Compressive interferometric acquisition: from lensless imaging to random beamforming in radioastronomy

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Introduction to random projections and compressive sensing

The multiple use of random projections in "data science"

Object

"$\boldsymbol{x}$"

signal
audio
volume
image
distribution
measure

$\boldsymbol{\Phi}$



Random projection

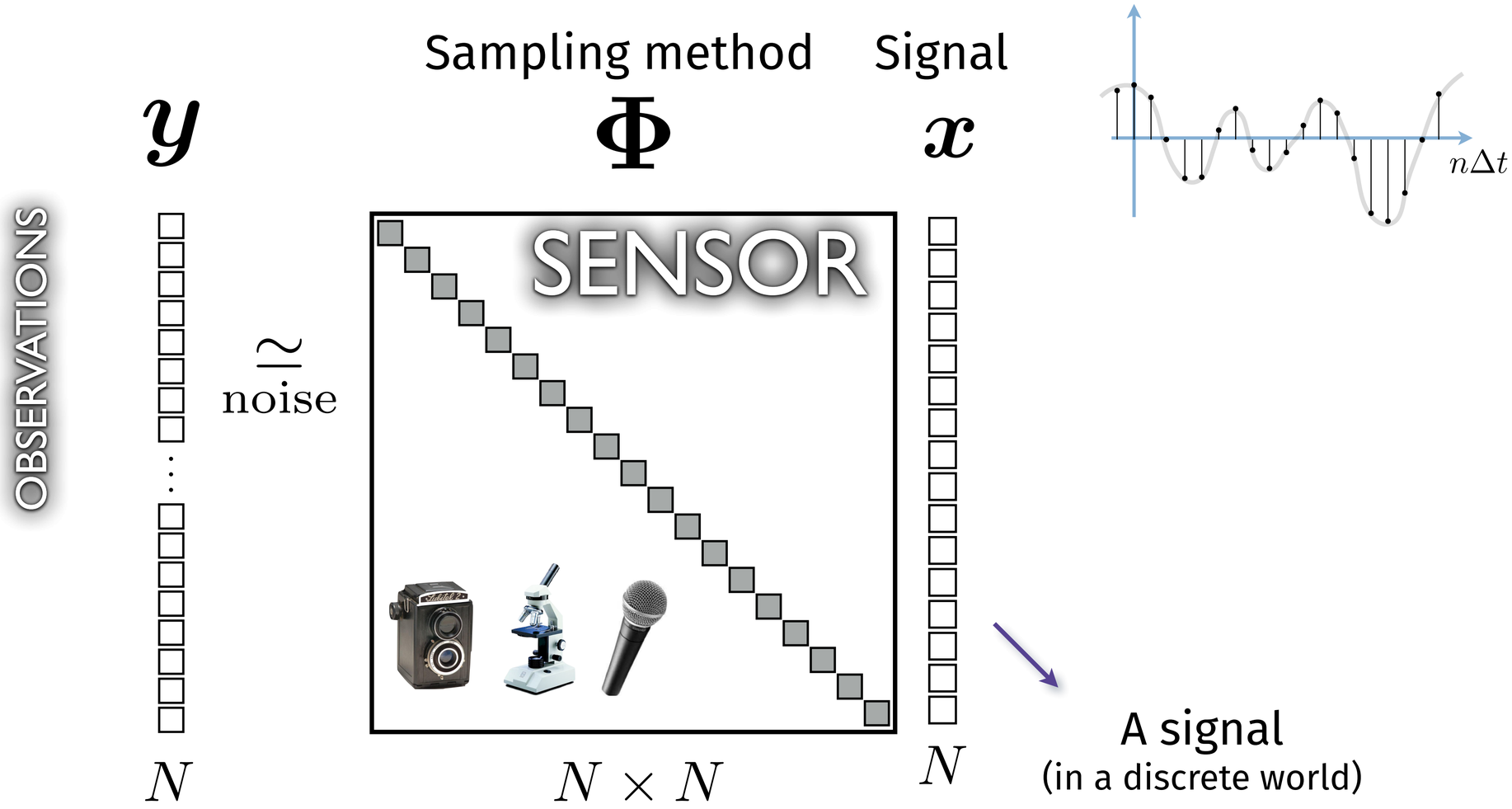
$\boldsymbol{y} = \boldsymbol{\Phi}(\boldsymbol{x})$

smaller dimension
physics-driven
distinctive
compressed
binarized

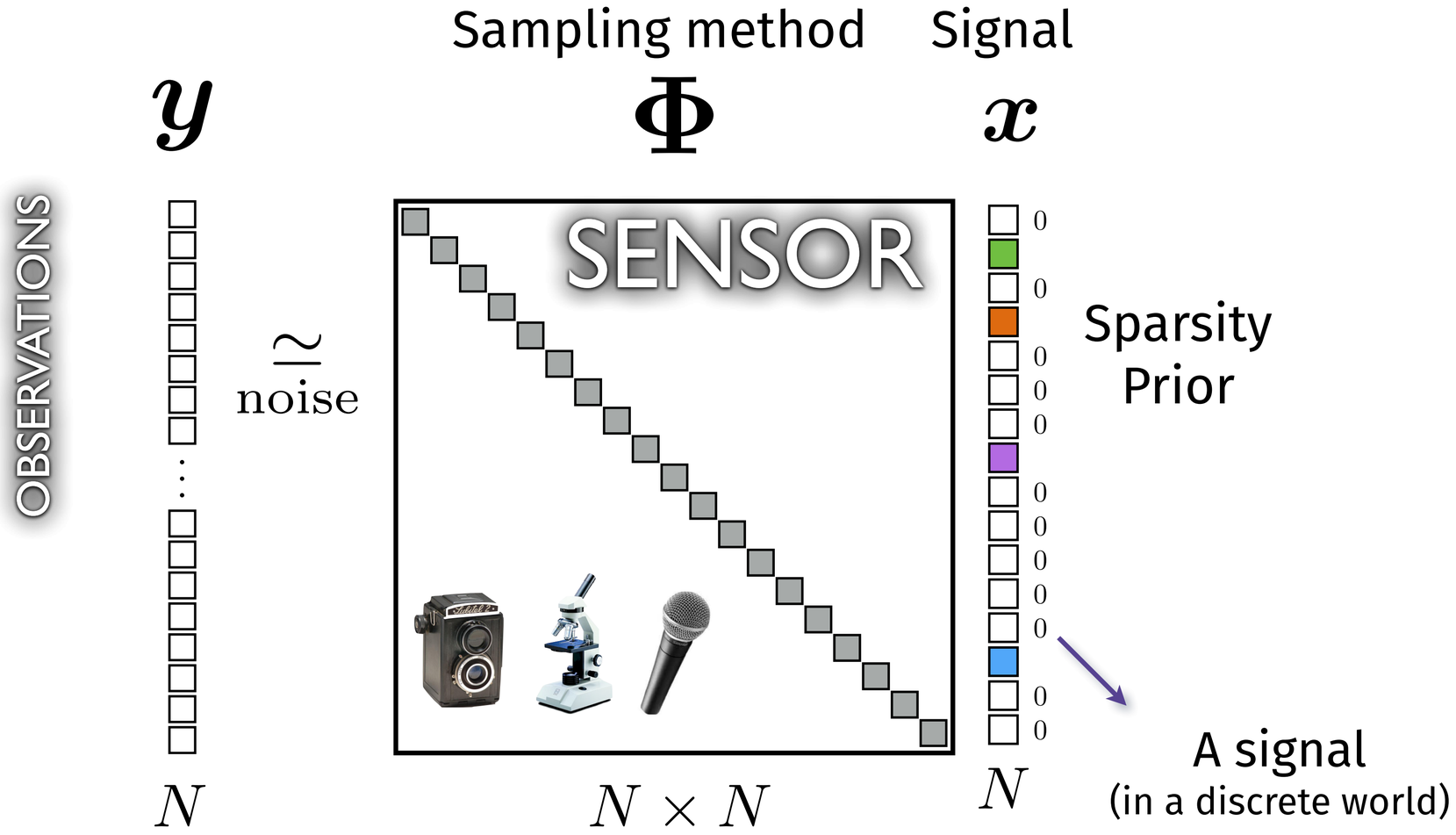
Random "projections" are ubiquitous in:

- Data mining & dimensionality reduction techniques
- Sensing and imaging methods (optics, astronomy, ...)
- Machine learning (sketching, explicit kernel, initialization, ...)
- Randomized numerical methods, ...

From “Nyquist” Sensing...

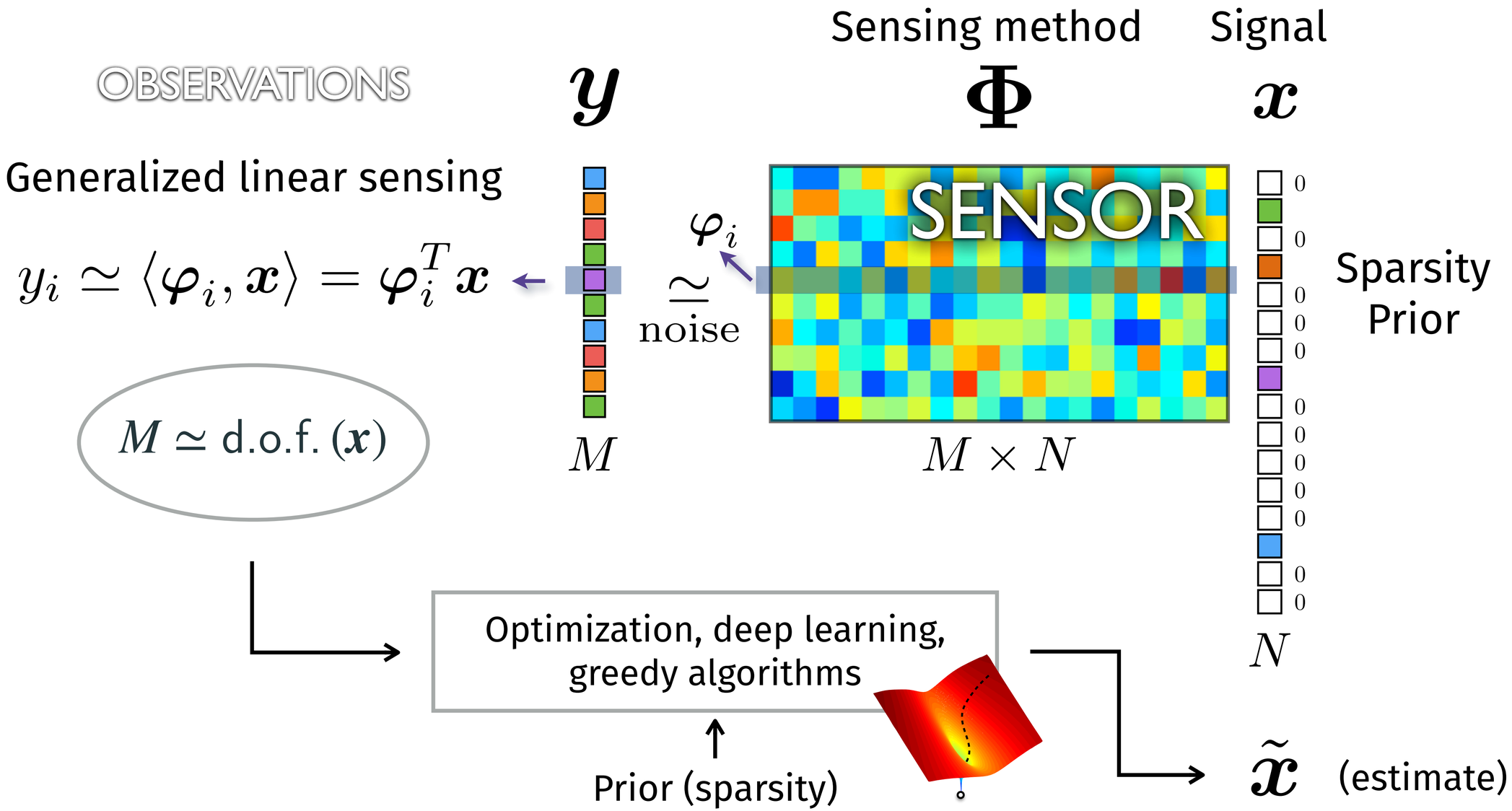


From “Nyquist” Sensing...



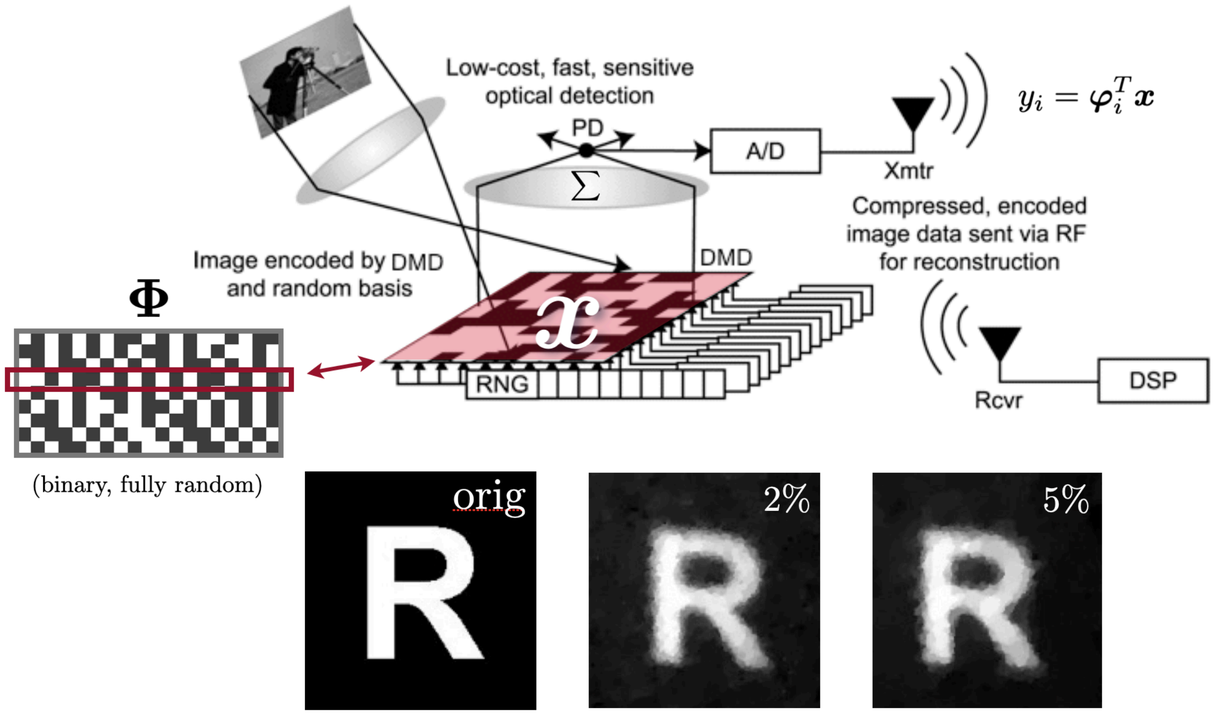
If $\boldsymbol{x}$ is *sparse*, $\boldsymbol{\Phi}$ is a wasteful sensing process!

… to compressive sensing (or computational sensing)

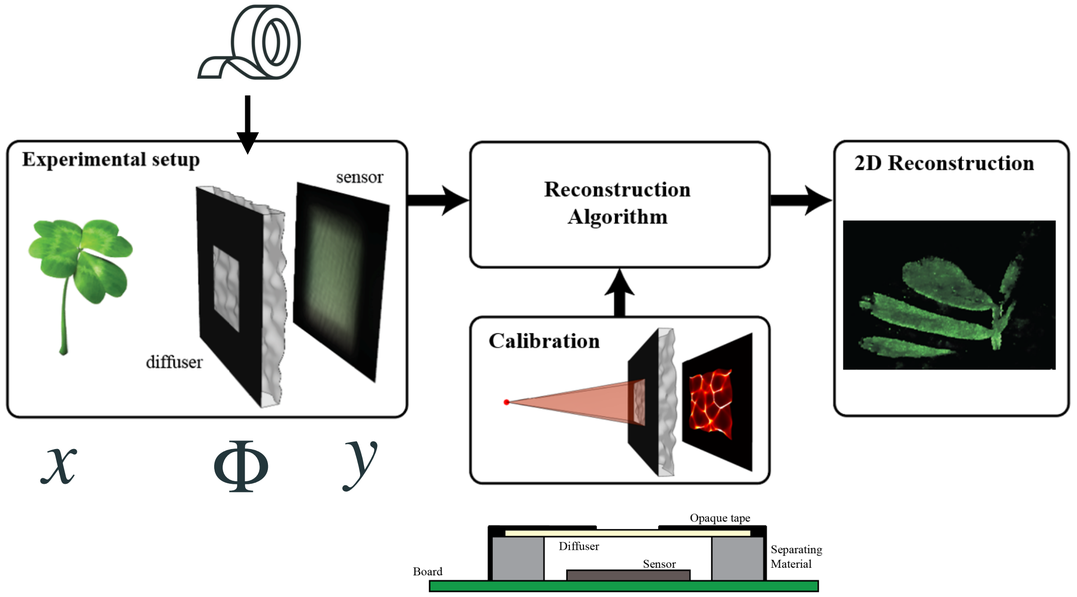


Examples of compressive sensors

“Rice One-pixel Camera”



“Diffuser Cam”



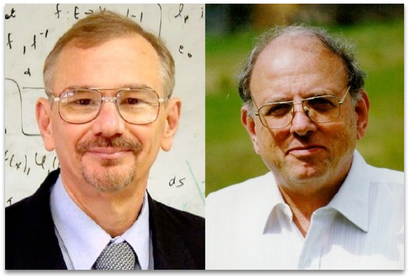
M. F. Duarte, M. A. Davenport, D. Takhar, J. N. Laska, T. Sun, K. F. Kelly and R. G. Baraniuk (2008) Single-pixel imaging via compressive sampling

N. Antipa, G. Kuo, R. Heckel, B. Mildenhall, E. Bostan, R. Ng, L. Waller, L. (2017). DiffuserCam: lensless single-exposure 3D imaging.

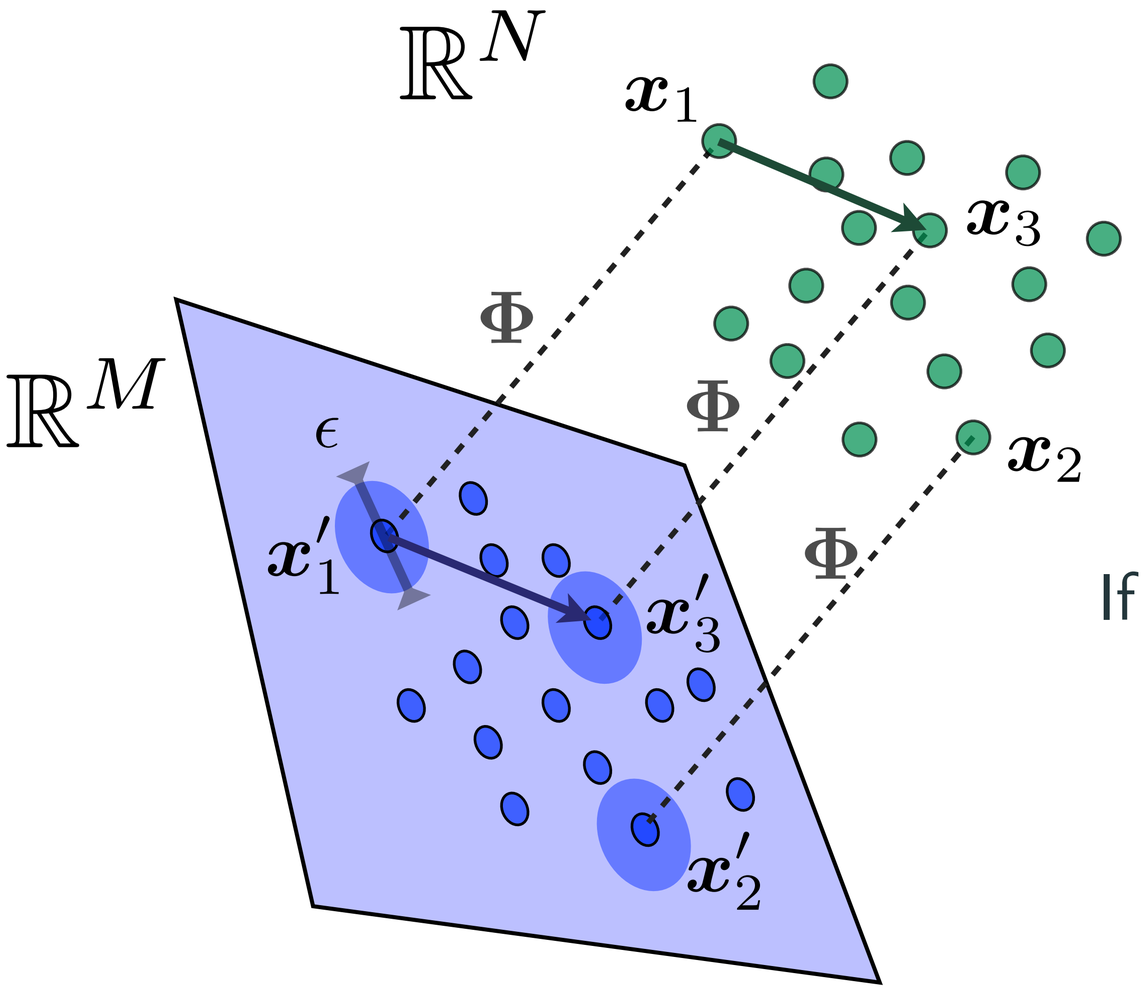
Why is it working? $\rightarrow$ embedding of low-complexity signals

Johnson-Lindenstrauss Lemma (1984)

For many random $M \times N$ matrices $\mathbf{\Phi}$ (e.g., Gaussian, Bernoulli, structured)



Johnson & Lindenstrauss



D points

$\mathcal{S} = \{\boldsymbol{x}_i : 1 \leqslant i \leqslant D\}$

If $M \geq C \log D$, then, with high probability,

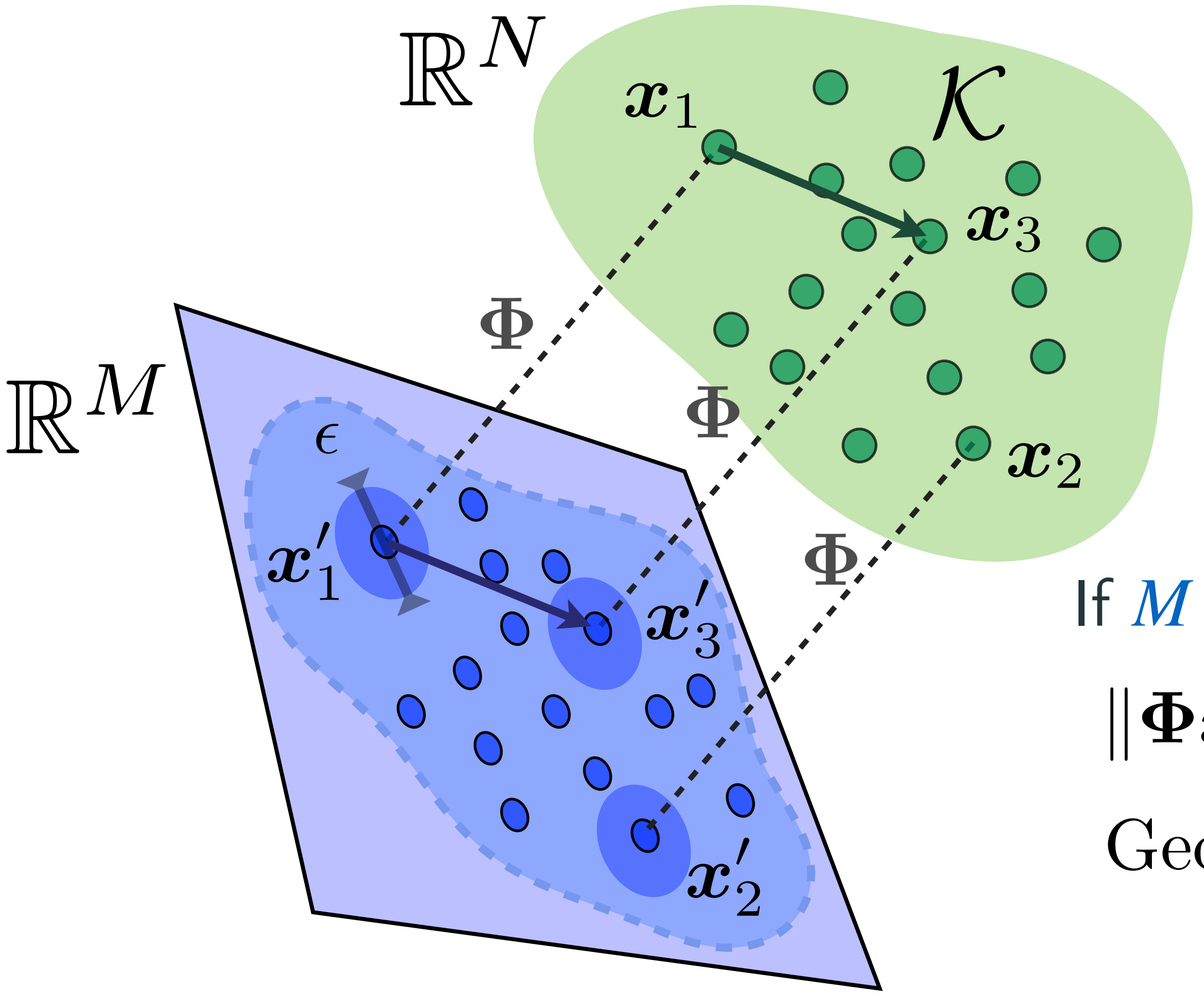
$$\|\mathbf{\Phi}\boldsymbol{x}_i - \mathbf{\Phi}\boldsymbol{x}_j\| \approx_\epsilon \|\boldsymbol{x}_i - \boldsymbol{x}_j\|, \ \forall i, j$$

Why is it working? $\rightarrow$ embedding of low-complexity signals

Generalization:

[Candès, Tao, Romberg, Donoho, ... 2006]

For many random $M \times N$ matrices $\mathbf{\Phi}$ (e.g., Gaussian, Bernoulli, structured)



$\mathcal{K} \equiv$ *low-complexity signal space*

(sparse signals, low-rank matrices, ...)

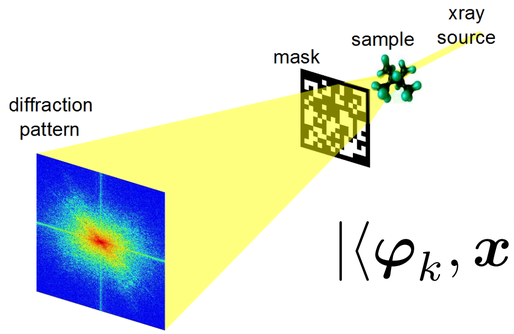
If $M \geq C\,\mathrm{dof}(\mathcal{K})$, then, with high probability,

$$\|\mathbf{\Phi}\boldsymbol{x} - \mathbf{\Phi}\boldsymbol{x}'\| \approx_\epsilon \|\boldsymbol{x} - \boldsymbol{x}'\|, \ \forall \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{K}$$

$$\text{Geometry of } \mathbf{\Phi}(\mathcal{K}) \approx \text{Geometry of } \mathcal{K}$$

Quadratic sensing? Phase retrieval and covariance matrix estimation

Phase retrieval



$$|\langle \boldsymbol{\varphi}_k, \boldsymbol{x} \rangle|^2 = \underbrace{\boldsymbol{\varphi}_k^* \overbrace{(\boldsymbol{x}\boldsymbol{x}^*)}^{\boldsymbol{X}} \boldsymbol{\varphi}_k}_{\boldsymbol{\Phi}(\boldsymbol{X})}$$

Covariance matrix estimation

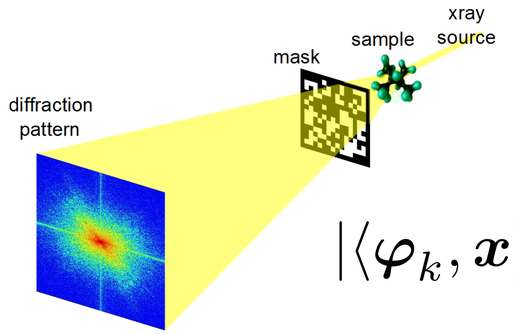
$$\boldsymbol{\Phi}(\mathbb{E}\boldsymbol{x}\boldsymbol{x}^\top) \approx \boldsymbol{\Phi}(\overbrace{\tfrac{1}{N}\textstyle\sum_k \boldsymbol{x}_k \boldsymbol{x}_k^\top}^{\boldsymbol{X}}) = \tfrac{1}{N}\textstyle\sum_k [(\boldsymbol{a}_j^\top \boldsymbol{x}_k)^2]_{j=1}^m$$

$$\text{for } \boldsymbol{x}_k \sim_{\text{iid}} \boldsymbol{x}$$

Chen & Goldsmith (2015) Cai & Zhang (2015)

Quadratic sensing? Phase retrieval and covariance matrix estimation

Phase retrieval



$$|\langle \boldsymbol{\varphi}_k, \boldsymbol{x} \rangle|^2 = \boldsymbol{\varphi}_k^*(\underbrace{\boldsymbol{x}\boldsymbol{x}^*}_{\boldsymbol{X}})\boldsymbol{\varphi}_k$$

$$\boldsymbol{\Phi}(\boldsymbol{X})$$

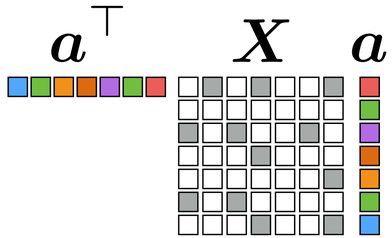
Covariance matrix estimation

$$\boldsymbol{\Phi}(\mathbb{E}\boldsymbol{x}\boldsymbol{x}^\top) \approx \boldsymbol{\Phi}(\underbrace{\tfrac{1}{N}\textstyle\sum_k \boldsymbol{x}_k\boldsymbol{x}_k^\top}_{\boldsymbol{X}}) = \tfrac{1}{N}\textstyle\sum_k [(\boldsymbol{a}_j^\top \boldsymbol{x}_k)^2]_{j=1}^m$$

for $\boldsymbol{x}_k \sim_{\text{iid}} \boldsymbol{x}$

$$\boldsymbol{y} := \boldsymbol{\Phi}(\boldsymbol{X}) := (\underbrace{\boldsymbol{a}_j^\top \boldsymbol{X} \boldsymbol{a}_j}_{\langle \boldsymbol{a}_j\boldsymbol{a}_j^\top, \boldsymbol{X}\rangle_F})_{j=1}^m \in \mathbb{R}^m$$

rank-one



Object to project = symmetric $n \times n$ matrices $\boldsymbol{X} \in \mathbb{R}^{n \times n}$

such as image, volume, covariance matrices, ...

Projection with m random vectors $\{\boldsymbol{a}_j \sim_{\text{iid}} \boldsymbol{a}\}_{j=1}^m \subset \mathbb{R}^n$

(e.g., Gaussian)

Chen & Goldsmith (2015) Cai & Zhang (2015)

Focus on two compressive sensing applications both related to interferometry

(compressive interferometry #1)

Lensless interferometry & rank-one projections



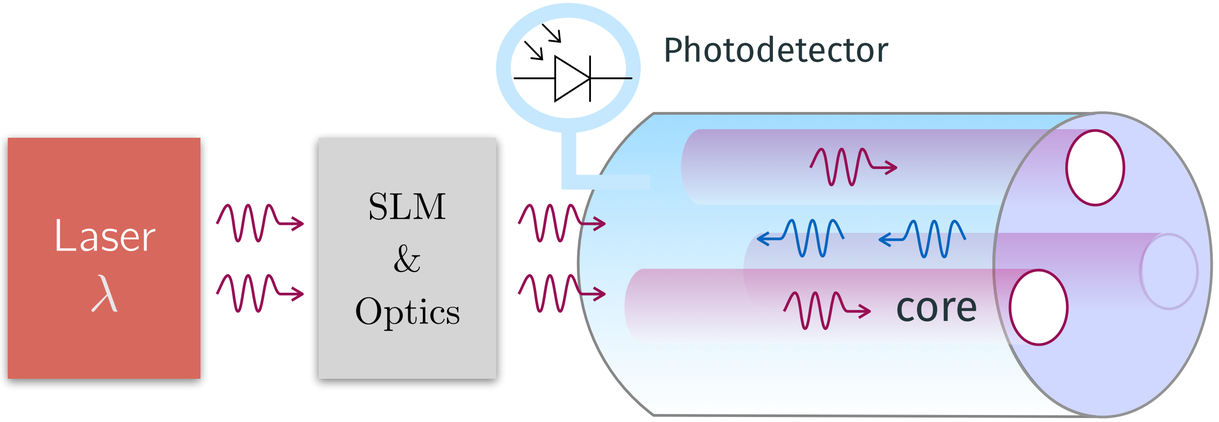




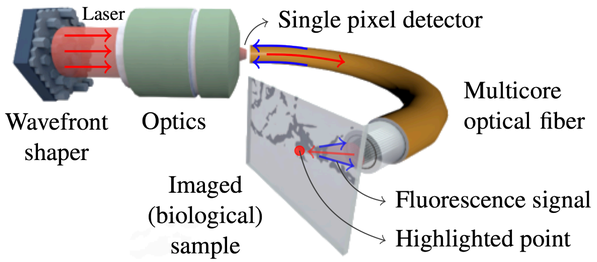
O. Leblanc* L. Jacques* M. Hofer† H. Rigneault† S. Sivankutty‡

*: ISPGroup, INMA, UCLouvain, Belgium. †: Institut Fresnel, France. ‡: PhLAM, France.

Lensless endoscopy: focused mode

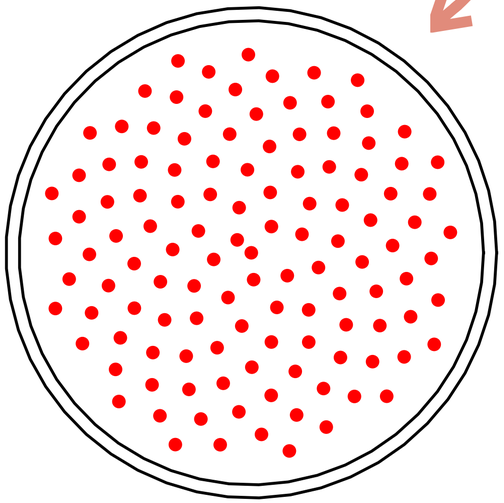


Multicore Fiber



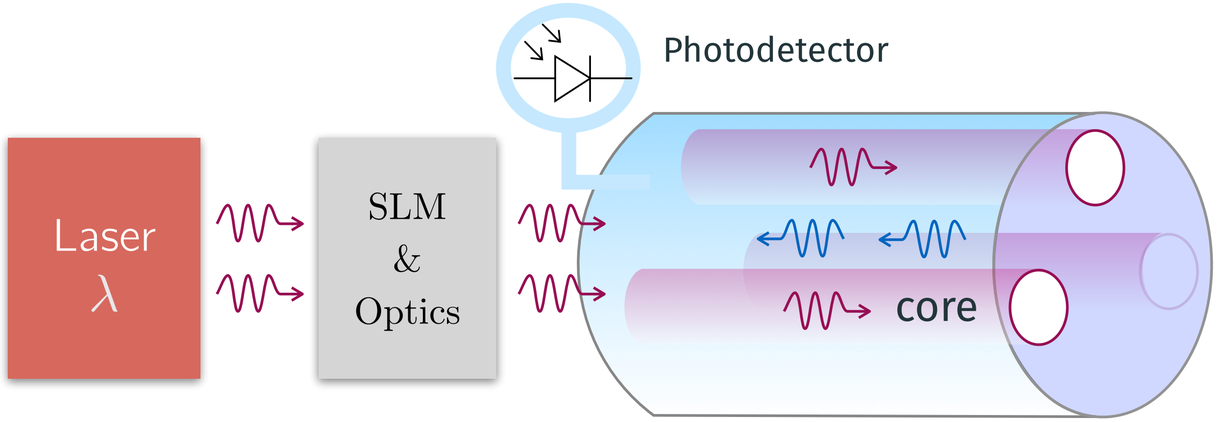
Fermat spiral

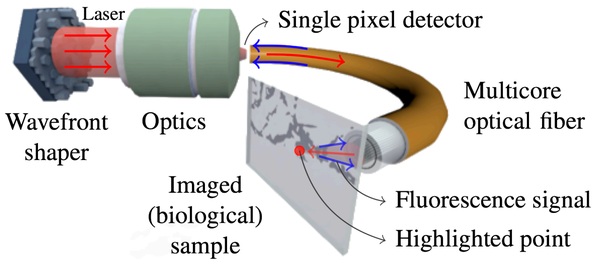




Andresen et al., 2016. Sivankutty et al., 2018.

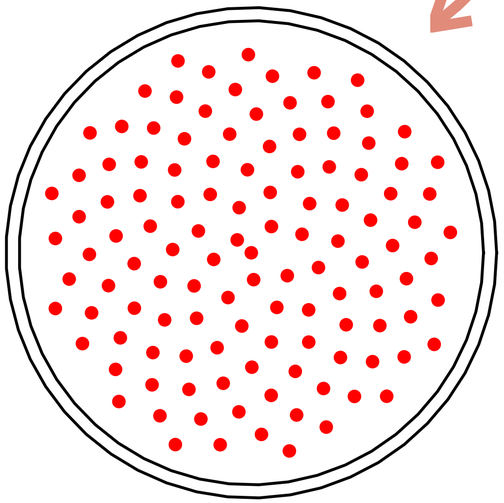
Lensless endoscopy: focused mode





Fermat spiral





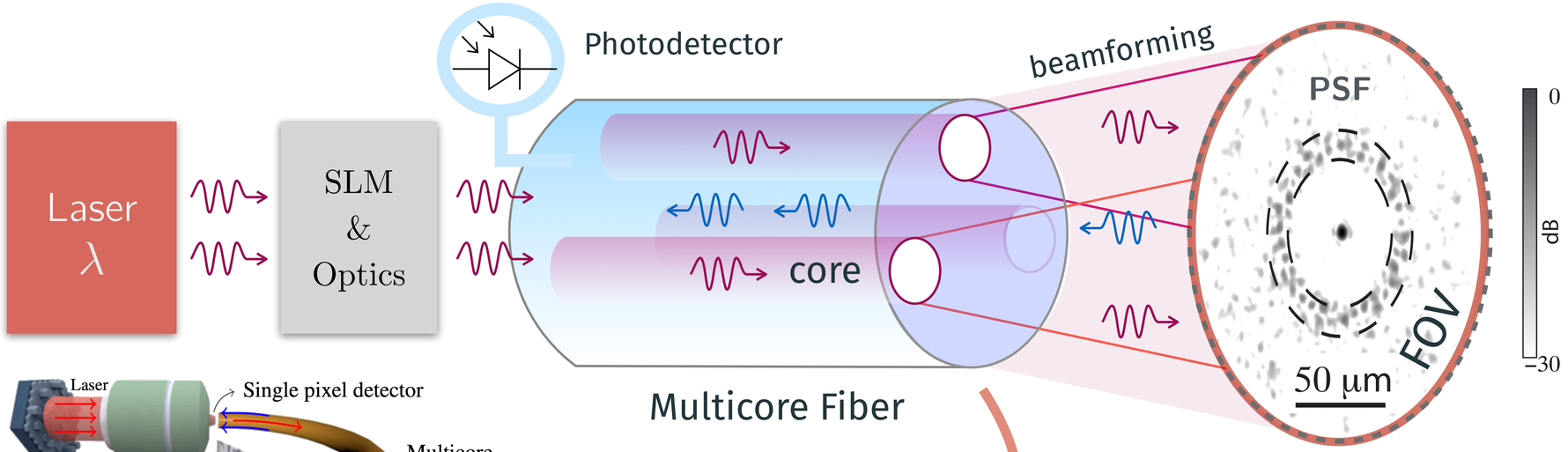


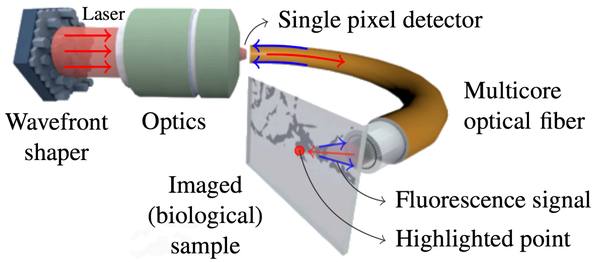
Ivo's spiral



Andresen et al., 2016. Sivankutty et al., 2018.

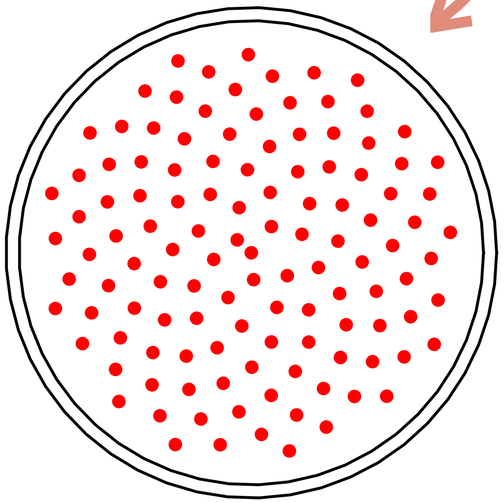
Lensless endoscopy: focused mode





Fermat spiral





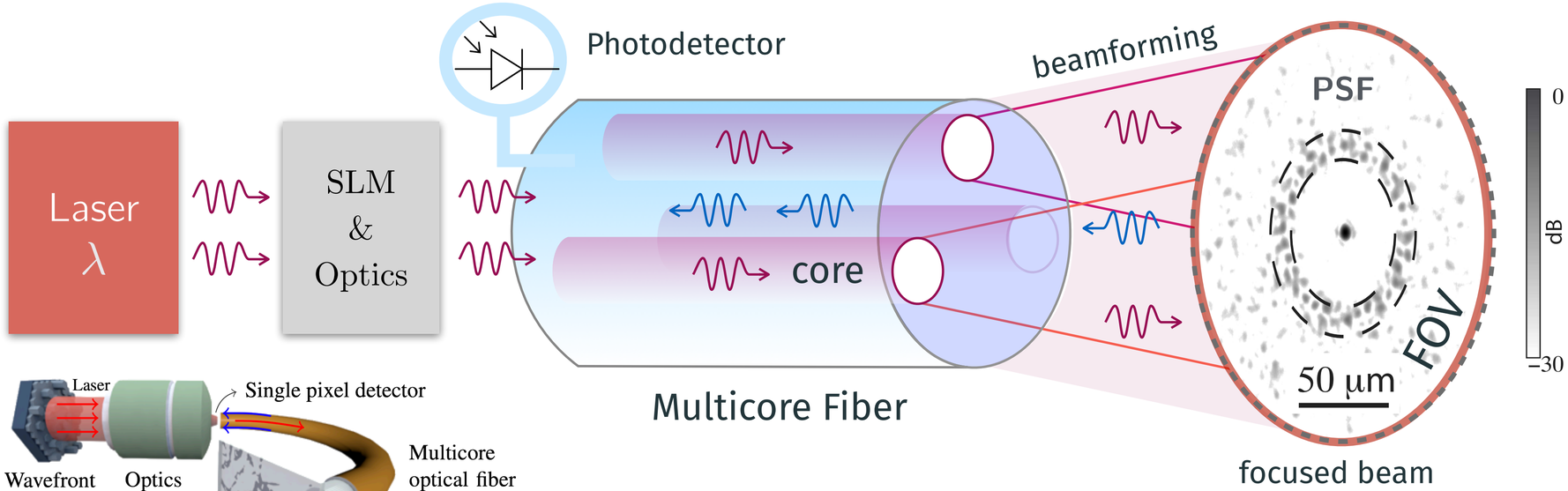


Ivo's spiral

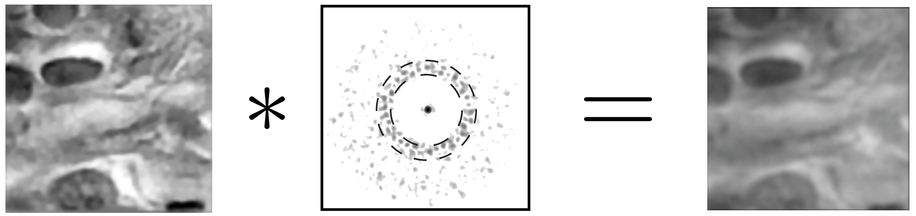


Andresen et al., 2016. Sivankutty et al., 2018.

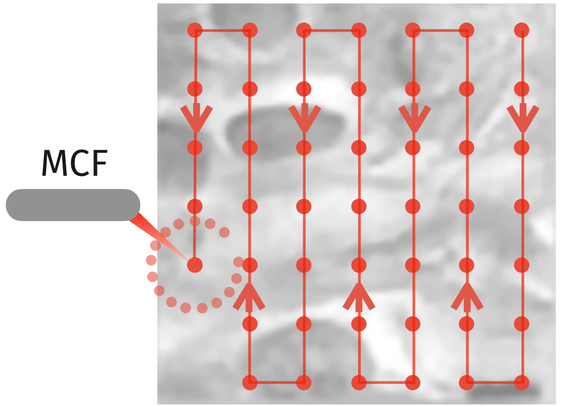
Lensless endoscopy: focused mode



Convolutive sensing model



Raster Scanning

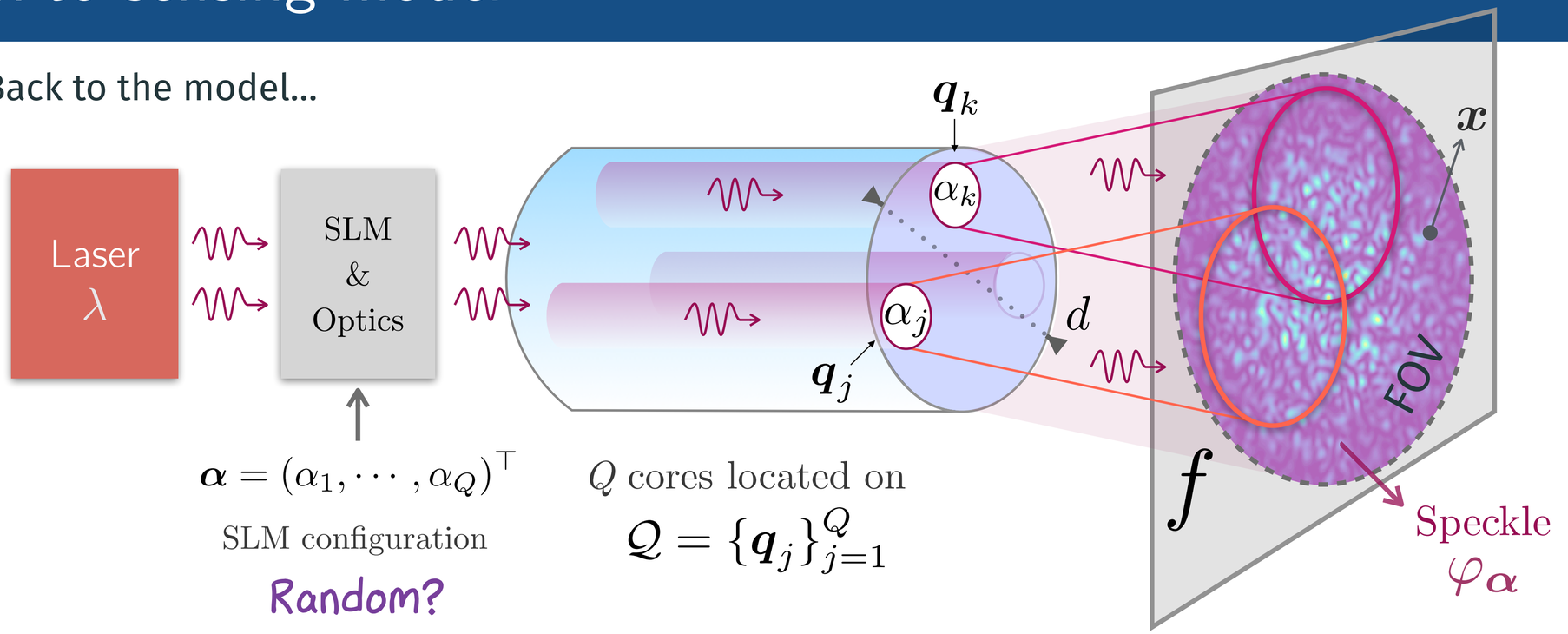


Biological sample

Andresen et al., 2016. Sivankutty et al., 2018.

A closer look to sensing model

Back to the model...



$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_Q)^\top$

SLM configuration

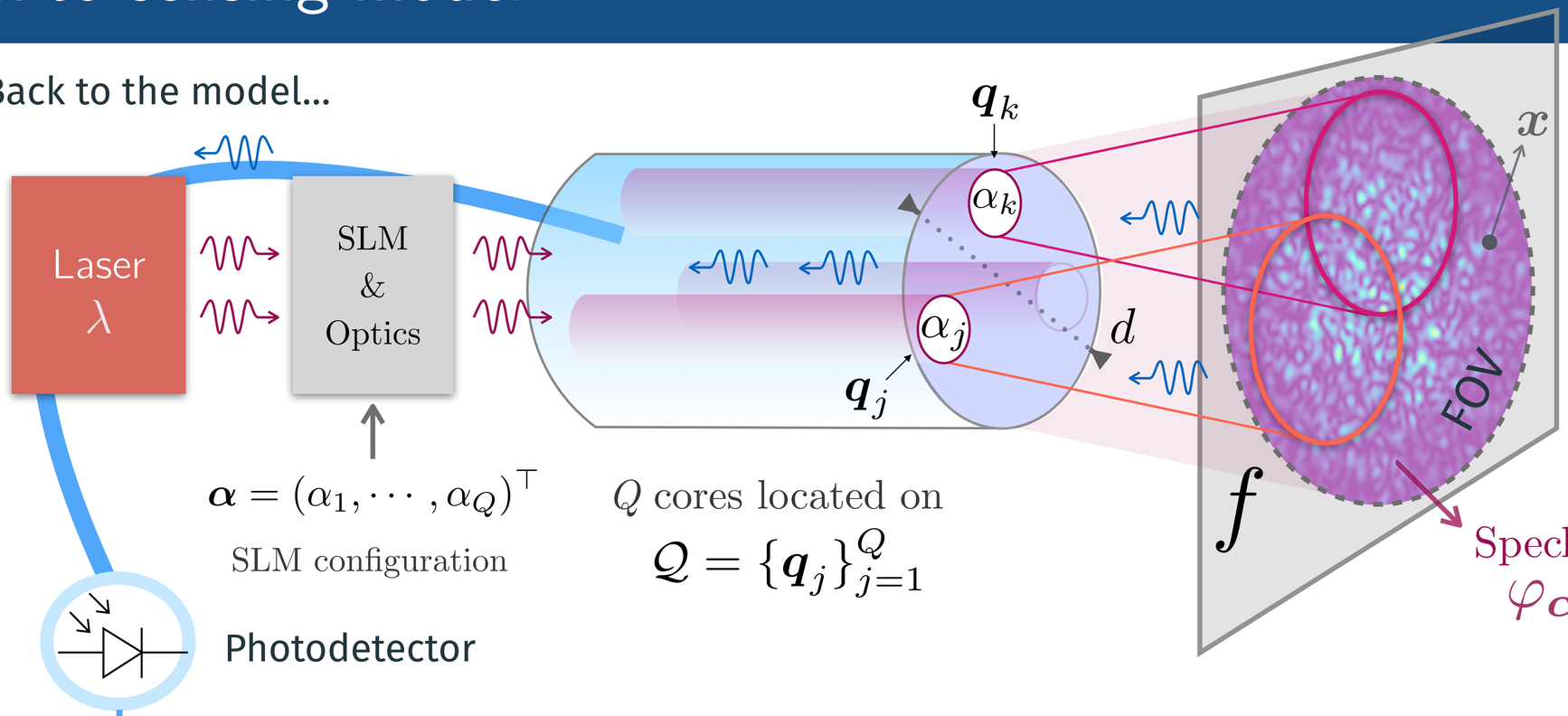
Random?

Q cores located on

$\mathcal{Q} = \{\boldsymbol{q}_j\}_{j=1}^{Q}$

A closer look to sensing model

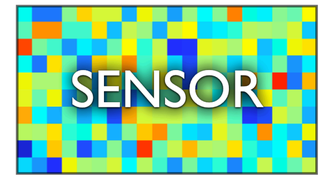
Back to the model...



$$y_{\boldsymbol{\alpha}} \propto \int_{\mathbb{R}^2} \varphi_{\boldsymbol{\alpha}}(\boldsymbol{x}) f(\boldsymbol{x}) \, \mathrm{d}^2\boldsymbol{x} = \langle \varphi_{\boldsymbol{\alpha}}, f \rangle$$

Speckle

$\boldsymbol{y}$ $\simeq$ $\boldsymbol{\Phi}$ $\boldsymbol{x}$

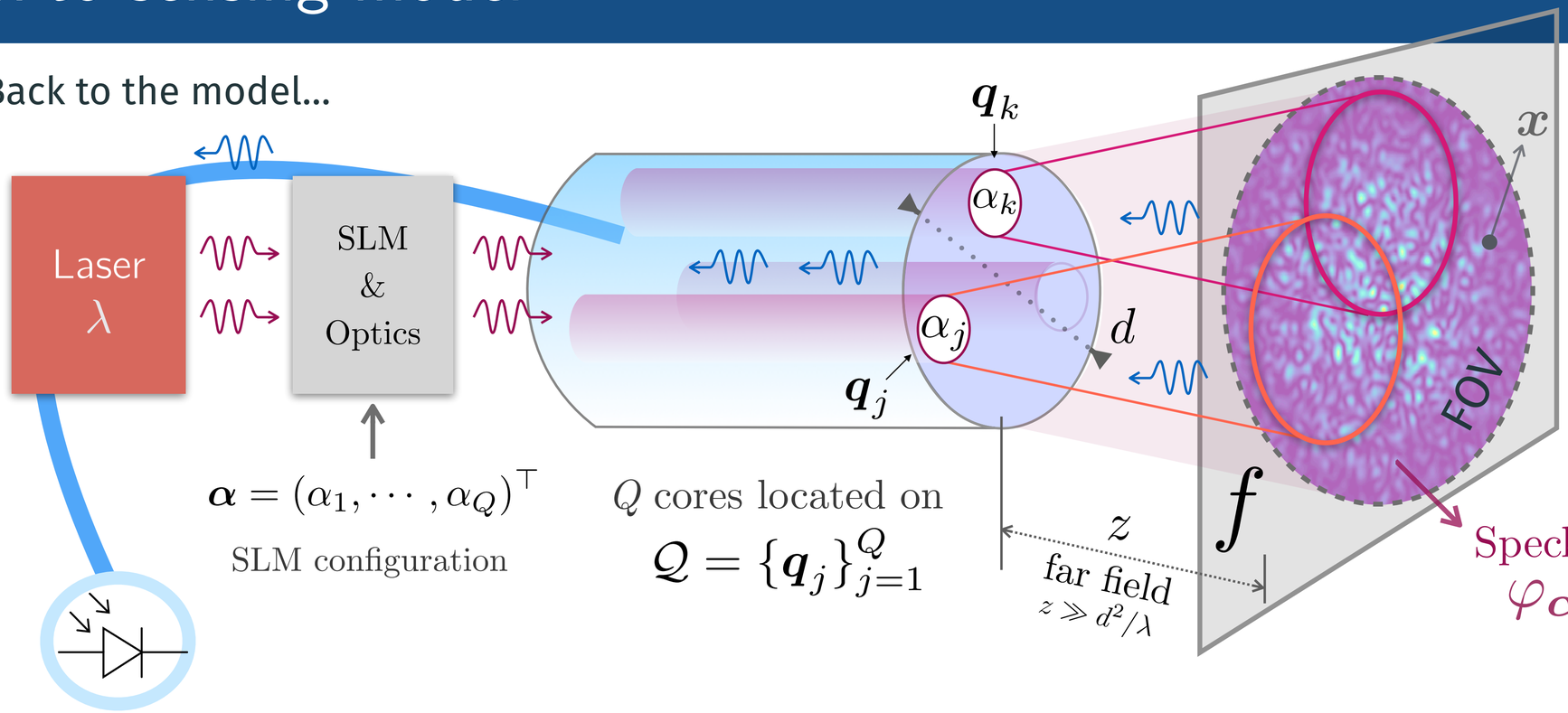


Possible compressive measurement model

S. Guérit, S. Sivankutty, J. A. Lee, H. Rigneault, L. Jacques (2021)

A closer look to sensing model

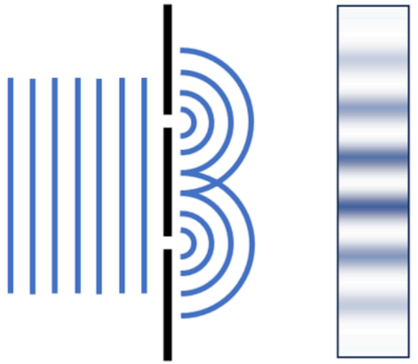
Back to the model...



$$y_\alpha = \langle \varphi_{\boldsymbol{\alpha}}, f \rangle$$

However, speckles are interferences: (Under far-field approximation)

$$\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \propto \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \underbrace{e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top \boldsymbol{x}}}_{\text{Core pair interference}}$$



Better compressive measurement model?

(noiseless) Interferometric sensing model

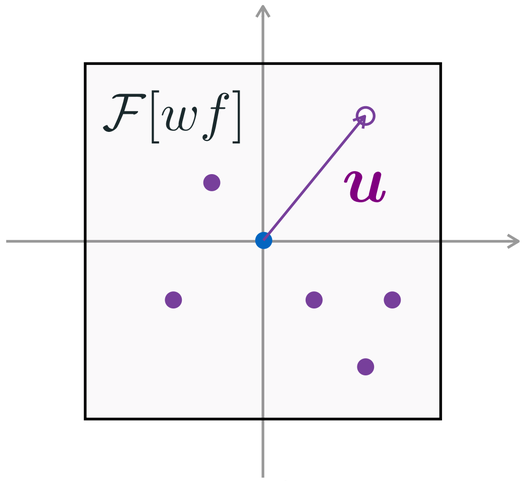
Therefore

$$y_{\boldsymbol{\alpha}} \propto \langle f, \varphi_{\boldsymbol{\alpha}} \rangle = \underset{\text{SLM}}{\boldsymbol{\alpha}^*} \underbrace{\boldsymbol{\mathcal{I}}[f]}_{\mathbb{C}^{Q\times Q}} \underset{\text{SLM}}{\boldsymbol{\alpha}}$$

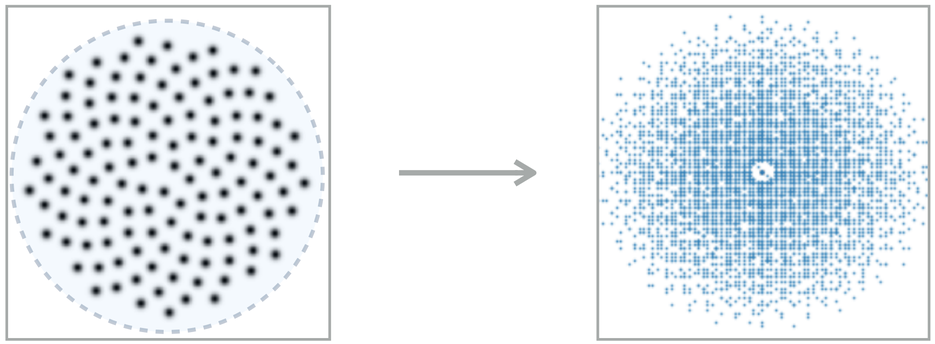
→ ROP!!

with the (Hermitian) *interferometric matrix* relying on the Fourier transform $\mathcal{F}$

$$(\boldsymbol{\mathcal{I}}[f])_{j,k} := \mathcal{F}[f]\big(\tfrac{1}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)\big)$$



$$\boldsymbol{u} \in \mathcal{V} := \tfrac{1}{\lambda z}(\mathcal{Q} - \mathcal{Q})$$



Core arrangement (Fermat spiral)

Core location differences

Interferometric sensing model

Composition of two sensing methods

$$\boldsymbol{y} = (y_{\boldsymbol{\alpha}_1}, \cdots, y_{\boldsymbol{\alpha}_P})^\top = \underbrace{\boldsymbol{\Phi}}_{P \times Q^2}(\underbrace{\boldsymbol{\mathcal{I}}[f]}_{Q \times Q}) + \underbrace{\text{noise}}_{\epsilon},$$

$$\text{with } \boldsymbol{\Phi}(\boldsymbol{M}) := \{\langle \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j^*, \boldsymbol{M} \rangle_{\mathrm{F}}\}_{j=1}^{P}.$$

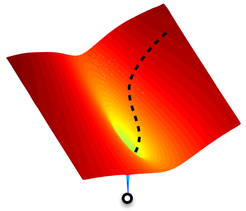
(under specific simplifying assumptions, with $\boldsymbol{f}$ = sampling of f)

If the $\{\boldsymbol{\alpha}_i\}$ are Gaussian, if $\boldsymbol{f}$ is K-sparse

and provided $P = O(K)$ and $Q^2 = O(K)$ (up to logs),

then, with high probability, one can compute† an estimate $\boldsymbol{f}^*$ of $\boldsymbol{f}$ with

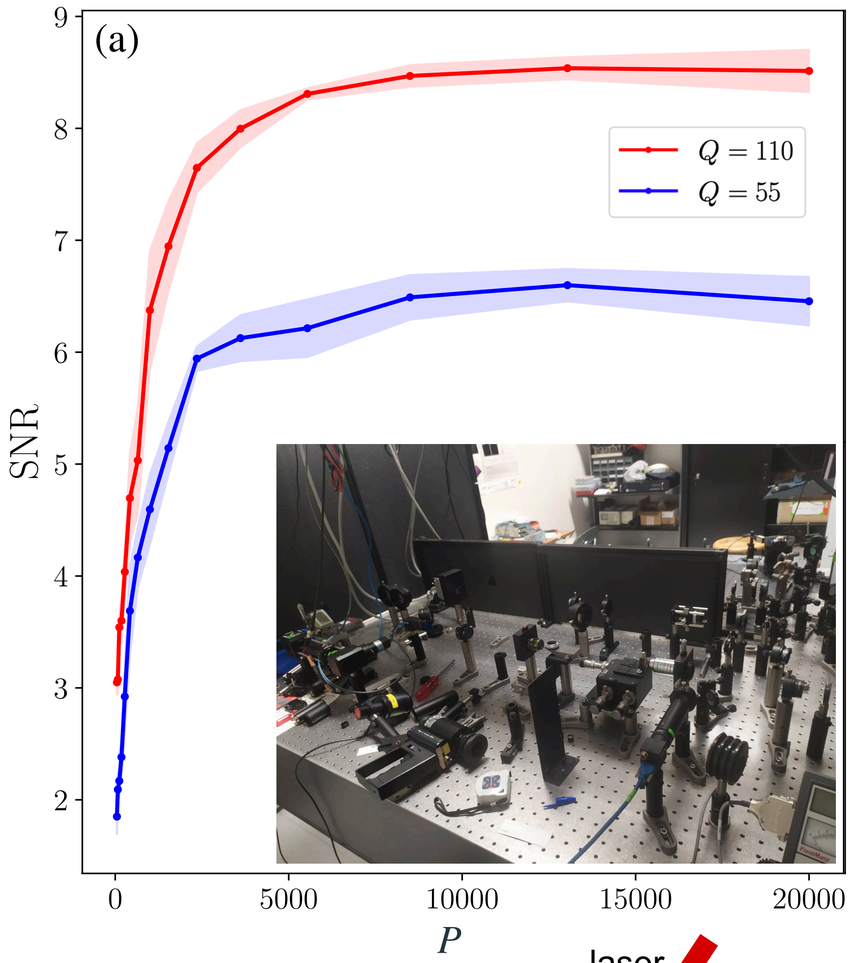
$$\|\boldsymbol{f} - \boldsymbol{f}^*\|_2 = O(\tfrac{\epsilon}{P})$$

†:  + Prior (sparsity)

O. Leblanc, M. Hofer, S. Sivankutty, H. Rigneault, L. Jacques (2023)

Experiments (in Institut Fresnel, France)

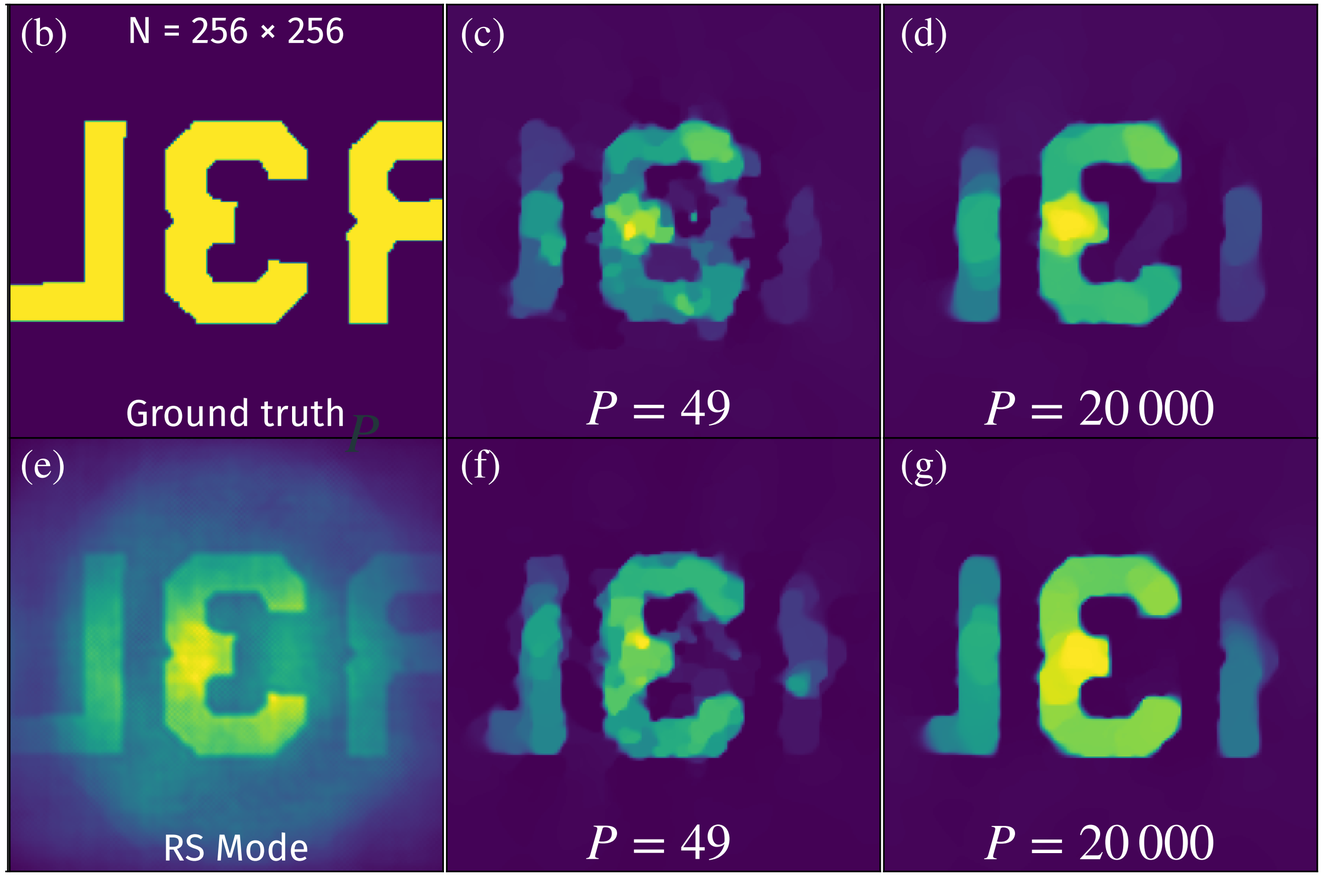
Average SNR (5 trials)

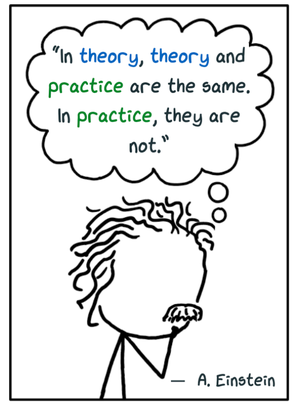


USAF target

$Q = 55$

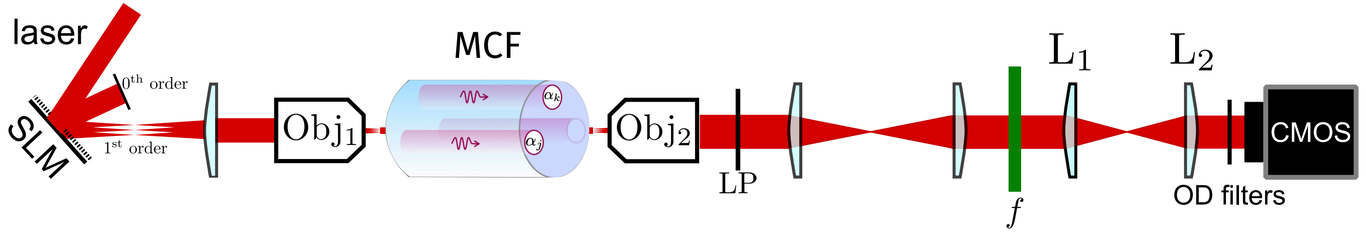
$Q = 110$





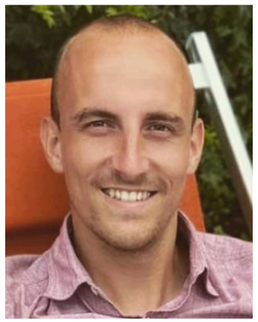
(Adapted from xkcd #1233)





(compressive interferometry #2)

Random beamforming in radio astronomy follows rank-one projection sensing


O. Leblanc*


L. Jacques*

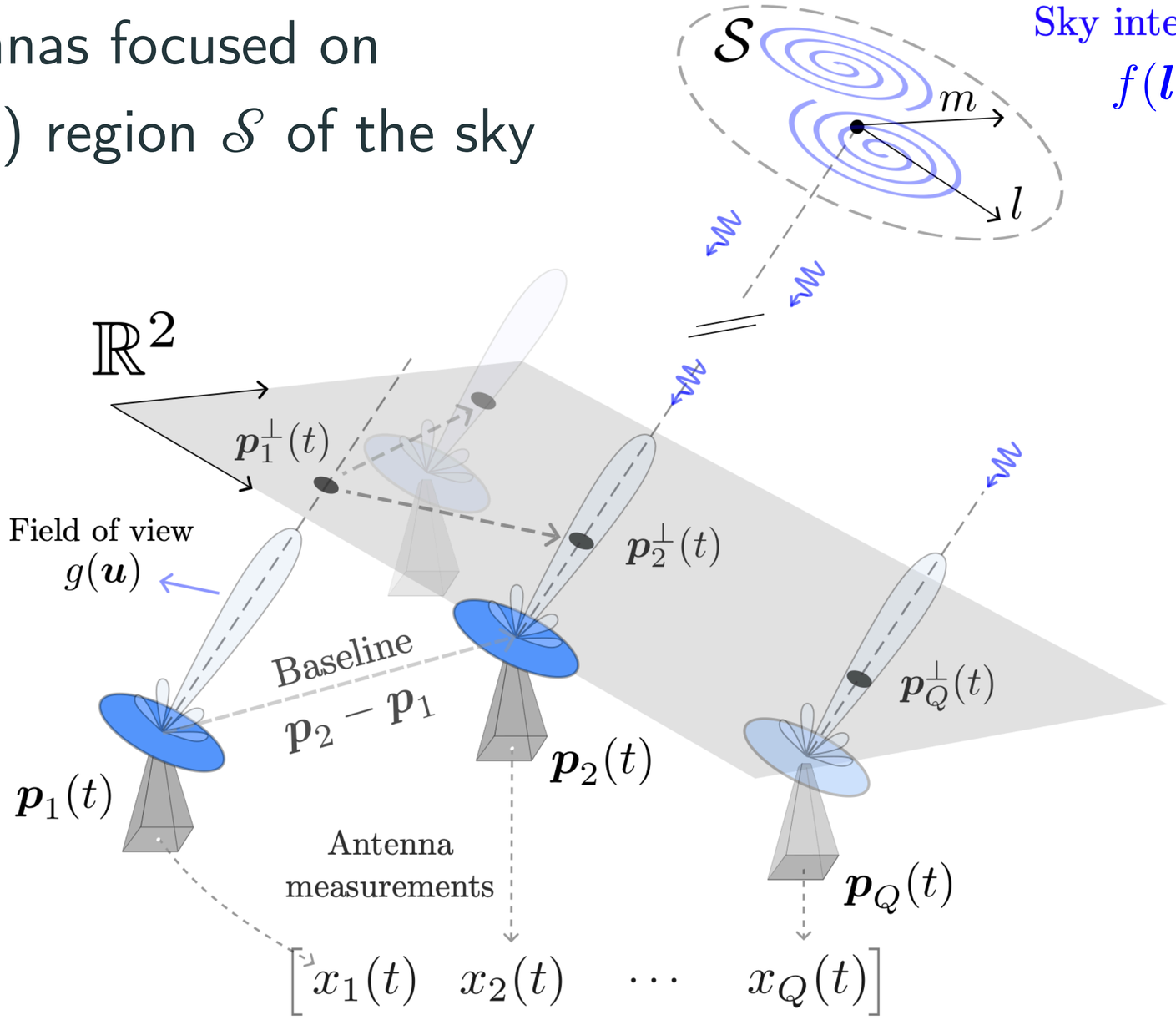

T. Chu†


Y. Wiaux†

*: ISPGroup, INMA, UCLouvain, Belgium. †: Heriot Watt, UK.

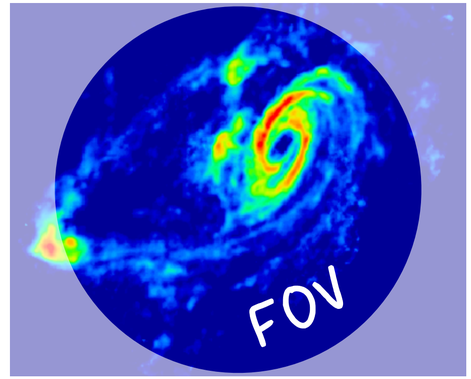
Radio interferometric sensing model

Q antennas focused on
a (small) region $\mathcal{S}$ of the sky



Sky intensity distribution

$$f(\boldsymbol{l}) = f(l, m)$$

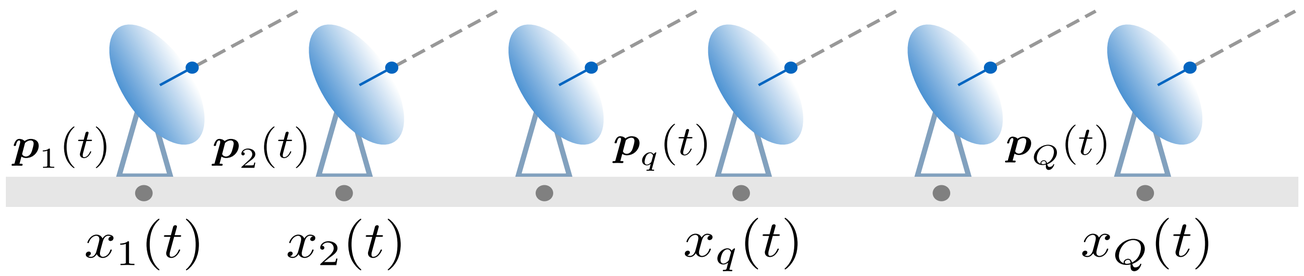


$f(\boldsymbol{l})$



Radio interferometric sensing model

Radio-interferometry principle: let's correlate/interfere all antenna signals



$$\boldsymbol{x}(t) = (x_1(t), \ldots, x_Q(t))^\top$$

$\rightarrow$

$$C_{ij}(t) := \mathbb{E}[x_i(t) x_j^*(t)]$$

$$\boldsymbol{C}(t) := \mathbb{E}[\boldsymbol{x}(t)\boldsymbol{x}^*(t)]$$

$Q \times Q$ covariance

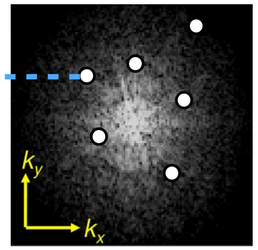
Practically $\rightarrow$ time averaging over B short time intervals (STI)

$$\mathbb{E}[\cdot] \approx \langle \cdot \rangle_{\text{time}}$$

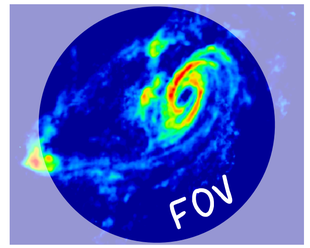
Radio interferometric sensing model

By the Van Cittert-Zernike theorem (VCZ): $\boldsymbol{x}(t) = (x_1(t), \ldots, x_Q(t))^\top$

$$\boldsymbol{C}(t) := \mathbb{E}[\boldsymbol{x}(t)\boldsymbol{x}^*(t)] = \quad \boldsymbol{\mathcal{I}}_{\Omega(t)}[f]$$



$\mathcal{F}[f](\nu)$



$f(\boldsymbol{l})$

visibilities $\boldsymbol{\nu}_{kj}$

2D Fourier

with $(\boldsymbol{\mathcal{I}}_{\Omega(t)}[f])_{jk} := \mathcal{F}[f]\Big(\underbrace{\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}}_{\in \mathcal{V} := \lambda^{-1}(\Omega - \Omega)}\Big)$

$\Omega(t) = \{\boldsymbol{p}_q^\perp(t)\}_{q=1}^Q$

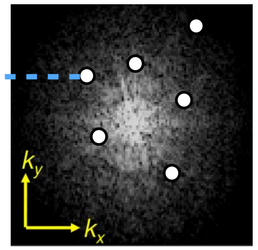


Radio interferometric sensing model

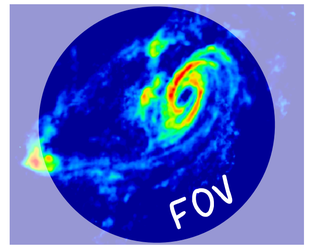
By the Van Cittert-Zernike theorem (VCZ): $\boldsymbol{x}(t) = (x_1(t), \ldots, x_Q(t))^\top$

$$\boldsymbol{C}_b := \underbrace{\frac{1}{I}\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]}_{\text{STI} \to \langle \boldsymbol{x}_b[\cdot]\boldsymbol{x}_b^*[\cdot]\rangle_{T_b}} \approx \quad \boldsymbol{\mathcal{I}}_b[f]$$

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2D Fourier

with $$(\boldsymbol{\mathcal{I}}_b[f])_{jk} := \mathcal{F}[f]\Big(\underbrace{\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}}_{\in \mathcal{V}_b := \lambda^{-1}(\Omega_b - \Omega_b)}\Big)$$

Practically $\to$ time averaging

- B short-time integration intervals T_b (STI)

  with I discrete time instants :

  $$\boldsymbol{x}(t) \to \boldsymbol{x}_b[i] \in \mathbb{R}^Q, \quad \mathbb{E}(\,\cdot\,) \approx \langle\,\cdot\,\rangle_{T_b}$$

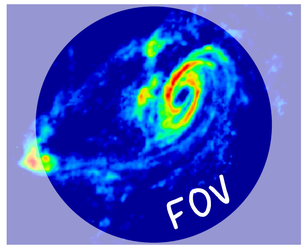
- Approximation:

  over each STI, Earth & visibilities are fixed

Radio interferometric sensing model

By the Van Cittert-Zernike theorem (VCZ): $\boldsymbol{x}(t) = (x_1(t), \ldots, x_Q(t))^\top$

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$f(\boldsymbol{l})$

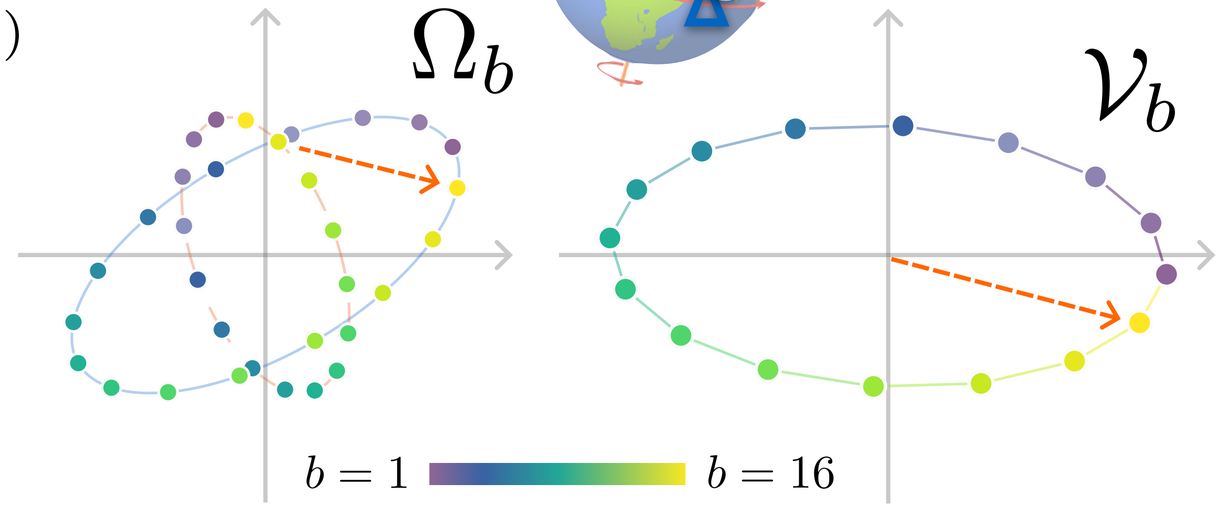


Eppur si muove



with $$(\boldsymbol{\mathcal{I}}_b[f])_{jk} := \mathcal{F}[f]\Big(\underbrace{\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}}_{\in \mathcal{V}_b := \lambda^{-1}(\Omega_b - \Omega_b)} \Big)$$

2D Fourier



Practically → time averaging

- B short-time integration intervals T_b (STI)
  with I discrete time instants :
  $\boldsymbol{x}(t) \to \boldsymbol{x}_b[i] \in \mathbb{R}^Q$, $\mathbb{E}(\,\cdot\,) \approx \langle\,\cdot\,\rangle_{T_b}$
- Approximation:
  over each STI, Earth & visibilities are fixed

$O(Q^2)$ visibilities per STI
$\to O(Q^2 B)$ in total

Challenges in radio-interferometry

Massive data stream 🦣**:**

- $\#\textit{visibilities}\ \mathscr{V} = \cup_{b=1}^{B} \mathscr{V}_b \rightarrow O(Q^2 B)$

  e.g., for the square-kilometer array (SKA)

  $Q = O(10^5)$, $B = O(100)$ $\rightarrow$ Storing $O(10^7)$ visibilities 😱

- Computing $\mathscr{F}[\sigma^\circ](\mathscr{V})$ via $\{\boldsymbol{C}_b\}_{b=1}^{B}$

  $\rightarrow O(IB\,Q^2) = O(10^9 \cdot 10^5) = O(10^{13})$



Challenges in radio-interferometry

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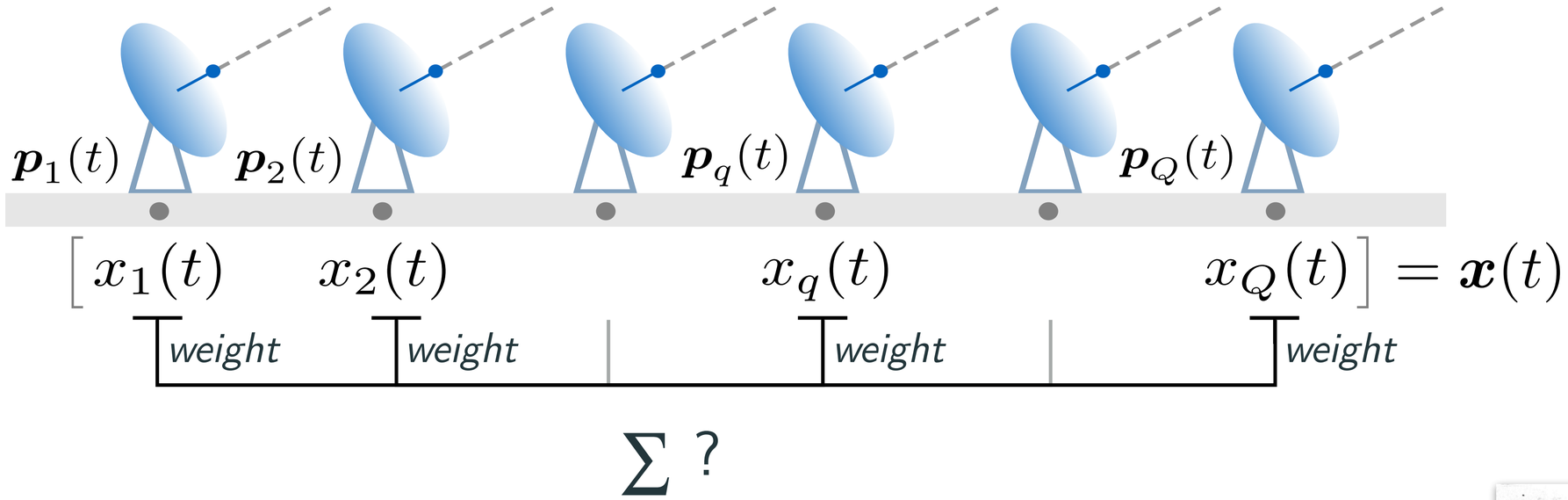


Solution: compressive radio-interferometric (RI) sensing scheme

- Dress up an old scheme, *beamforming*, in new clothes
- Compress measurements at antenna & reconstruction levels
- Supported by theoretical guarantees (under a few simplifications).

Beamforming ≡ rank-one projections of covariance matrix

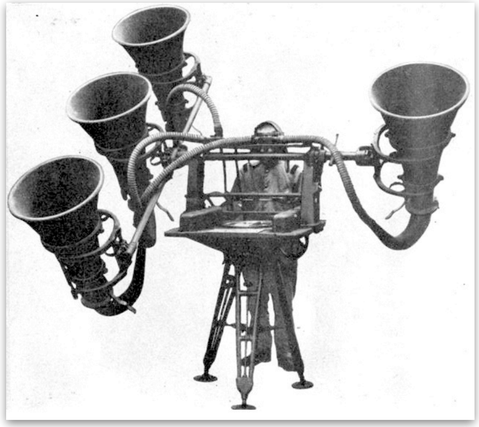
What if we create ***virtual*** antennas? Let's do beamforming



Beamforming ≡

creating virtual antenna for

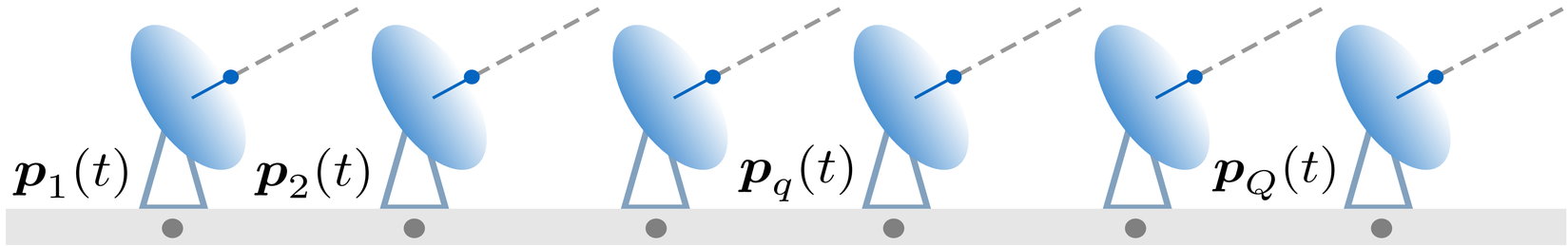
- spatial filtering,
- directionality, ...



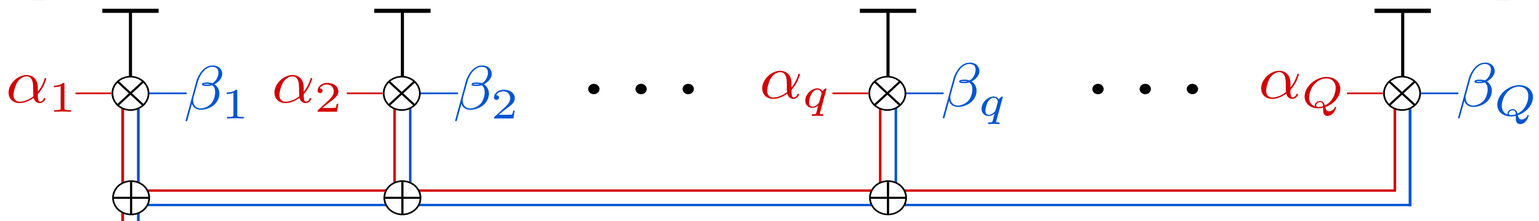
Acoustic radar, Japan (1930)

Beamforming ≡ rank-one projections of covariance matrix

What if we create ***virtual*** antennas? Let's do beamforming



$$\begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_q(t) & \cdots & x_Q(t) \end{bmatrix} = \boldsymbol{x}(t)$$



2 virtual antennas $\mu(t)$, $\nu(t)$

$$\mu(t) = \sum_{q=1}^{Q} \alpha_q^* x_q(t) = \langle \boldsymbol{\alpha}, \boldsymbol{x}(t) \rangle$$

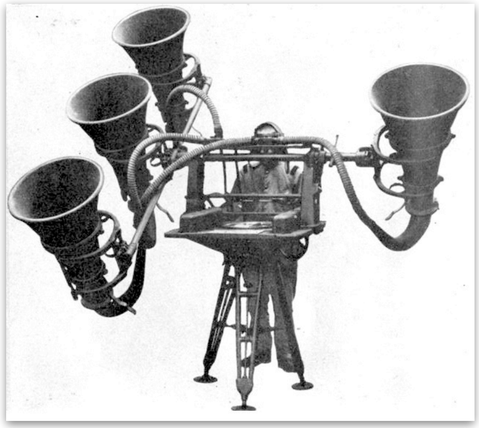
$$\nu(t) = \sum_{q=1}^{Q} \beta_q^* x_q(t) = \langle \boldsymbol{\beta}, \boldsymbol{x}(t) \rangle$$

Beamforming ≡

optimising α_q, β_q for, e.g.,

- spatial filtering,
- directionality, ...

Given Q weights α_q, β_q

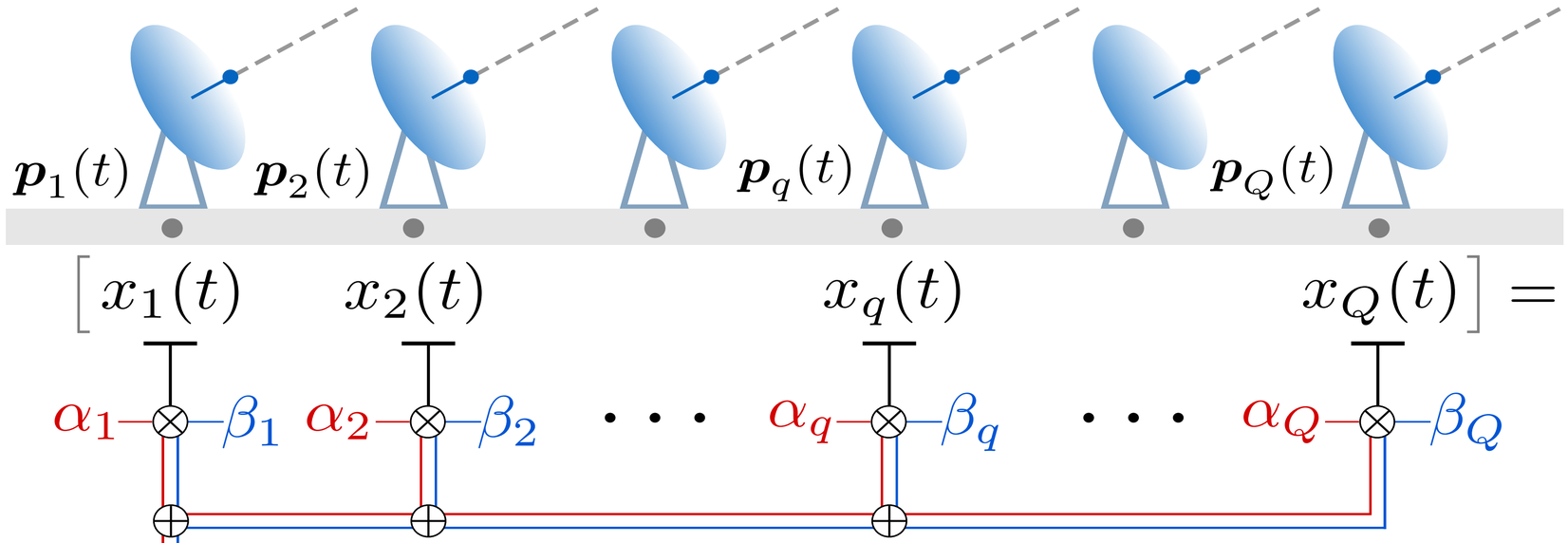


Acoustic radar, Japan (1930)

Beamforming ≡ rank-one projections of covariance matrix

What if we create ***virtual*** antennas? Let's do ~~beamforming~~

sketch-forming,
lazy-forming



Given Q random weights α_q, β_q

2 random virtual antennas
$\mu(t)$, $\nu(t)$

$$\mu(t) = \sum_{q=1}^{Q} \alpha_q^* x_q(t) = \langle \boldsymbol{\alpha}, \boldsymbol{x}(t) \rangle$$

$$\nu(t) = \sum_{q=1}^{Q} \beta_q^* x_q(t) = \langle \boldsymbol{\beta}, \boldsymbol{x}(t) \rangle$$

Van Cittert Zernike →

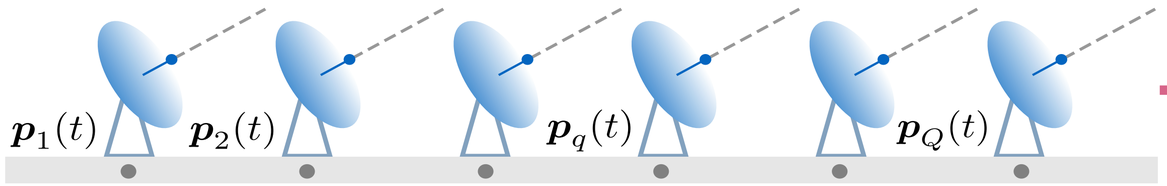
$$\mathbb{E}\mu\nu^* = \boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}_\Omega[f]\boldsymbol{\beta}$$

with $\left(\boldsymbol{\mathcal{I}}_\Omega[f]\right)_{jk} = \mathcal{F}[f]\left(\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}\right)$



This is a (single) rank-one projection (ROP)!

Beamforming ≡ rank-one projections of covariance matrix

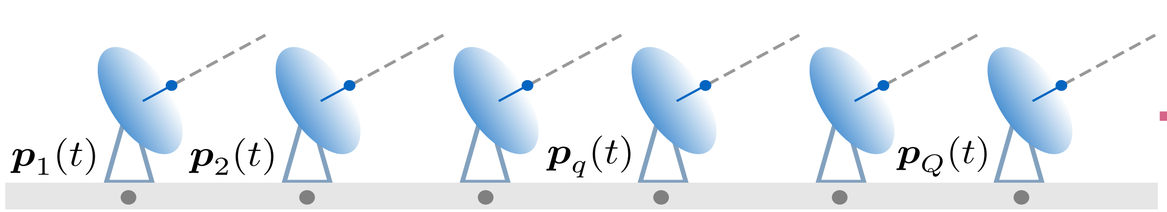


$$\rightarrow \quad \boldsymbol{x}_b[i] \in \mathbb{C}^Q \quad \rightarrow \quad \boldsymbol{X}$$

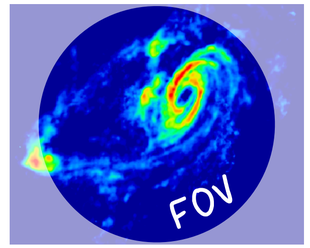
Practically $\rightarrow$ time averaging over B intervals T_b, and P ROPs per interval

$$\boldsymbol{\alpha}_p^* \Big[\underbrace{\tfrac{1}{I} \textstyle\sum_{i=1}^{I} \boldsymbol{x}_b[i] \boldsymbol{x}_b^*[i]}_{=: \boldsymbol{C}_b} \Big] \boldsymbol{\beta}_p \;=\; \langle \mu(\cdot; \boldsymbol{\alpha}_p) \nu^*(\cdot; \boldsymbol{\beta}_p) \rangle_{T_b}$$

Beamforming ≡ rank-one projections of covariance matrix



$$\rightarrow \quad \boldsymbol{x}_b[i] \in \mathbb{C}^Q \quad \rightarrow \quad \boldsymbol{X}$$



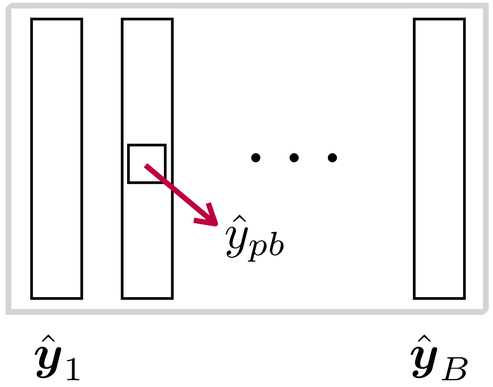
$f(\boldsymbol{l})$

Practically $\rightarrow$ time averaging over B intervals T_b, and P ROPs per interval

$$\boldsymbol{\alpha}_p^* \Big[\underbrace{\tfrac{1}{I}\textstyle\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]}_{=:\boldsymbol{C}_b} \Big] \boldsymbol{\beta}_p \;=\; \langle \mu(\cdot;\boldsymbol{\alpha}_p)\nu^*(\cdot;\boldsymbol{\beta}_p)\rangle_{T_b} \underset{\text{VCZ}}{\approx} \boldsymbol{\alpha}_p^* \boldsymbol{\mathcal{I}}_b[f]\, \boldsymbol{\beta}_p$$

ROPs

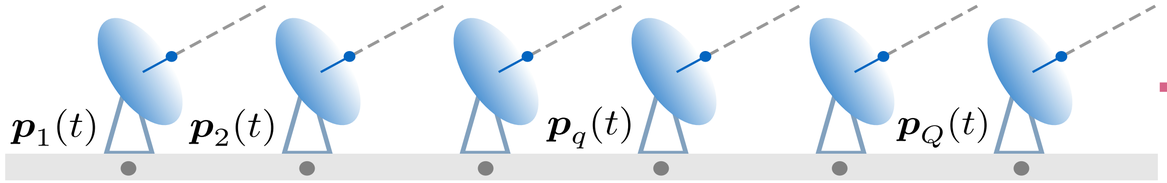
$Q^2B \rightarrow PB$



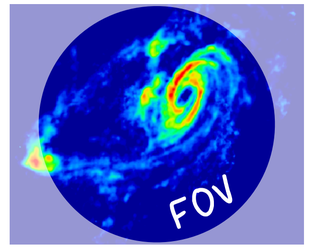
1-st compression

Compression (acquisition side)

Beamforming ≡ rank-one projections of covariance matrix



$$\rightarrow \boldsymbol{x}_b[i] \in \mathbb{C}^Q \rightarrow \boldsymbol{X}$$



$f(\boldsymbol{l})$

Practically $\rightarrow$ time averaging over B intervals T_b, and P ROPs per interval

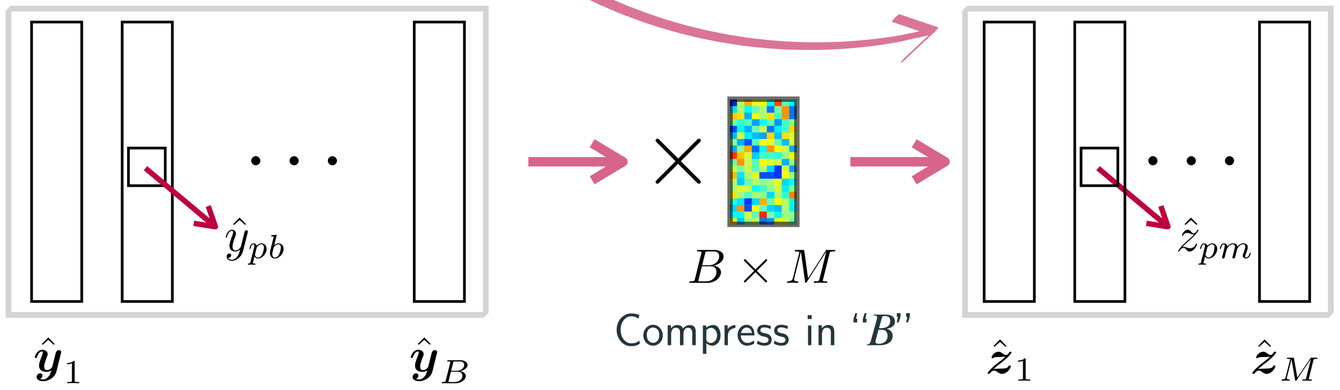
$$\boldsymbol{\alpha}_p^* \Big[\underbrace{\tfrac{1}{I}\textstyle\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]}_{=:\boldsymbol{C}_b} \Big] \boldsymbol{\beta}_p = \langle \mu(\cdot;\boldsymbol{\alpha}_p)\nu^*(\cdot;\boldsymbol{\beta}_p) \rangle_{T_b} \underset{\text{VCZ}}{\approx} \boldsymbol{\alpha}_p^* \boldsymbol{\mathcal{I}}_b[f] \boldsymbol{\beta}_p$$

ROPs

$Q^2B \rightarrow PB$

$Q^2B \rightarrow PM$

M modulations

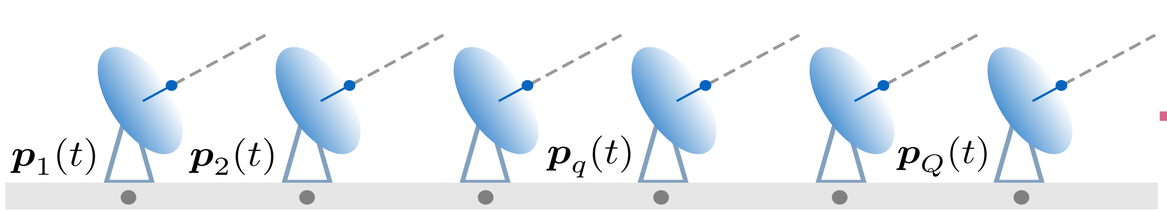


2-nd compression (modulation & averaging)

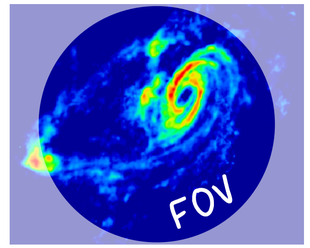
Compression (acquisition side)

$$\{\hat{\boldsymbol{z}}_m\}_{m=1}^{M}$$

Beamforming ≡ rank-one projections of covariance matrix



$$\rightarrow \boldsymbol{x}_b[i] \in \mathbb{C}^Q \rightarrow \boldsymbol{X}$$



$f(\boldsymbol{l})$

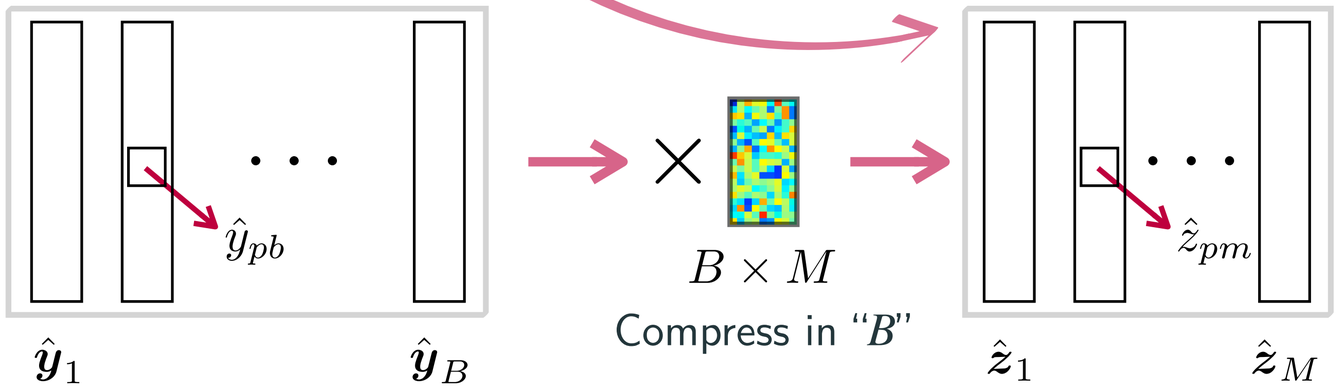
Practically $\rightarrow$ time averaging over B intervals T_b, and P ROPs per interval

$$\boldsymbol{\alpha}_p^* \Big[\underbrace{\tfrac{1}{I}\textstyle\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]}_{=:\boldsymbol{C}_b} \Big] \boldsymbol{\beta}_p \;=\; \langle \mu(\cdot;\boldsymbol{\alpha}_p)\nu^*(\cdot;\boldsymbol{\beta}_p) \rangle_{T_b} \;\approx\; \boldsymbol{\alpha}_p^* \boldsymbol{\mathcal{I}}_b[f] \boldsymbol{\beta}_p$$

ROPs

$Q^2B \rightarrow PB$

$Q^2B \rightarrow PM$



Similarly...

Compression (imaging side)

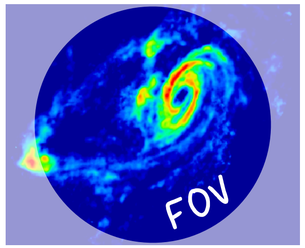
$$\{\boldsymbol{z}_m\}_{m=1}^{M}$$

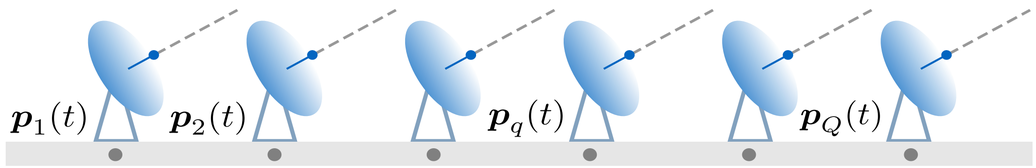
$Q^2B \rightarrow PM$

Compression (acquisition side)

$$\{\hat{\boldsymbol{z}}_m\}_{m=1}^{M}$$

On the reconstruction side...







$\boldsymbol{X}$ → Correlation + ROPs ×

Fourier + ROPs + sampling

$f'(\boldsymbol{l})$

$\approx$

$$\boldsymbol{f}^* \leftarrow \min_{\boldsymbol{f}'} \underbrace{\lambda \mathcal{R}(\boldsymbol{f}')}_{\text{prior}} + \underbrace{(\sum_{m=1}^{M} \| \hat{\boldsymbol{z}}_m[\boldsymbol{X}] - \boldsymbol{z}_m[\boldsymbol{f}'] \|^2}_{\text{Fidelity term (likelihood)}}$$

Pick the *prior* you like:

- L1, TV, learned, plug-and-play, unrolled, ...
- μSARA "Sparsity Averaging Reweighted" (Terris et al, 2022)
  or R2D2 (learned matching pursuit, aka CLEAN)

Reconstruction guarantees? Theory for a simplified scenario (no modulation)

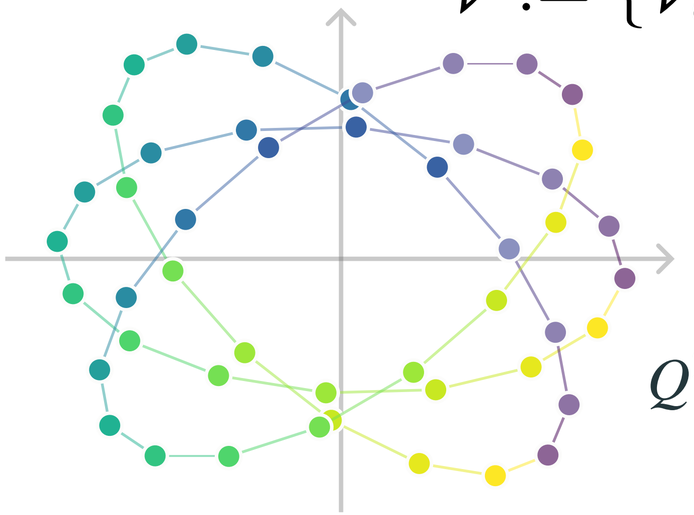
ROP model: for P random virtual antennas for each of the B short-time-intervals

Q antennas

B times



$$\mathcal{V} := \{\mathcal{V}_b\}_{b=1}^{B}$$



Q^2 ellipses

Q^2B visibilities

(under specific simplifying assumptions, no modulation, $M = 1$)

If the $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}_{p=1,b=1}^{P,B}$ are Gaussian, if $\boldsymbol{f}$ is K-sparse

and provided $P = O(K)$ and $Q^2B = O(K)$ (up to logs),

then, with high probability, one can compute an estimate $\boldsymbol{f}^*$ of $\boldsymbol{f}$ with

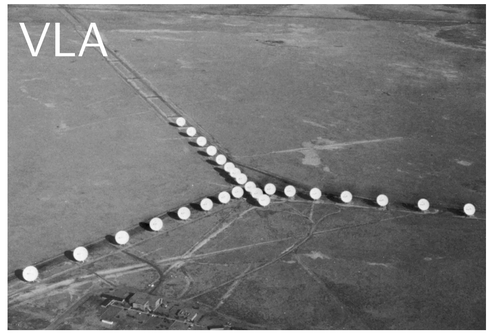
$$\|\boldsymbol{f} - \boldsymbol{f}^*\|_2 = O(\tfrac{\epsilon}{P})$$

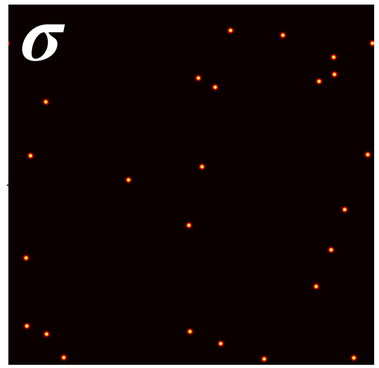
Noisy visibilities
(noise bound ϵ)

Reconstruction guarantees? Simulations with modulations

Modulated ROP model (MROP):

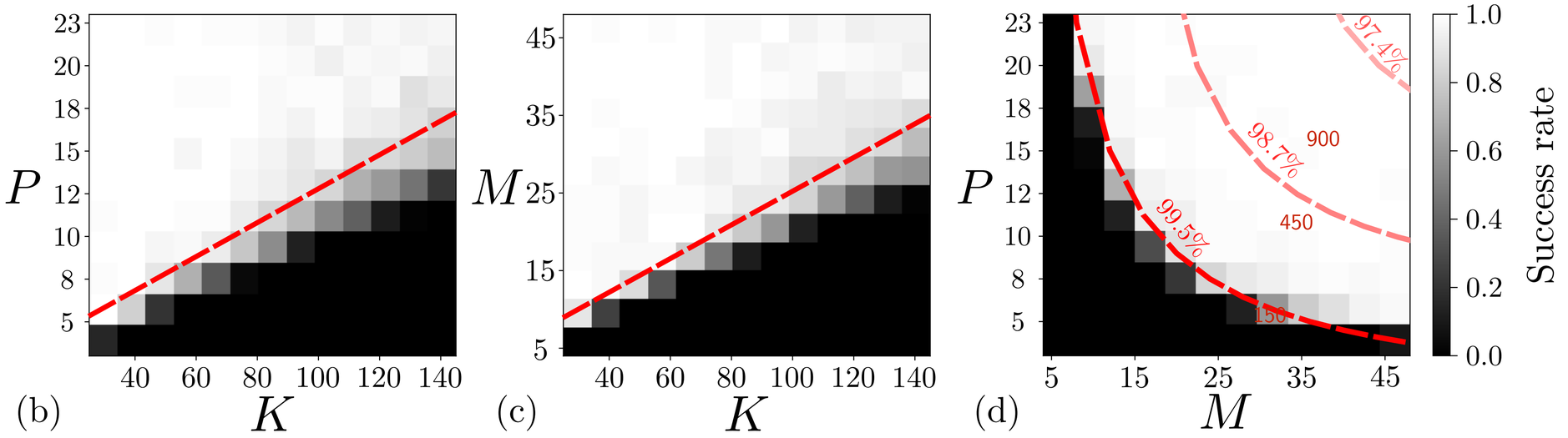
- Monte Carlo simulations
- $N = 10^4$, $B = 100$, $Q = 27$
- Various K, P, M
- Very Large Array (VLA) visibility/frequency coverage





$N = 100 \times 100$

Phase transition diagrams (success if SNR > 40 dB)



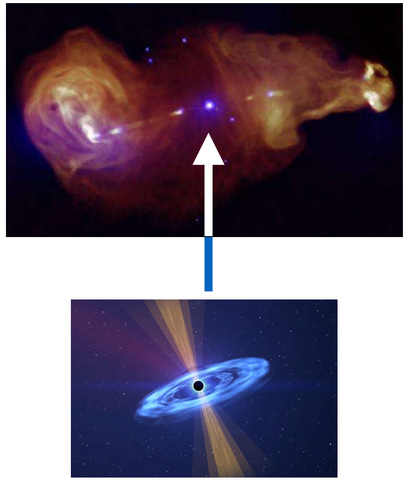
High reconstruction success as soon as $PM \geqslant CK$, with $C \simeq 5$.

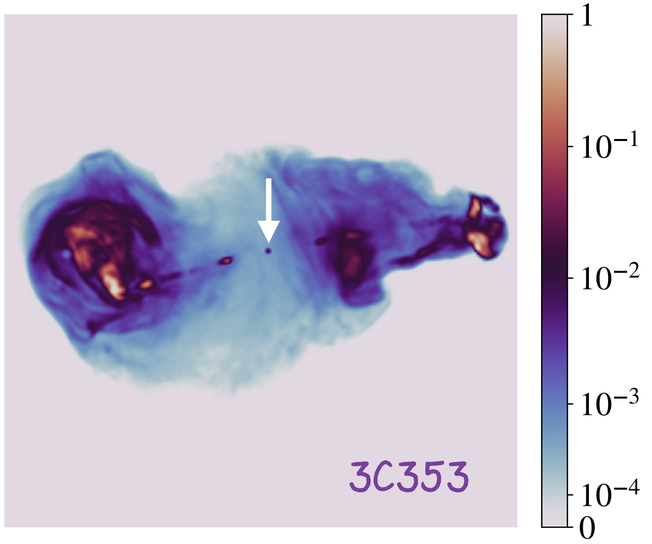
$$\rightarrow \quad PM \ll Q^2 B = 70\,200$$

Radio Galaxy 3C353, uSARA reconstruction

Reconstruction for the image of the Radio Galaxy 3C353

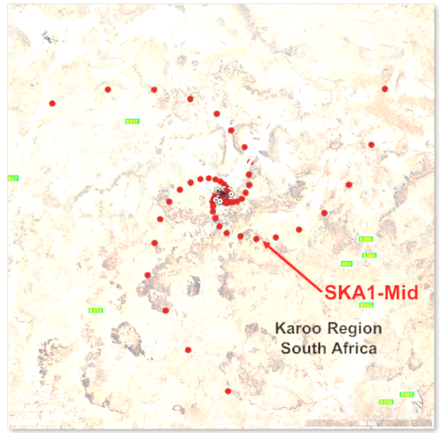
- Classical reconstruction ≡ all visibilities
- Basic subsampled visibilities
- Baseline dependent averaging (BDA)
- and, ours, the MROP sensing models

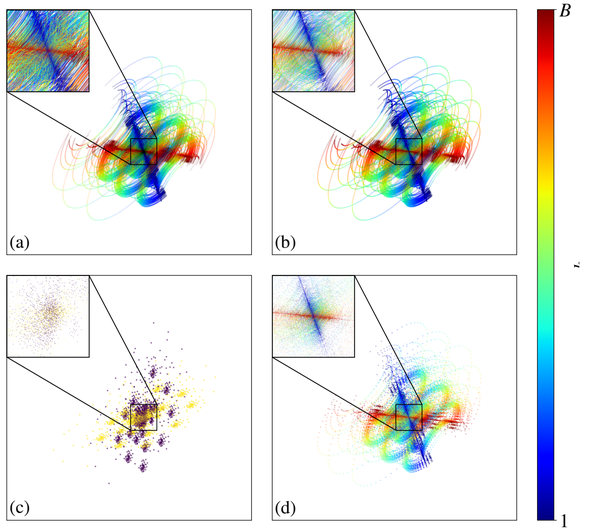




Parameters:

- #Visibilities ($V = Q^2B$) / #Pixels (N) = 14.5
- Total number of measurements: $D = PM$
- ROP compressions levels: D/N
  $\in \{9.8\ 10^{-1}, 3.8\ 10^{-1}, 6.1\ 10^{-2}\}$
- Simulated MeerKat visibilities (Q=64)
- uSARA reconstruction (Terris et al, 2022)
  (inverse problem solving with "average sparsity" prior)





MeerKat visibilities

Radio Galaxy 3C353, uSARA reconstruction

Two compression metrics

D/VB and D/N

Total # measurements:

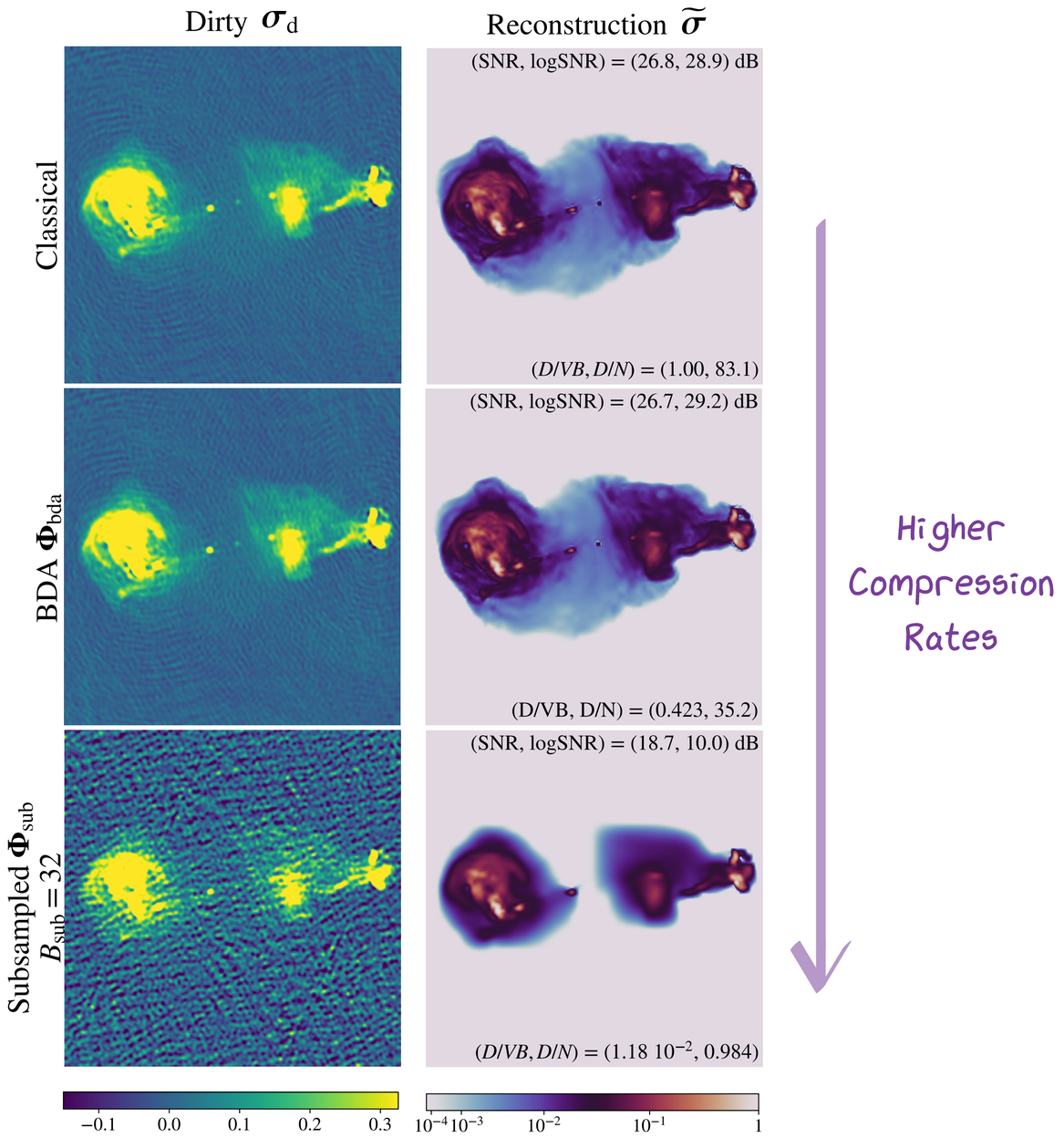
$D = PM$

Total # visibilities:

$V = Q^2 B$

Total # pixels:

N



Higher Compression Rates

Radio Galaxy 3C353, uSARA reconstruction

Two compression metrics

D/VB and D/N

Total # measurements:

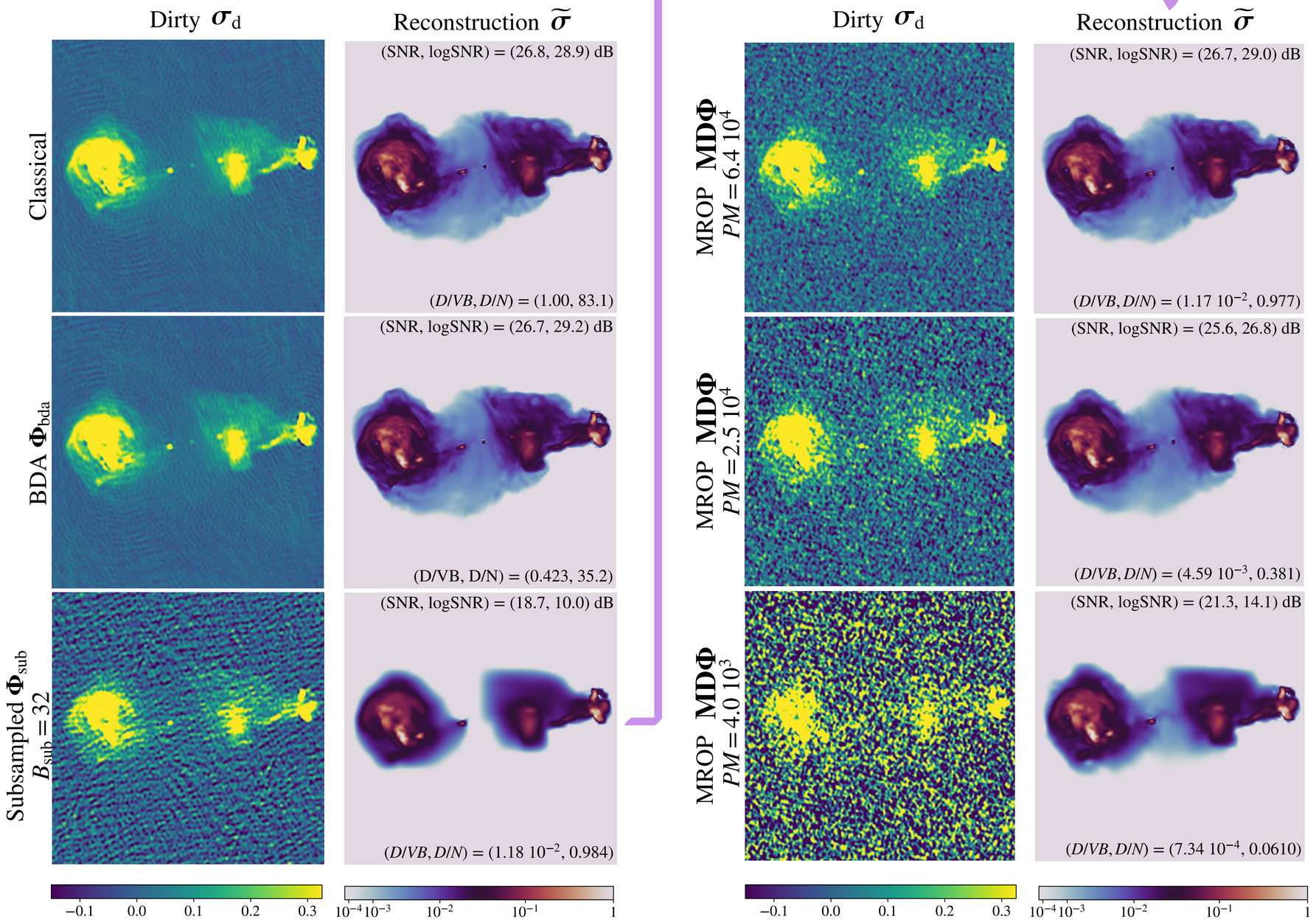
$D = PM$

Total # visibilities:

$V = Q^2 B$

Total # pixels:

N

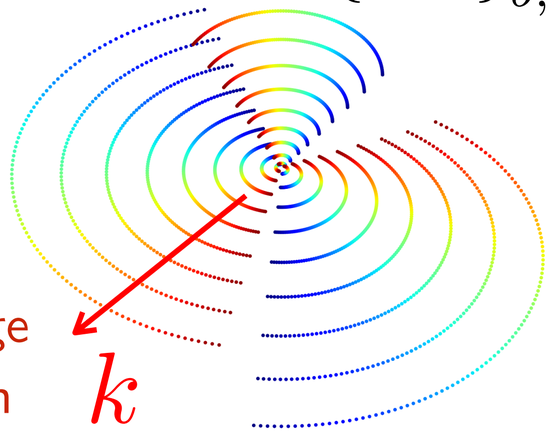


Perspective?

(Ongoing) increased compression over frequencies for panchromatic image

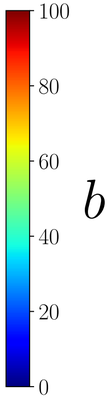
MeerKat antenna location

$\{\Omega_{bk}\}_{b,k=1}^{B,K}$



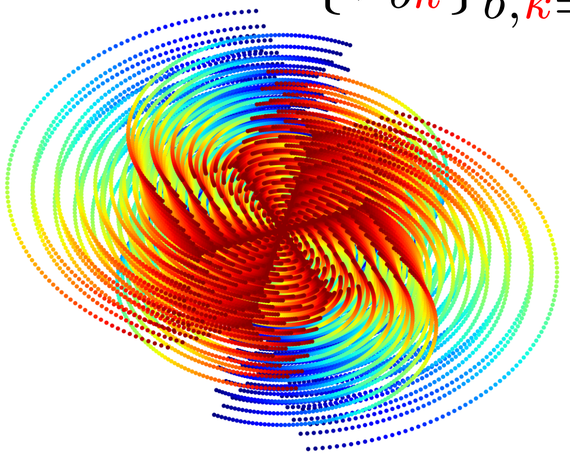
Frequency change = radial dilation

k



b

$\{\mathcal{V}_{bk}\}_{b,k=1}^{B,K}$



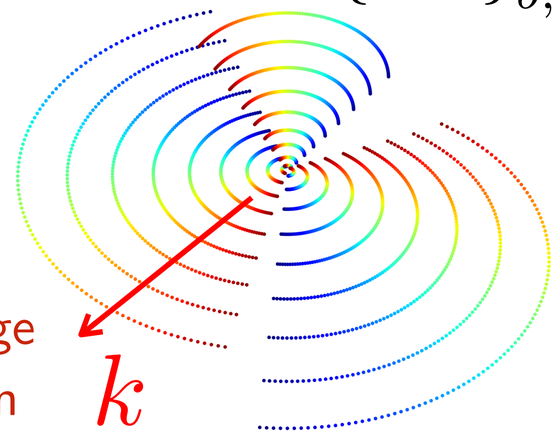
MeerKat visibilities

Perspective?

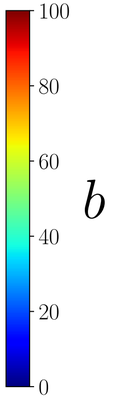
(Ongoing) increased compression over frequencies for panchromatic image

MeerKat antenna location

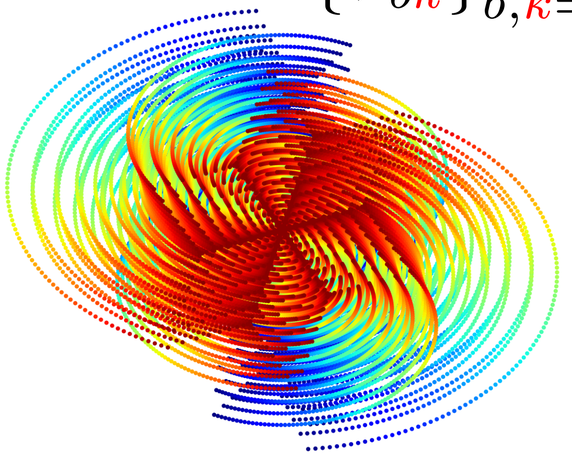
$\{\Omega_{bk}\}_{b,k=1}^{B,K}$



Frequency change = radial dilation k



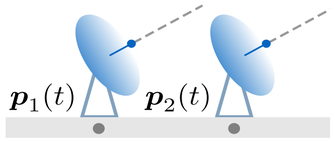
$\{\mathcal{V}_{bk}\}_{b,k=1}^{B,K}$



MeerKat visibilities

$$\boldsymbol{\alpha}_p^* \Big[\underbrace{\tfrac{1}{I}\textstyle\sum_{i=1}^{I} \boldsymbol{x}_{bk}[i]\boldsymbol{x}_{bk}{}^*[i]}_{=:\boldsymbol{C}_{bk}} \Big] \boldsymbol{\beta}_p \;\approx\; \boldsymbol{\alpha}_p^* \boldsymbol{\mathcal{I}}_{bk}[f]\, \boldsymbol{\beta}_p$$

Generalised compression scheme

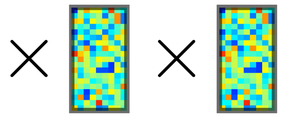


$\boldsymbol{X}$

$[Q^2BK]$



Correlation + ROPs





$\{\hat{\boldsymbol{z}}_{m_B m_K}\}_{m_B=1,m_K=1}^{M_B,M_K}$

$[PM_BM_K]$

Take-away messages

- In 2004-2006, birth of compressive sensing

  $\Rightarrow$ random constructions & inverse problem solving & optimization

- Since last 20 years, many extensions to
  - Compressive imaging techniques
  - Non-linear sensing models (one-bit, phase retrieval, ...)
- Compressive interferometric models are possible
  - Data compression through randomness injected in SLM, weights, ...
  - Rank-one projections combined with partial Fourier sensing.

Thank you for your attention!

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