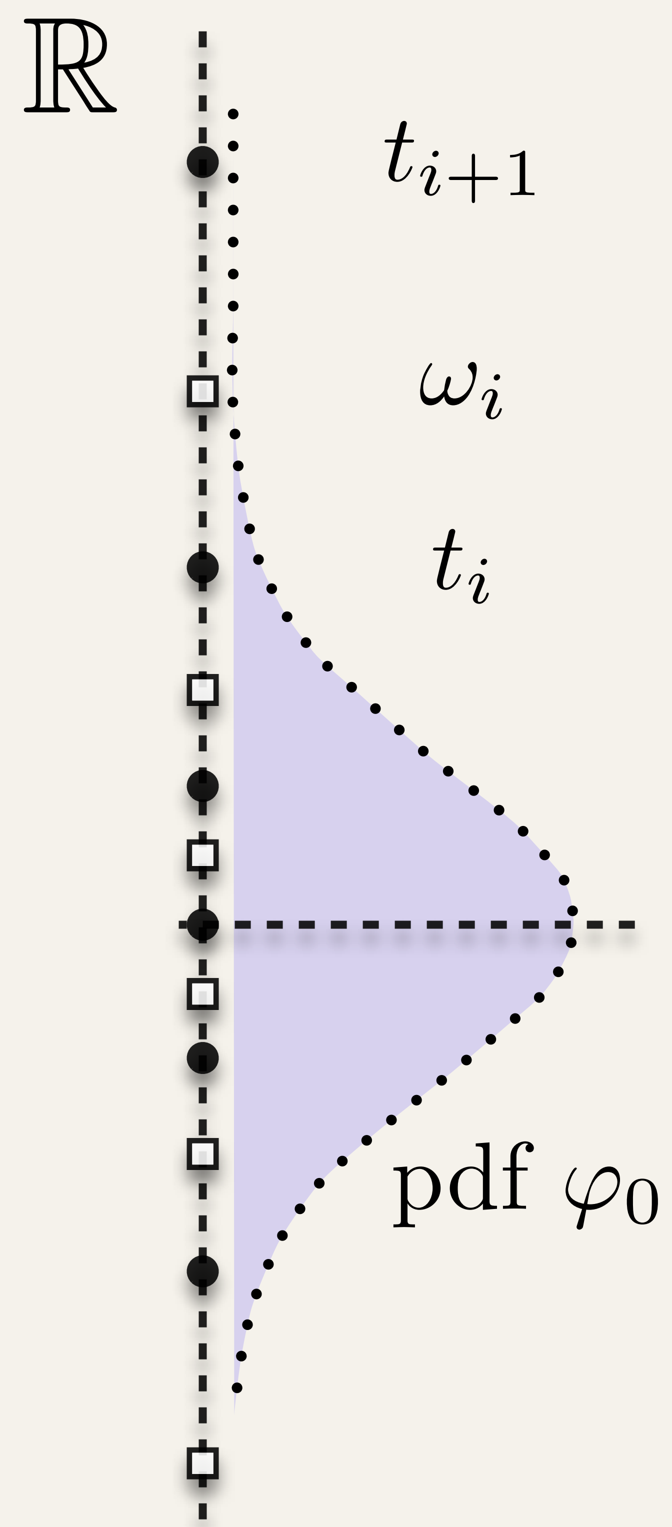
$$\boxed{M = O\big(\,(K \log N/K\,)^{\underline{p/2}}\,\big)}$$

$\Longrightarrow$ Another reading: limited range of valid $p$ for a given $M$ (and $K$)!

# 3. Non-Uniform Generalization through Compander Theory

# Compander approach

Non-uniform $B$ bits quantizer $\mathcal{Q}$ .



# Compander approach

$$\boxed{\mathcal{Q} = \mathcal{G}^{-1} \circ \mathcal{Q}_\alpha \circ \mathcal{G}}$$

Under High Resolution Assumption (HRA)

= high $B$

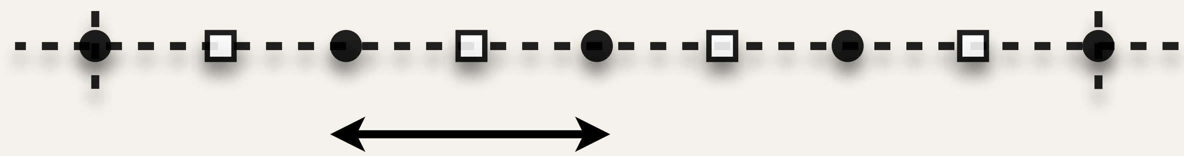
Non-uniform $B$ bits quantizer $\mathcal{Q}$.

Compressor

$\mathcal{G} : \overline{\mathbb{R}} \to [0, 1]$

$0$ $t'_i$ $\omega'_i$ $t'_{i+1}$ $1$

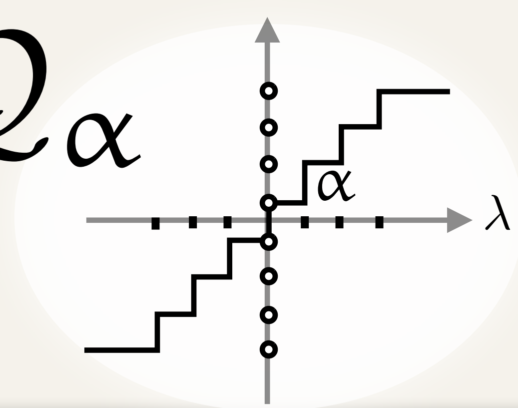
$B$ bits



$\alpha = 2^{-B}$

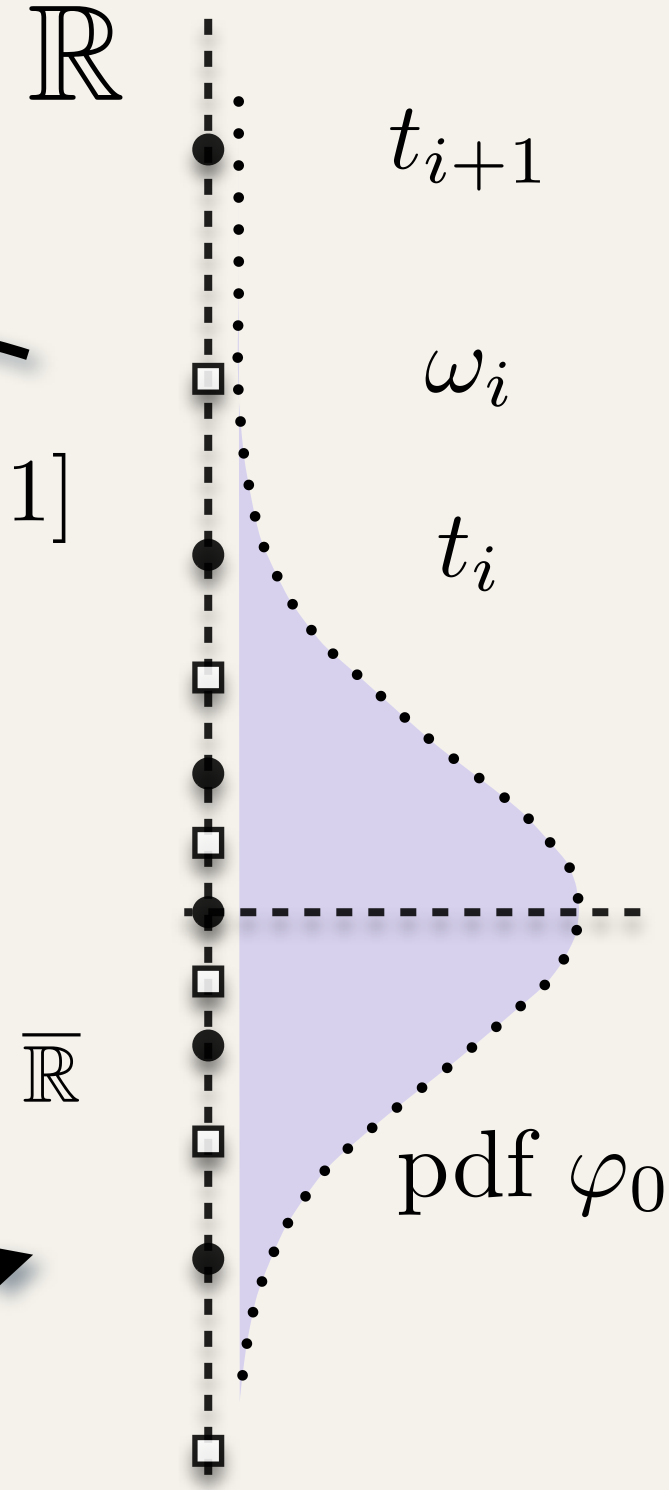
Uniform quantizer

$\mathcal{Q}_\alpha$




$\mathcal{G}^{-1} : [0, 1] \to \overline{\mathbb{R}}$

Expander



# Compander approach

Optimal quantization: $\mathcal{Z} \sim \varphi_0$, $\mathbb{E}|\mathcal{Z} - \mathcal{Q}(\mathcal{Z})|^2$ min.

$$\Rightarrow \quad \tfrac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{G}(\lambda) \;:=\; \Big[\int_{\mathbb{R}} \varphi_0^{1/3}(t)\,\mathrm{d}t\Big]^{-1} \varphi_0^{1/3}(\lambda)$$

Example:

If $\varphi_0 = \gamma_{0,\sigma_0} \sim \mathcal{N}(0,\sigma_0^2)$,
$\mathcal{G}'(t) = \gamma_{0,\sqrt{3}\,\sigma_0}(t)$ and
$\mathcal{G}(t) = G_{0,\sqrt{3}\,\sigma_0}(t)$.

Distortion: [Panter & Dite, 1951]

$$\boxed{\mathbb{E}|\mathcal{Z} - \mathcal{Q}(\mathcal{Z})|^2 \;\underset{B}{\simeq}\; \frac{2^{-2B}}{12} \int_{\mathbb{R}} \mathcal{G}'(t)^{-2}\,\varphi_0(t)\,\mathrm{d}t \;=\; \overbrace{\frac{2^{-2B}}{12}\,\|\varphi_0\|_{1/3}}^{\alpha^2}.}$$

Gaussian: $\frac{1}{12}\|\varphi_0\|_{1/3} \simeq 2.721\,\sigma_0^2$

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Gaussian: $\frac{1}{12}\|\varphi_0\|_{1/3} \simeq 2.721\,\sigma_0^2$

Therefore: if $\boldsymbol{z} \sim \mathcal{N}^{M\times 1}(0,1)$, two observations:

$(c \simeq 2.721)$

1. "Distortion Constraint" (**DC**): $\|\mathcal{Q}[\boldsymbol{z}] - \boldsymbol{z}\|_2 \ \underset{M,B}{\simeq} \ \sqrt{c\,M}\,2^{-B} = \sqrt{c\,M}\,\alpha.$

2. "Quantization Constraint" (**QC**): $\|\mathcal{G}(Q[\boldsymbol{z}]) - \mathcal{G}(\boldsymbol{z})\|_\infty \ \underset{M}{\simeq} \ \frac{1}{2}2^{-B} = \frac{1}{2}\alpha.$

# Compander approach

- Can we see **QC** and **DC** as

  2 instances of a more general set of constraints?
- Is it useful for reconstructing from QCS ?

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  2 instances of a more general set of constraints?
- Is it useful for reconstructing from QCS ?

YES!

- How? Again with a $\ell_p$-norm but ...

1. Generalize levels (not the thresholds, fixed by quantization)
2. Weight the bin contributions

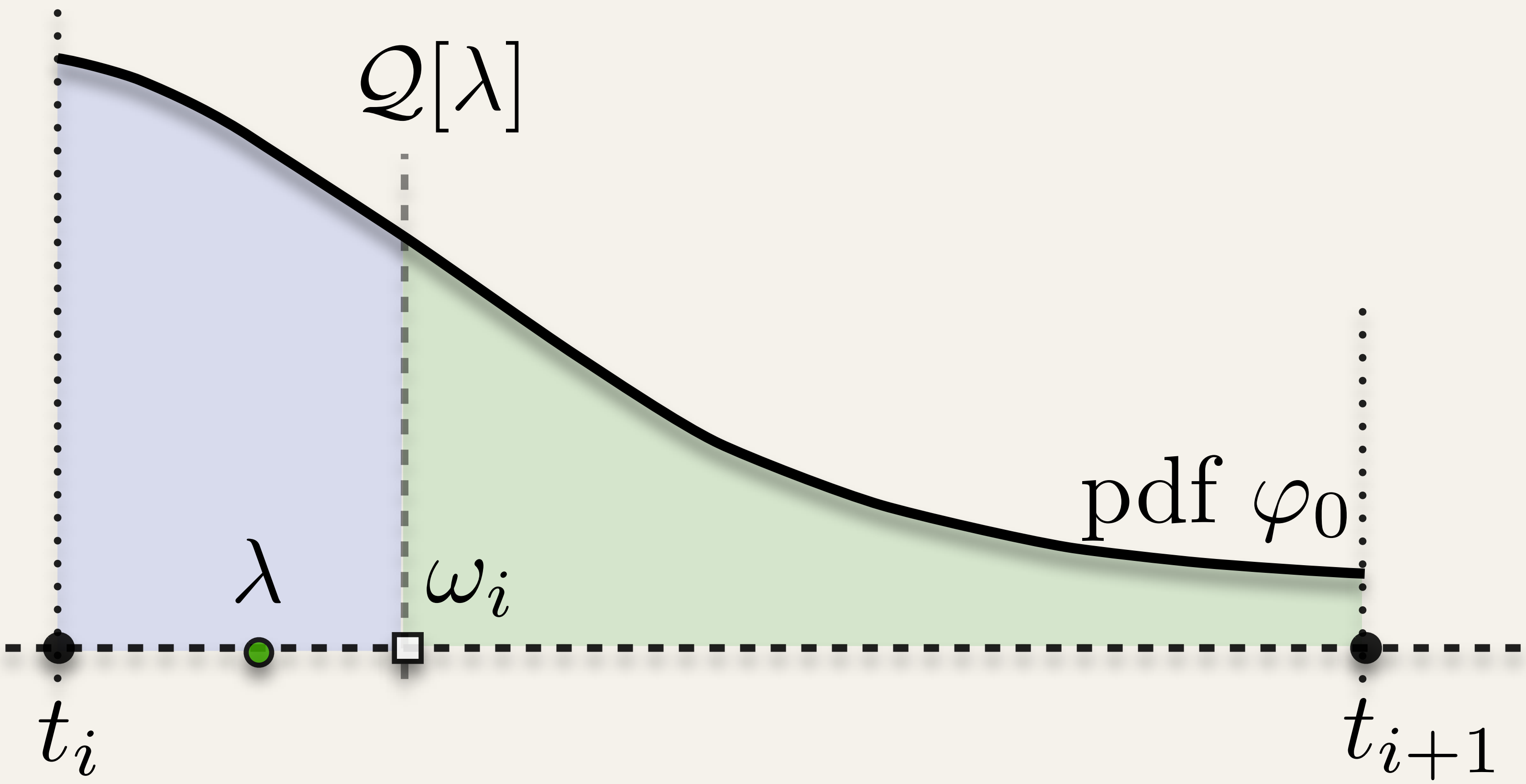
... see next 2 slides

# 1. Generalizing levels

We had ...

For 2 - optimal level: Lloyd-Max

$$\boxed{\omega_i := \arg\min_{\lambda \in \mathcal{R}_i} \Big( \int_{\mathcal{R}_i} |\lambda - t|^2 \, \varphi_0(t) \, \mathrm{d}t \Big)^{1/2}}$$

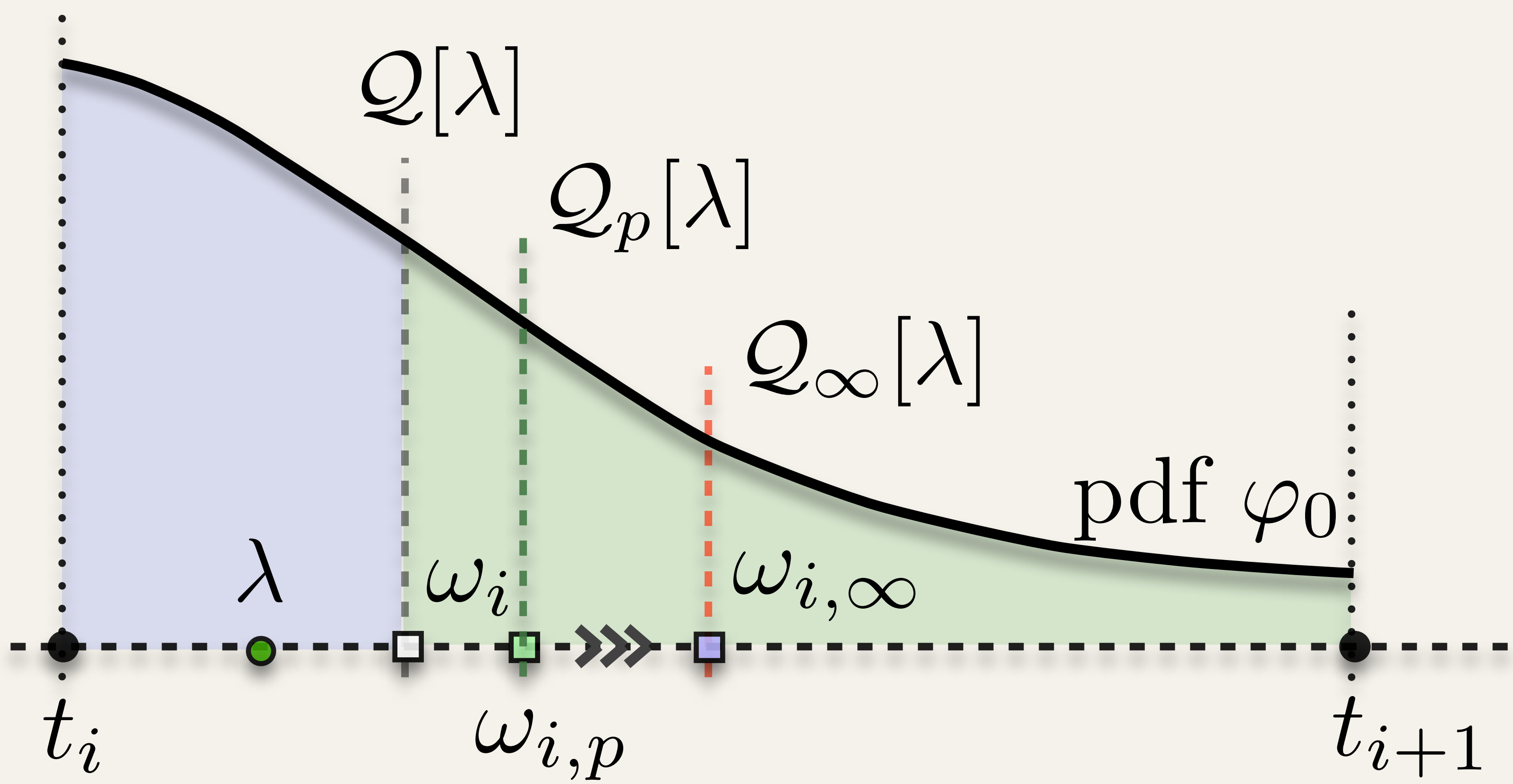


# 1. Generalizing levels

Now we define ...

For $p$ - optimal level:

$$\boxed{\omega_{i,p} := \arg\min_{\lambda\in\mathcal{R}_i} \Big( \int_{\mathcal{R}_i} |\lambda - t|^p \varphi_0(t)\, \mathrm{d}t \Big)^{1/p}}$$




Lloyd-Max, Companders: $\omega_i = \omega_{i,2}$

Bin mid-point: $\omega_{i,\infty} = \frac{1}{2}(t_i + t_{i+1}) = \lim_{p\to+\infty} \omega_{i,p}$

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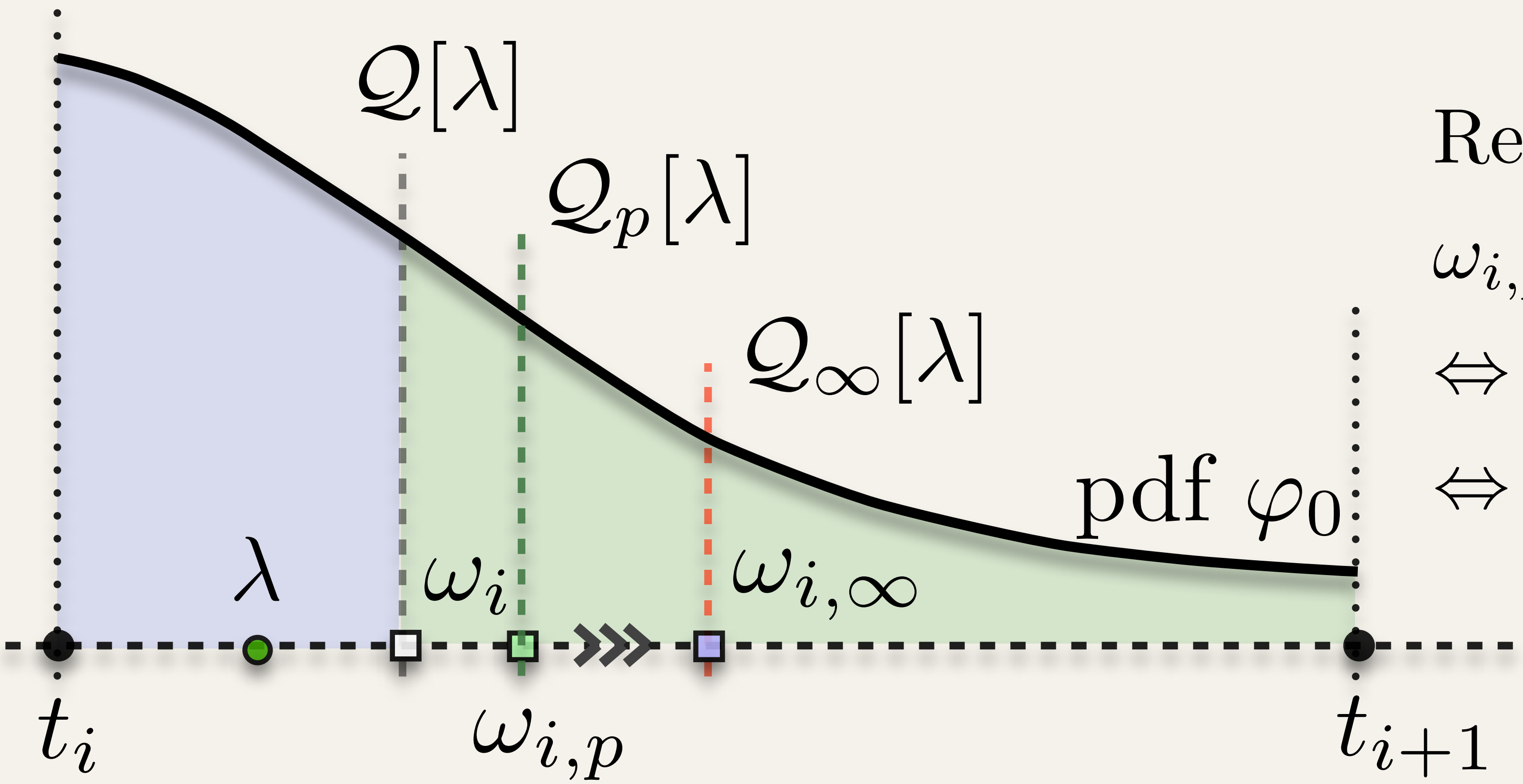


Requantization: $\mathcal{Q} \to \mathcal{Q}_p$?

$\omega_{i,p} = \mathcal{Q}_p[\lambda] = \mathcal{Q}_p[\mathcal{Q}[\lambda]]$,

$\Leftrightarrow \lambda$ in $\mathcal{R}_i$,

$\Leftrightarrow \omega_{i,2} = \mathcal{Q}[\lambda] \in \mathcal{R}_i$.

Lloyd-Max, Companders: $\omega_i = \omega_{i,2}$

Bin mid-point: $\omega_{i,\infty} = \frac{1}{2}(t_i + t_{i+1}) = \lim_{p\to+\infty} \omega_{i,p}$

# 2. Weighting the bin contribution

Given $p \geqslant 2$ and $\boldsymbol{z} \in \mathbb{R}^M$,

Define $\boldsymbol{w} \in \mathbb{R}_+^M : w_i(p) := \mathcal{G}'\big(\mathcal{Q}_p(z_i)\big)^{(p-2)/p}$

and $\|\cdot\|_{p,\boldsymbol{w}} := \|\mathrm{diag}(\boldsymbol{w}) \cdot \|_p$.

Consider the $\ell_{p,\boldsymbol{w}}$−distortion: $\boxed{\|\mathcal{Q}_p(\boldsymbol{z}) - \boldsymbol{z}\|_{p,\boldsymbol{w}}^p}$

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For $p = 2$, $\boldsymbol{w} = \mathbf{1} \Rightarrow$ Panter-Dite formula (**DC**)

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For $p = 2$, $\boldsymbol{w} = \mathbf{1} \Rightarrow$ Panter-Dite formula **(DC)**

For $p \to \infty$, then $w_i \to \mathcal{G}'(\mathcal{Q}_p(z_i)) \simeq_B \frac{\alpha}{\alpha_{k(i)}}$

$\Rightarrow \|\mathcal{Q}_p(\boldsymbol{z}) - \boldsymbol{z}\|_{p,\boldsymbol{w}} \to \max_i \frac{\alpha}{\alpha_{k(i)}}|\mathcal{Q}_p(z_i) - z_i| \leqslant \alpha/2$ **(QC)**

# Distortion Estimator in $\ell_{p,\boldsymbol{w}}$

If $\boldsymbol{z} \sim \mathcal{N}^{M\times 1}(0, \sigma_0^2)$, writing $\|\cdot\|_{p,\boldsymbol{w}} := \|\mathrm{diag}(\boldsymbol{w}) \cdot\|_p$,

$$\boxed{\|\mathcal{Q}_p(\boldsymbol{z}) - \boldsymbol{z}\|_{p,\boldsymbol{w}}^p \underset{B,M}{\simeq} M \, \frac{2^{-Bp}}{(p+1)\, 2^p} \, \|\varphi_0\|_{1/3} \; =: \; \epsilon_p^p}$$

with $\epsilon_2^2 = \frac{M\, 2^{-2B}}{12} \|\varphi_0\|_{1/3}$ and $\epsilon_\infty = \alpha/2$ !

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*Equi-distortion principle:* (asymptotically in B and M)

Validation of $p$-levels and weights at any $p \geqslant 2$ ? For $\boldsymbol{z} \sim \mathcal{N}^{M\times 1}(0, \sigma_0)$, equal contribution of each bin to global $\ell_{p,\boldsymbol{w}}$−distortion $\|\mathcal{Q}_p[\boldsymbol{z}] - \boldsymbol{z}\|_{p,\boldsymbol{w}}$!

$\rightarrow$ bin contribution: $\|\varphi_0\|_{1/3}\,\frac{2^{-B(p+1)}}{(p+1)2^p}$

# 4. Reconstructing ...

# New $\ell_{p,\boldsymbol{w}}-$ Constraints

Given $\boldsymbol{x} \in \mathbb{R}^N$, testing $\boldsymbol{u} \in \mathbb{R}^N$ with

$$\|\boldsymbol{\Phi u} - \mathcal{Q}_p(\boldsymbol{\Phi x})\|_{p,\boldsymbol{w}} \ \leqslant \ \epsilon_p \qquad (\mathbf{D}_p\mathbf{C})$$

<u>Lemma</u>: Asymptotically in $B$ and $M$:

$$\mathrm{D_2C} \ \equiv \ \mathrm{DC} \quad \text{and} \quad \mathrm{D_\infty C} \ \equiv \ \mathrm{QC}.$$

# New Reconstructions

Generalized Basis Pursuit DeNoise (GBPDN):

$$\boxed{\operatorname{argmin}_{\boldsymbol{u} \in \mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{y} - \boldsymbol{\Phi u}\|_{p,\boldsymbol{w}} \leqslant \epsilon}$$

Dequantization: $\mathcal{Q}_p[\boldsymbol{\Phi x}]$ $\quad w_i(p) = \mathcal{G}'^{(p-2)/p}(\mathcal{Q}_p[z_i])$

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Stabilization: $\boldsymbol{y}=\boldsymbol{\Phi x}+\boldsymbol{n},\quad n_i\sim_{\text{iid}}\mathrm{GGD}(0,\alpha_i,p)\propto e^{-|t/\alpha_i|^p}$

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Dequantization: $\mathcal{Q}_p[\boldsymbol{\Phi}\boldsymbol{x}]$ $\quad w_i(p) = \mathcal{G}'^{(p-2)/p}(\mathcal{Q}_p[z_i])$

Stabilization: $\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{n}, \quad n_i \sim_{\text{iid}} \mathrm{GGD}(0, \alpha_i, p) \propto e^{-|t/\alpha_i|^p}$

Forgetting $\boldsymbol{y}$ and $\boldsymbol{w}$ origins,
is GBPDN provably robust/stable?

# Need more general tools... (1/2)

- Well ...

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- Well ... a General $\mathrm{RIP}(\ell_{p,\boldsymbol{w}}, \ell_2 | K, \delta, \mu)$ ☺

$\exists\ \mu > 0,\ \ \delta \in (0,1)$

$$\sqrt{1-\delta}\,\|\boldsymbol{v}\|_2 \ \leqslant\ \tfrac{1}{\mu}\|\boldsymbol{\Phi}\boldsymbol{v}\|_{p,\boldsymbol{w}} \ \leqslant\ \sqrt{1+\delta}\,\|\boldsymbol{v}\|_2$$

for all $K$ sparse signals $\boldsymbol{v}$.

# Need more general tools... (2/2)

- Existence? Well, again Gaussian $\boldsymbol{\Phi} \in \mathcal{N}^{M\times N}(0,1)$
  - Converging Moment (CM):

$$\exists\ 0 < \rho_p^{\min} < \rho_p^{\max} < \infty : \boxed{\rho_p^{\min} \leqslant M^{-1/p}\|\boldsymbol{w}\|_p \leqslant \rho_p^{\max}}$$

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$$\boxed{M^{2/p} \geqslant c\,\delta^{-2} \underbrace{\left(\frac{\rho_\infty^{\max}}{\rho_p^{\min}}\right)^2}_{\theta_p} K \log N/K}$$

and $M \geqslant 2\,(2\theta_p)^p$.

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- BTW, is it consistent?

for $p = 2$ and $w_i = 1$ with $\Pr = q$, and $0$ with $\Pr = 1 - q$

$$\boxed{\theta_p \simeq 1/q \;\Rightarrow\; qM = O(K \log N/K)} \qquad \text{(good!)}$$

# GBPDN Robustness

If $\boldsymbol{\Phi}$ is $\mathrm{RIP}(\ell_{p,\boldsymbol{w}}, \ell_2 | s, \delta_s, \mu)$ for $s \in \{K, 2K, 3K\}$

Then, $\boxed{\|\boldsymbol{x} - \boldsymbol{x}^*\|_2 \leqslant A_p\, \epsilon/\mu + B_p\, \|\boldsymbol{x} - \boldsymbol{x}_K\|_1/\sqrt{K}}$

GBPDN solution

$$A_p = 4\sqrt{1+\delta_{2K}}/(1-\delta_{2K}-C_p) \geqslant 4$$
$$B_p = 2(1+C_p-\delta_{2K})/(1-\delta_{2K}-C_p) \geqslant 2$$

$$\begin{cases} C_p = O\big(\sqrt{(\delta_{2K}+\delta_{3K})(p-2)}\big), & p \gg 2 \\ C_p = \delta_{3K} + O(p-2), & p \simeq 2 \end{cases}$$

with $1 - \delta_{2K} - C_p > 0$

# GBPDN and QCS

Given $\mathcal{Q}_p[\cdot]$, $\boldsymbol{w}(p) \in \mathbb{R}^M_+$ and $\epsilon_p$ as before,
GBPDN robustness provides:

$$\|\boldsymbol{x}^* - \boldsymbol{x}\| \underset{B,M}{\lesssim} 4\,c'\,\frac{2^{-B}}{\sqrt{p+1}} + 2\,e_0(K),$$

with $c' = (9/8)(e\pi/3)^{1/2} < 1.8981$.

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- Of course, no free lunch:

  Gaussian RIP$(\ell_{p,\boldsymbol{w}}, \ell_2|K,\delta,\mu)$ imposes

$$M = O\big((\theta_p\,K \log N/K)^{p/2}\big)$$

$$\theta_p^{p/2} \simeq_{B,M} \sqrt{(p+1)/3}$$

# 5. Numerical Experiments

# Solving GBPDN

$$\text{GBPDN} \quad \Leftrightarrow \quad \min_{\boldsymbol{u}\in\mathbb{R}^N} f(\boldsymbol{u}) + g(\boldsymbol{L}u)$$

Non smooth *lsc* convex functions

- Proximal methods and operator splitting.
- Relaxed Arrow-Hurwicz algorithm [Chambolle, Pock, 2010]
- Projection on $\ell_p-$balls $(p \geqslant 2)$ (Newton method)

# Results

$N = 1024,\ K = 16,\ 2^4$ bits, avg. over 50 trials

Bernoulli-Gaussian $K$-sparse signal model.

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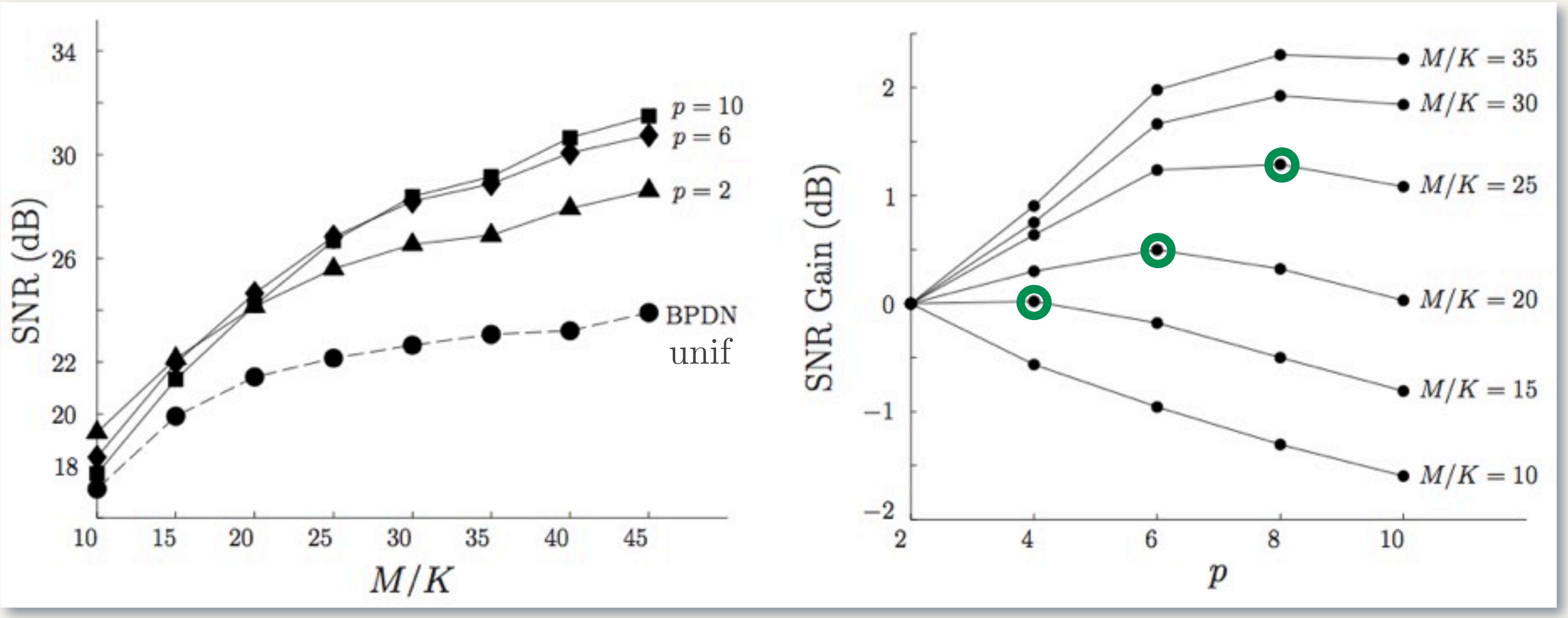

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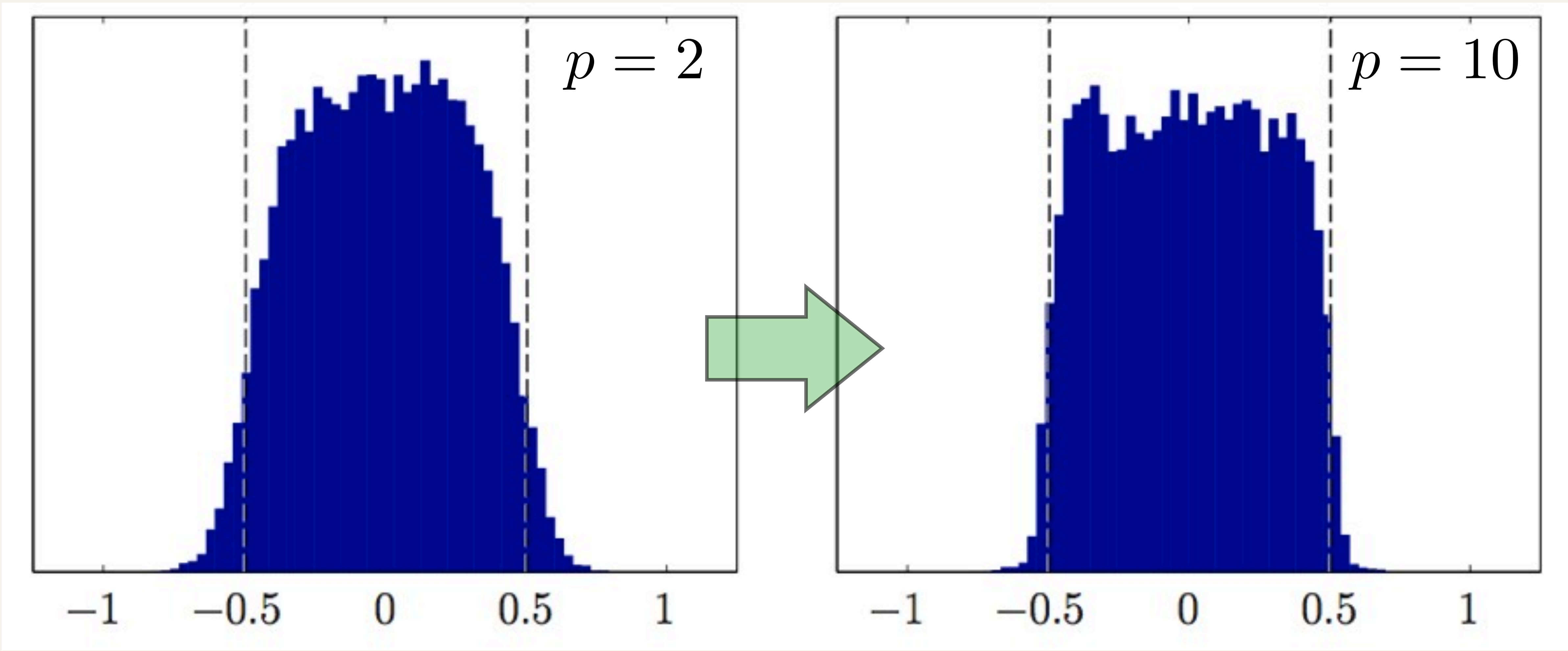
$$\alpha^{-1}\left(\mathcal{G}(\boldsymbol{\Phi x}^*) - \mathcal{G}(\boldsymbol{y})\right)$$



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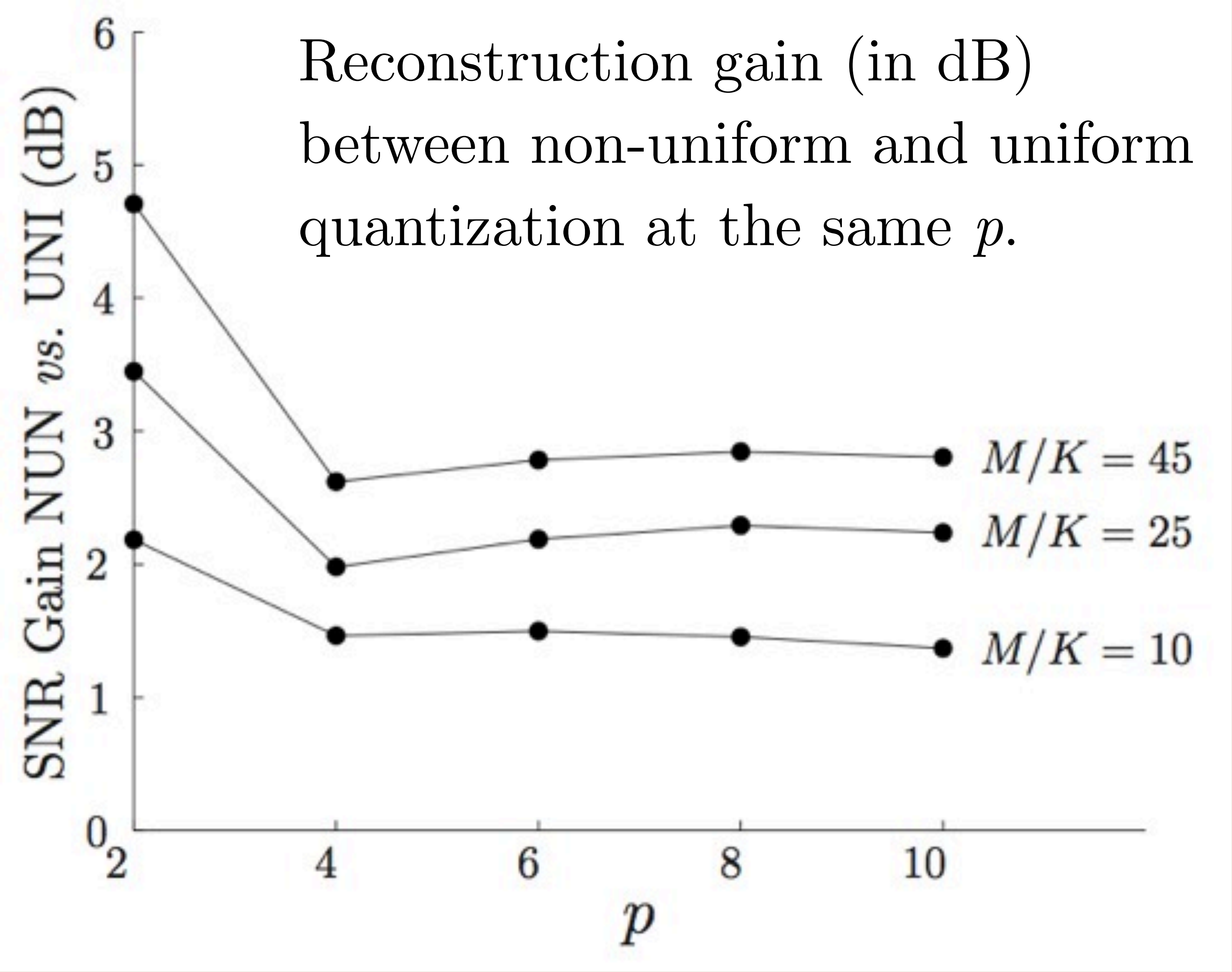

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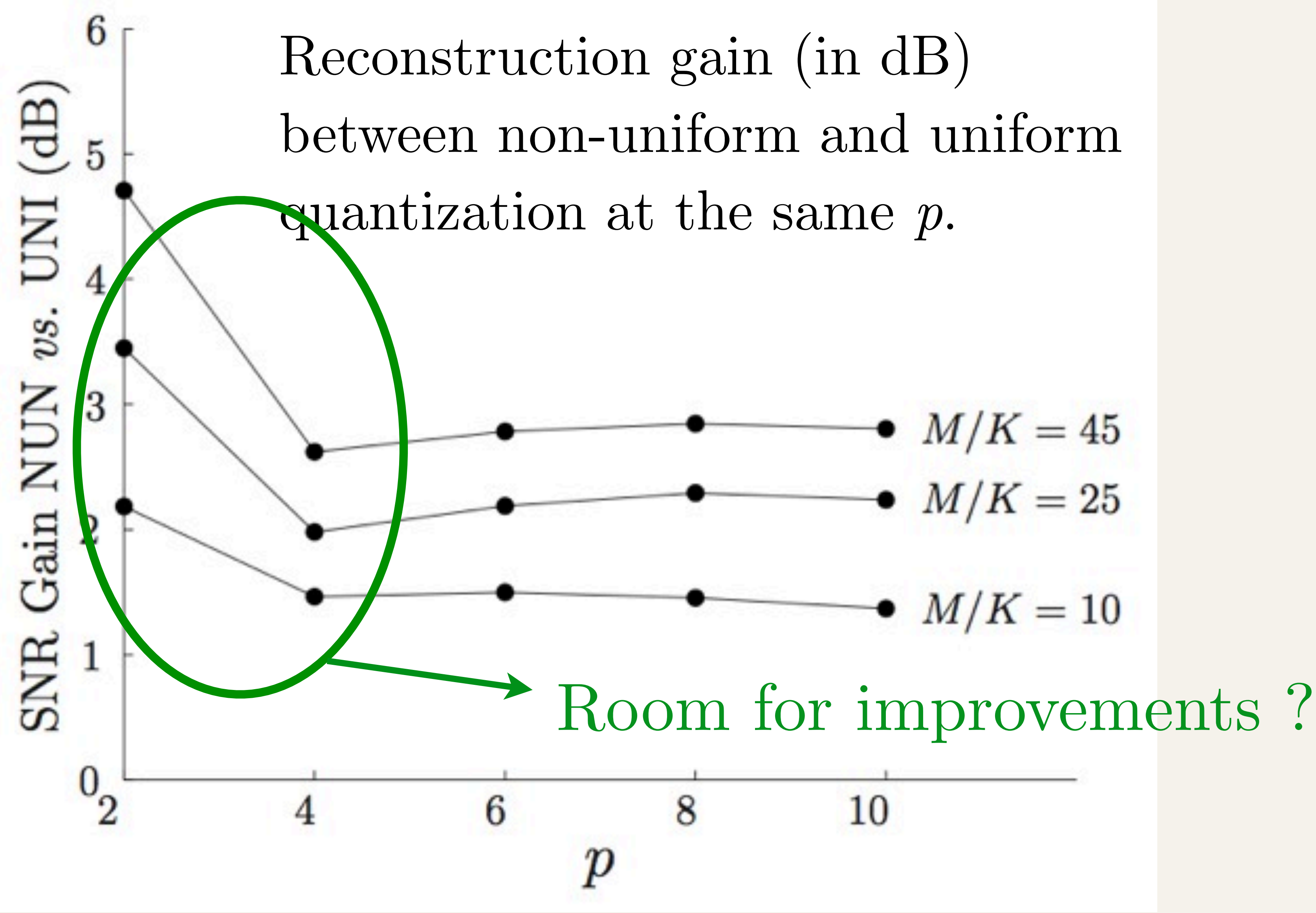

# Conclusions

# Conclusions and perspectives

- Compander formalism $\Rightarrow$ new $\ell_{p,\boldsymbol{w}}$ constraints
- Precise new estimator bound
  given fixed quantization! ( $\ell_2$ optimal)
- New reconstruction: GBPDN
  + use in heteroscedastic GGD noise stabilization
- QCS oversampling principle ...
- <u>Future work</u>:
  - optimize quantizer thresholds
    given $\ell_{p,\boldsymbol{w}}$ distortion
  - correlated measurements?

Thank you.