Computational Fourier Transform Interferometry for Hyperspectral Imaging

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In collaboration with

LAMBDA-X ADVANCED VISION & METROLOGY

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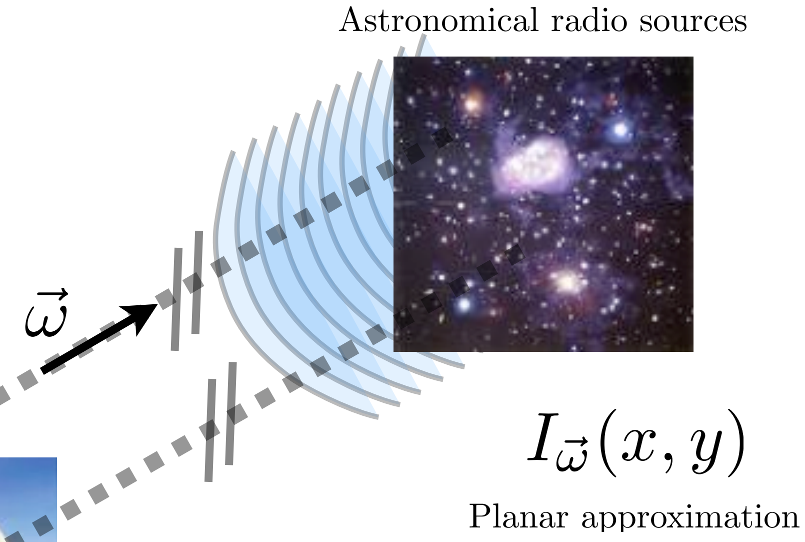
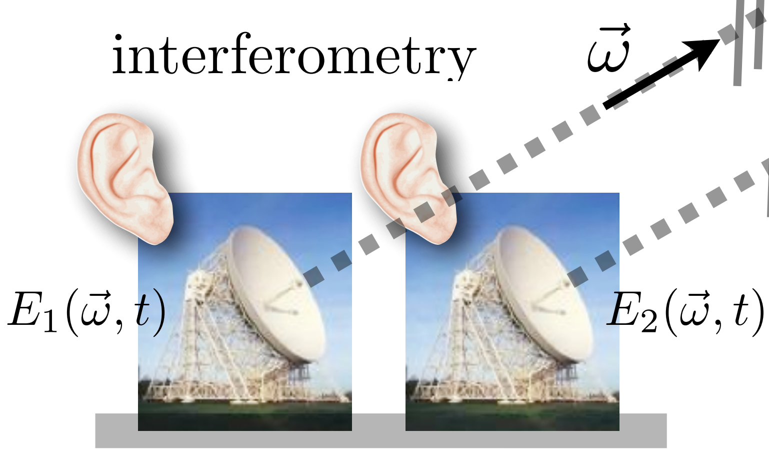
2: Instituto de Telecomunicações, Portugal

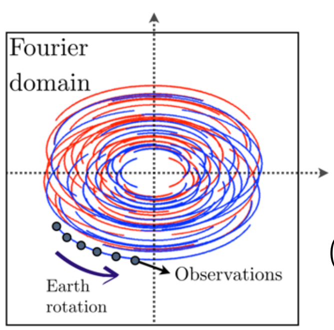
ISP GROUP Image & Signal Processing UCLouvain icteam fnrs LA LIBERTÉ DE CHERCHER instituto de telecomunicações

Interferometric imaging

Aperture Synthesis in Radio-Astronomy







Inversion
(CLEAN, CS, ...)

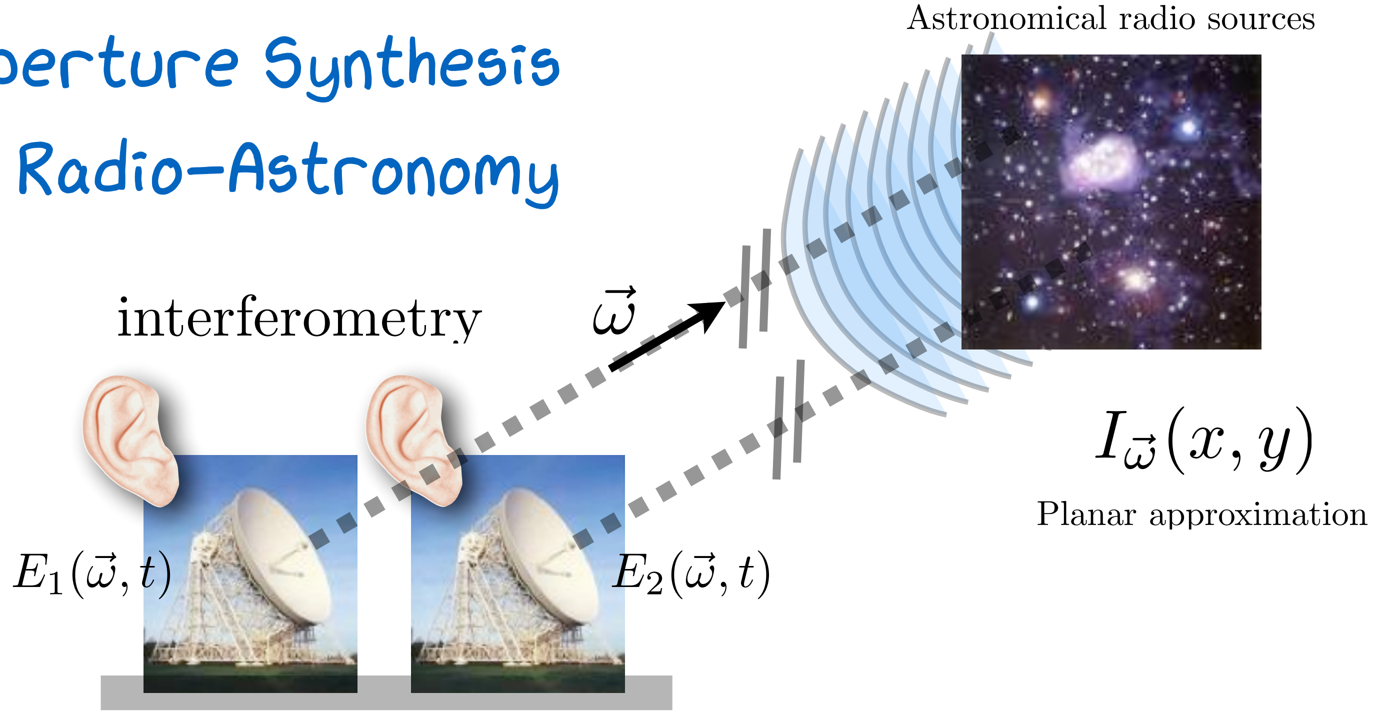
Optical Interferometry

Synthetic Aperture Radar

Phase Shifting in Optics
(e.g., material surface imaging)

Interferometric imaging

Aperture Synthesis in Radio-Astronomy



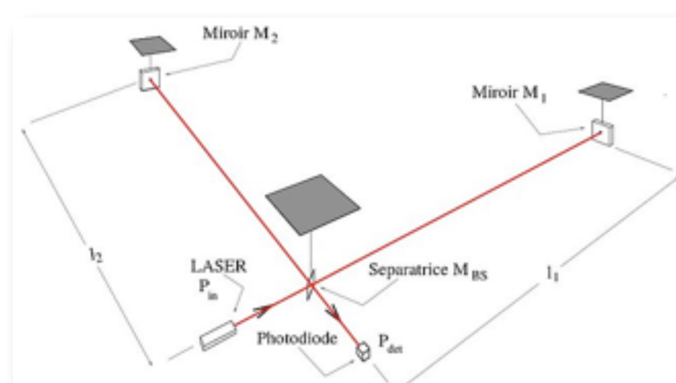
Optical Interferometry

Synthetic Aperture Radar

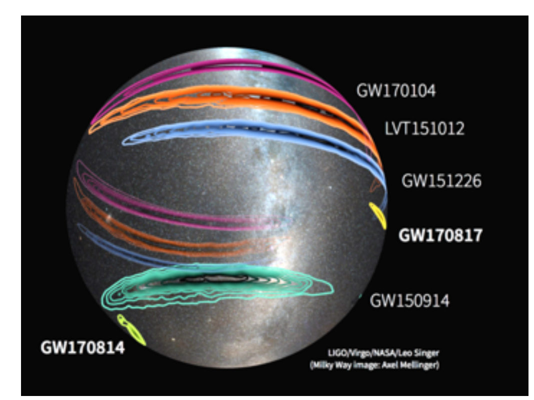
Phase Shifting in Optics
(e.g., material surface imaging)

and maybe soon ...

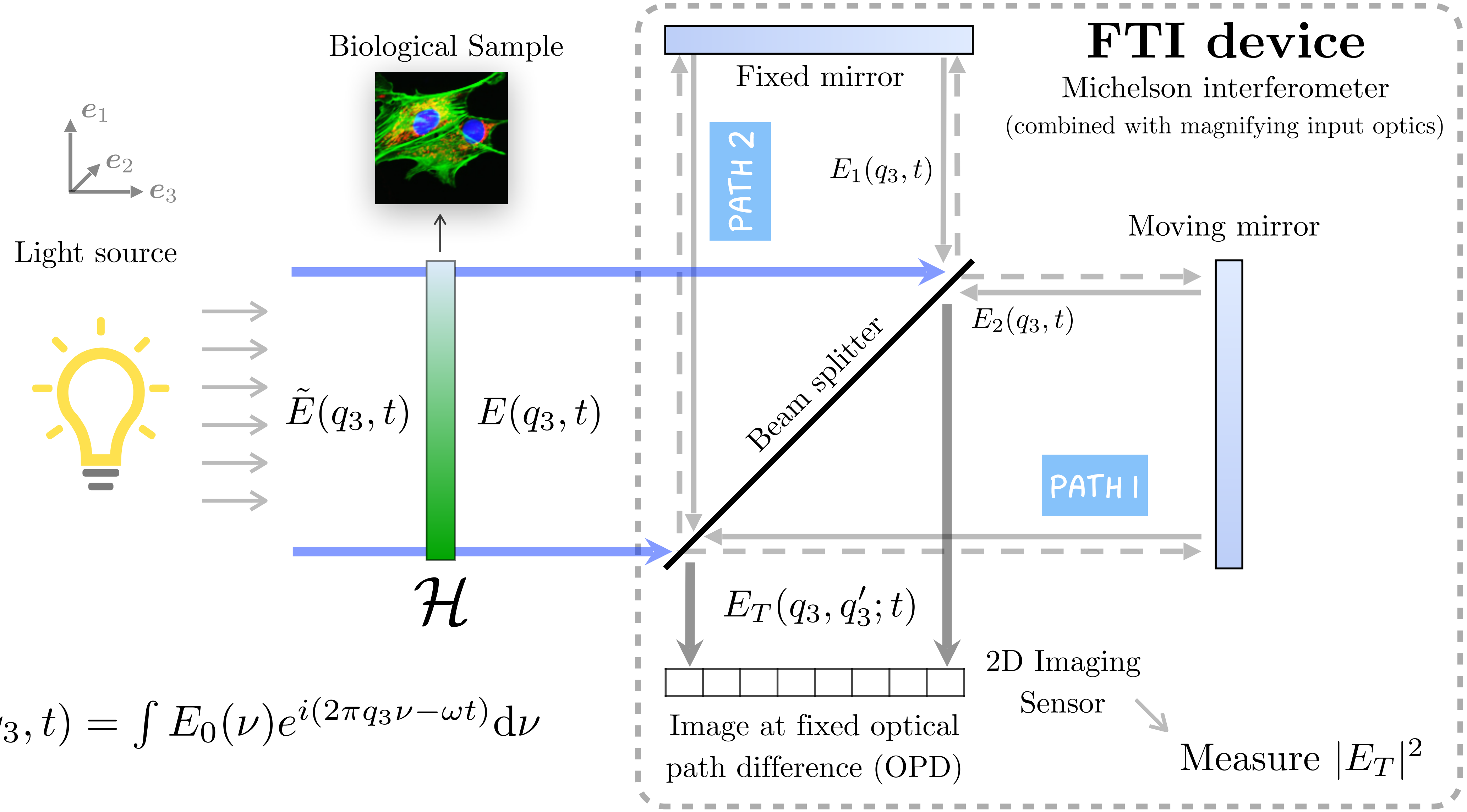
HR gravitational wave imaging



Virgo (source Wikipedia)

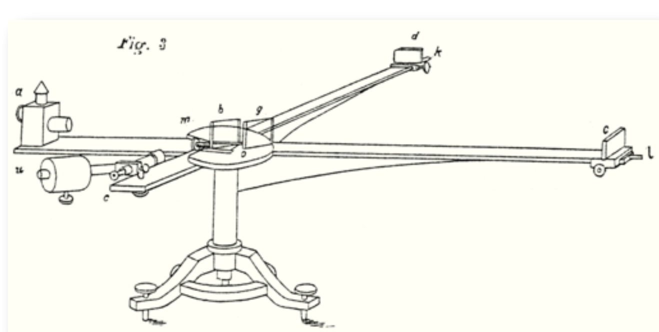


Fourier Transform Interferometry (in biological imaging)

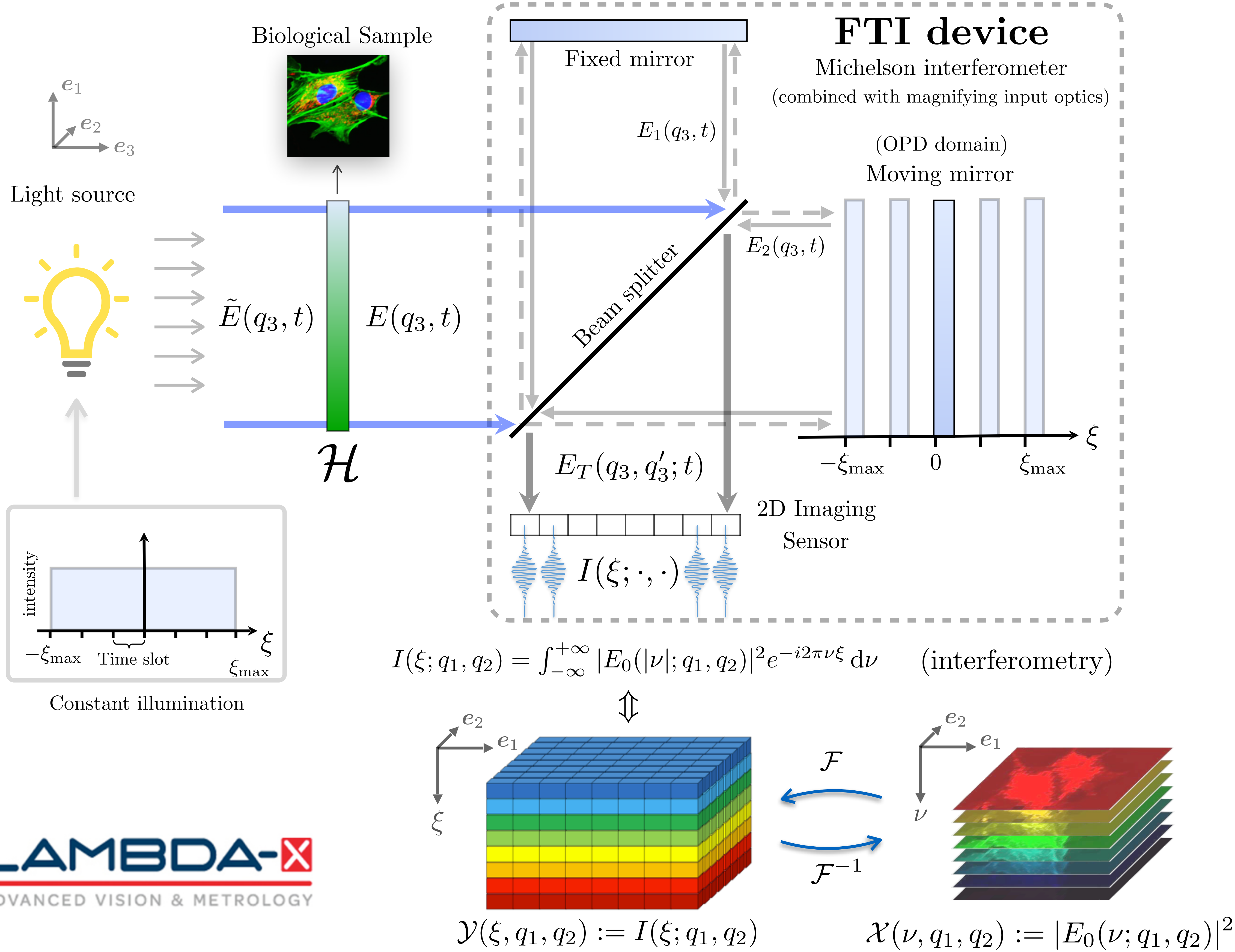


$$E(q_3, t) = \int E_0(\nu) e^{i(2\pi q_3 \nu - \omega t)} \mathrm{d}\nu$$

A. A. Michelson
1852-1931



Fourier Transform Interferometry (in biological imaging)



$$I(\xi;q_1,q_2) = \int_{-\infty}^{+\infty} |E_0(|\nu|;q_1,q_2)|^2 e^{-i2\pi\nu\xi}\,\mathrm{d}\nu \qquad \text{(interferometry)}$$

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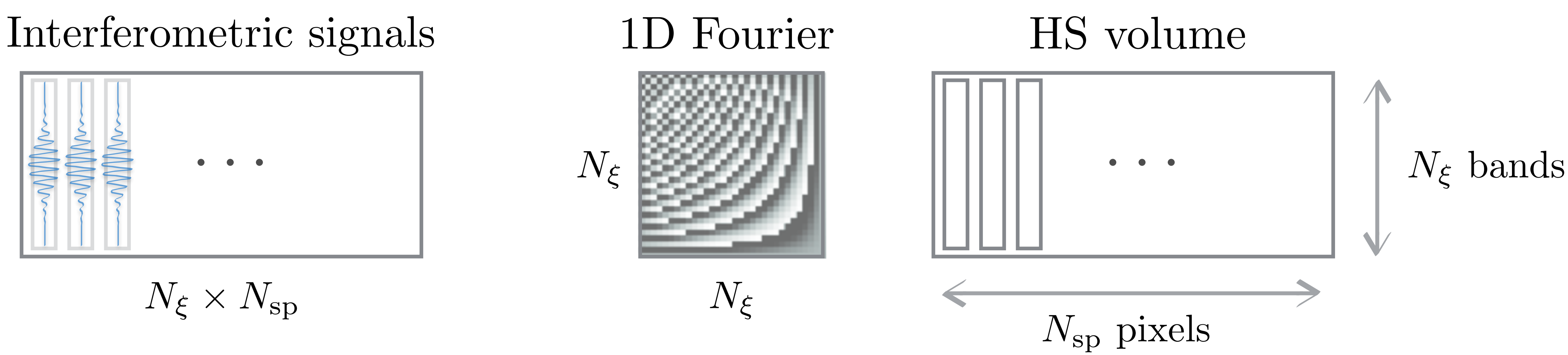
Fourier Transform Interferometry

- FTI discrete sensing model

  (essentially, 1-D Fourier sensing at Nyquist, on each pixel)

$$\overset{\text{observations}}{\boldsymbol{Y}} = \boldsymbol{F}\boldsymbol{X} + \overset{\text{noise}}{\boldsymbol{N}}$$

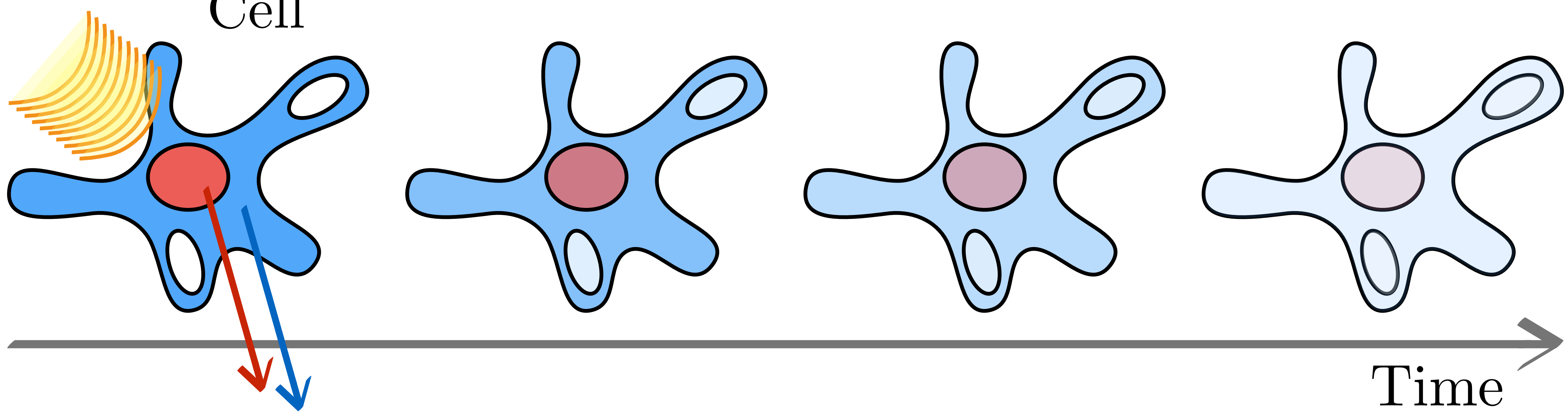


No spatial mixing!

Fourier Transform Interferometry

- Classical reconstruction: Fourier inversion $\hat{\boldsymbol{X}} = \boldsymbol{F}^{-1}\boldsymbol{Y}$

longer acquisition $\Longrightarrow \begin{cases} ✔ \text{ higher spectral resolution} \\ \quad \text{(related to OPD discretization: O(100-1000) bands!)} \\ ✗ \text{ more photo-bleaching} \end{cases}$



Fourier Transform Interferometry

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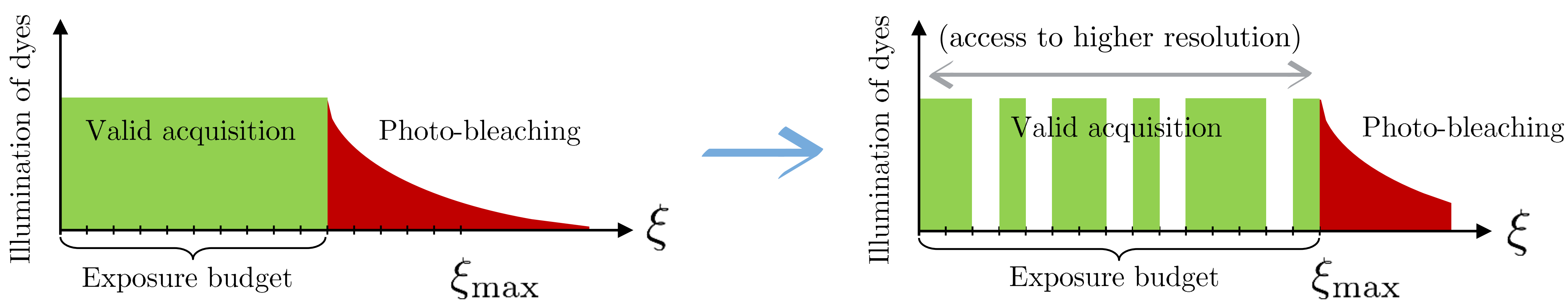
longer acquisition $\Longrightarrow$ ✔ higher spectral resolution (related to OPD discretization: O(100-1000) bands!) / ✗ more photo-bleaching

- Solution: OPD subsampling

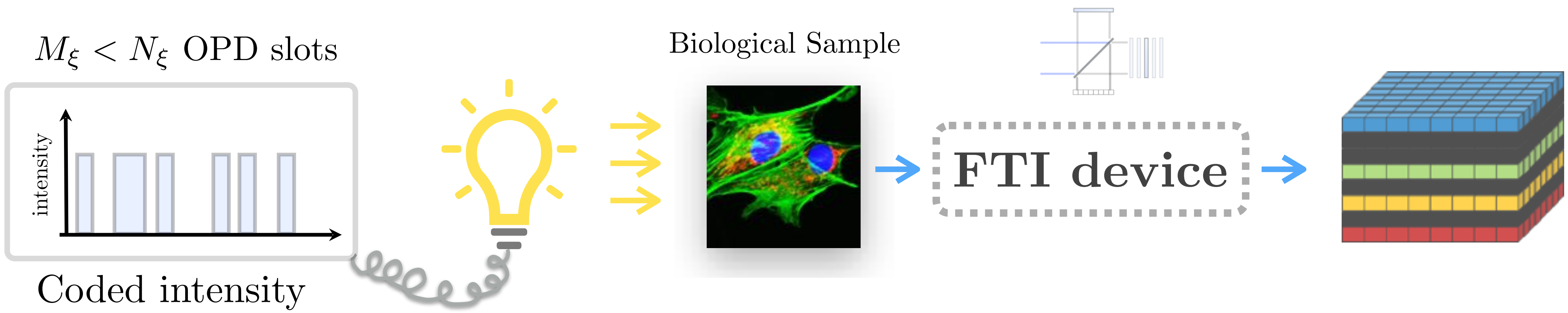


- How? Using compressive sensing theory

Compressive FTI

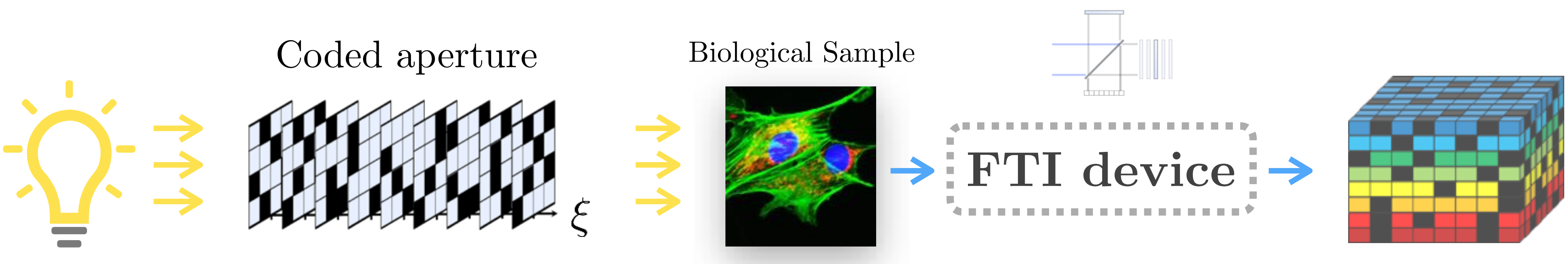
Two possible **system modifications**

Coded Illumination (CI-FTI):



Number of measurements $= M_\xi N_{\rm sp}$

Structured Illumination (SI-FTI):



$M < N := N_{\rm sp} N_\xi$ unmasked pixels
$=$ number of measurements

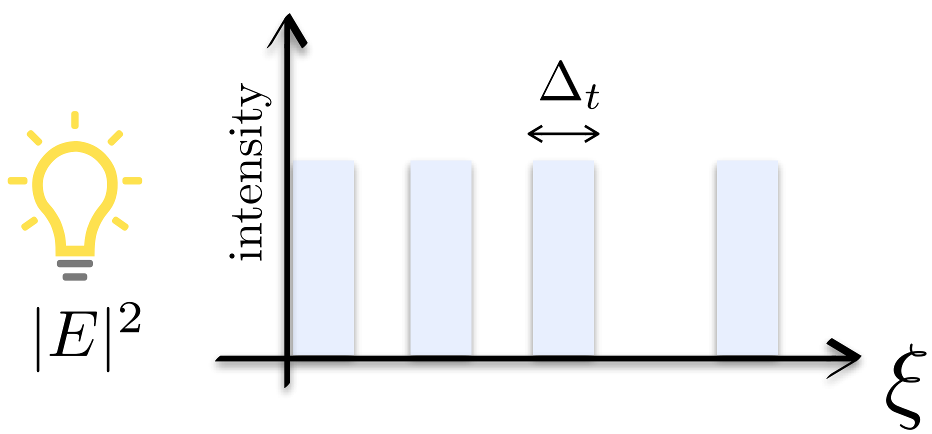
Compressive FTI

Two possible **intensity modes**

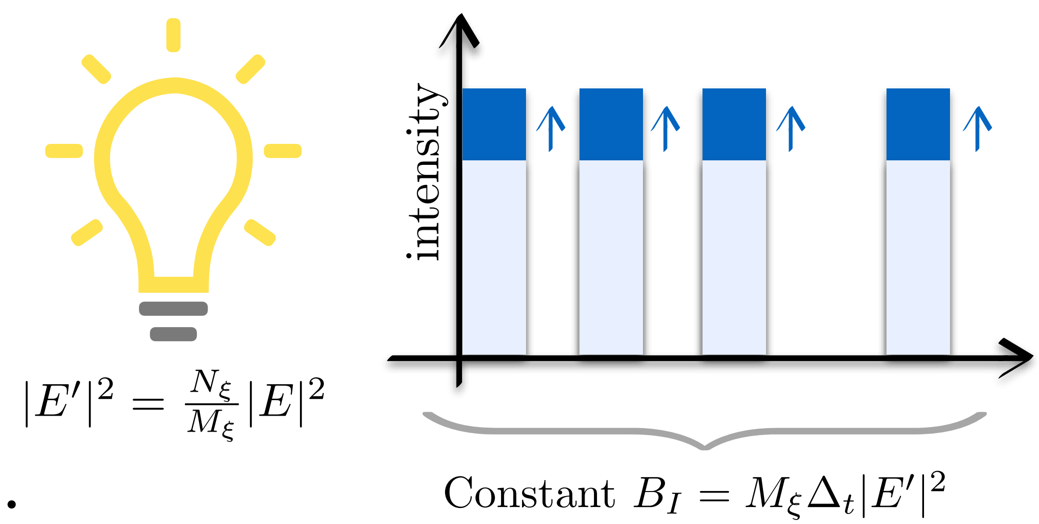
Total intensity is constant

Coded Illumination (CI-FTI):

Unconstrained



Constrained



Structured Illumination (SI-FTI):

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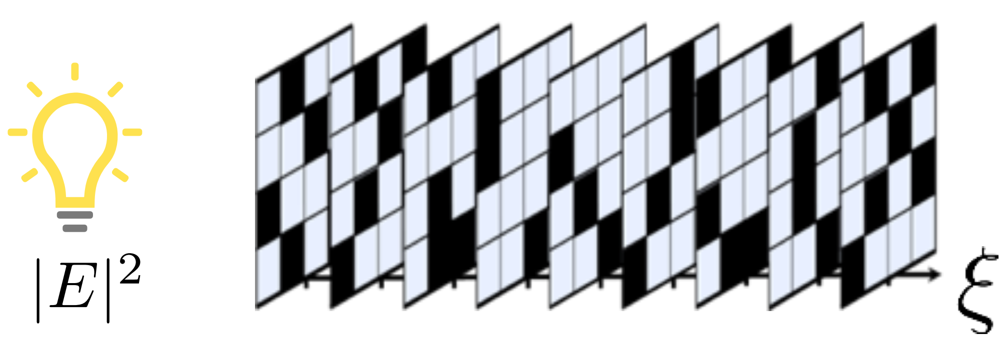

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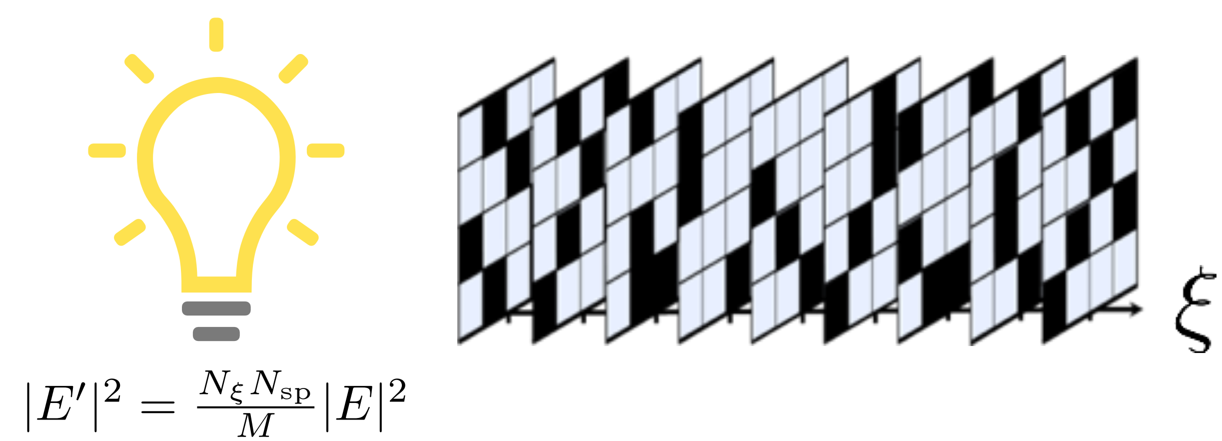

Input intensity is constant

Uniform Density Sampling (UDS) in CS

- Reconstruction in CS theory: (use the restricted isometry property — RIP)

If $\boldsymbol{A} \in \mathbb{C}^{M \times N}$ is RIP$(\delta < \sqrt{2} - 1, \Sigma_{2K})$, *i.e.*, $\|\boldsymbol{A}\boldsymbol{x}\|^2 \simeq_\delta \|\boldsymbol{x}\|^2$ for all $\boldsymbol{x} \in \Sigma_{2K}$,
Then, for $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n}$, $\|\boldsymbol{n}\| \leqslant \epsilon$, and $\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1$ s.t. $\|\boldsymbol{A}\boldsymbol{u} - \boldsymbol{y}\| \leqslant \epsilon$,

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(K^{-1/2}\|\boldsymbol{x} - \boldsymbol{x}_K\|_1 + \epsilon).$$

Combines both
the sensing matrix
and the sparsity basis

$$\boldsymbol{y} = \boldsymbol{\Phi}(\boldsymbol{\Psi}\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$$

Uniform Density Sampling (UDS) in CS

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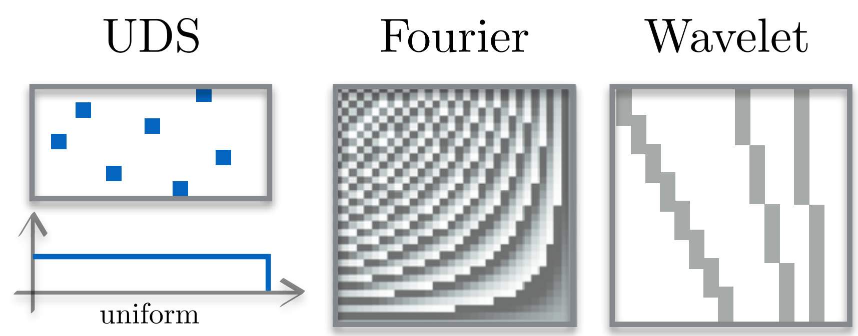
> $$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(K^{-1/2}\|\boldsymbol{x} - \boldsymbol{x}_K\|_1 + \epsilon).$$

- Partial random ONB sensing:

$$\boldsymbol{A} = \sqrt{\tfrac{N}{M}}\boldsymbol{R}_\Omega \boldsymbol{U}^* \boldsymbol{\Psi} \in \mathbb{C}^{M\times N}$$

e.g.,



with $\boldsymbol{U}, \boldsymbol{\Psi}$ ONB, $\Omega = \{\Omega_1, \cdots, \Omega_M\}$, $\Omega_i \sim_{\rm iid} \mathcal{U}(\{1, \cdots, N\})$. (sampling without replacement)

RIP? $\boxed{M \geqslant C\delta^{-2}\mu^2 K \ln^3(K)\ln(N)} \overset{\text{w.h.p.}}{\Rightarrow} \boldsymbol{A}$ is RIP(δ, Σ_k)

[Candès, Romberg, Rauhut, Foucart, ...]

Uniform Density Sampling (UDS) in CS

- Reconstruction in CS theory: (use the restricted isometry property — RIP)

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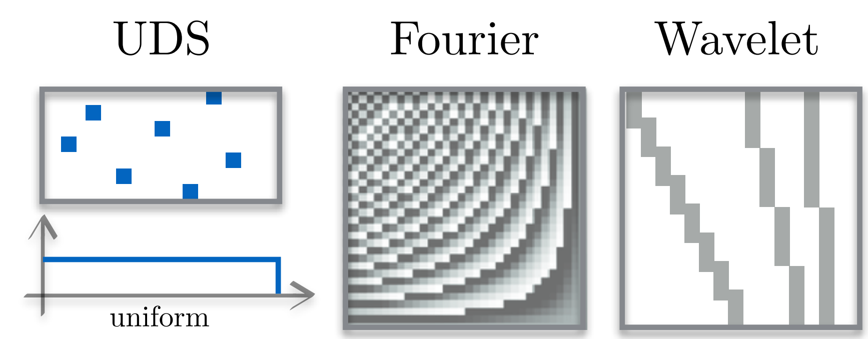

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[Candès, Romberg, Rauhut, Foucart, ...]

- **Issue**: $\mu = \mu(\boldsymbol{U}^*\boldsymbol{\Psi}) := \sqrt{N}\max_{ij} |(\boldsymbol{U}^*\boldsymbol{\Psi})_{ij}| \simeq \sqrt{N}$ for Fourier/Wavelet

$$\Rightarrow M \geqslant CN!$$

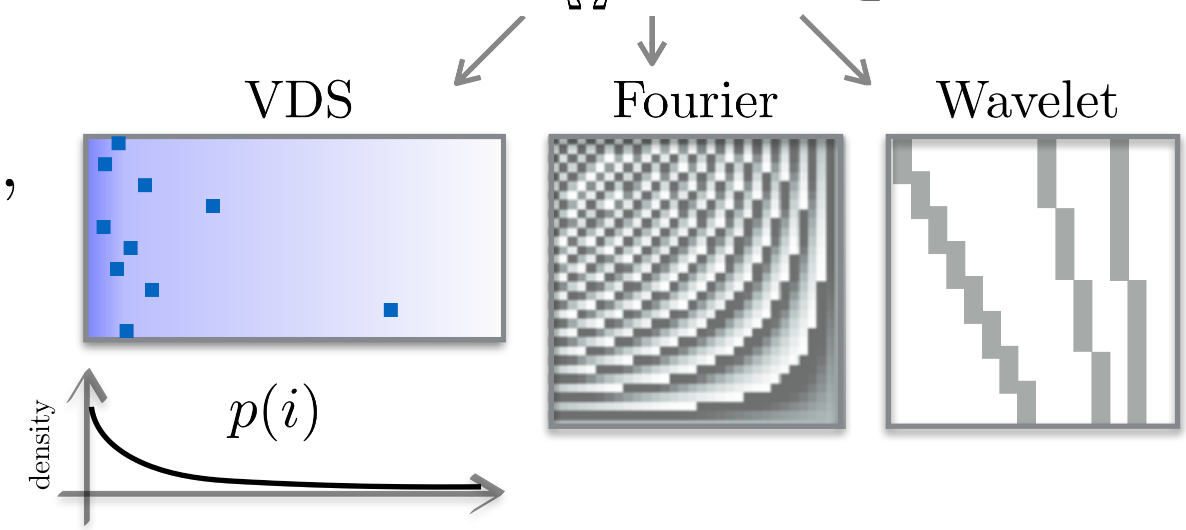
No compressive FTI !?

Variable Density Sampling (VDS) in CS

- VDS:

$$\boldsymbol{A} = \boldsymbol{R}_\Omega \boldsymbol{U}^* \boldsymbol{\Psi} \in \mathbb{C}^{M\times N}$$

e.g.,



with $\boldsymbol{U}, \boldsymbol{\Psi}$ ONB, $\Omega = \{\Omega_1, \cdots, \Omega_M\}$, $\Omega_j \sim_{\text{iid}} \beta$, with

$$\mathbb{P}(\beta = i) = p(i), \quad i \in \{1, \cdots, N\}).$$

(sampling without replacement)

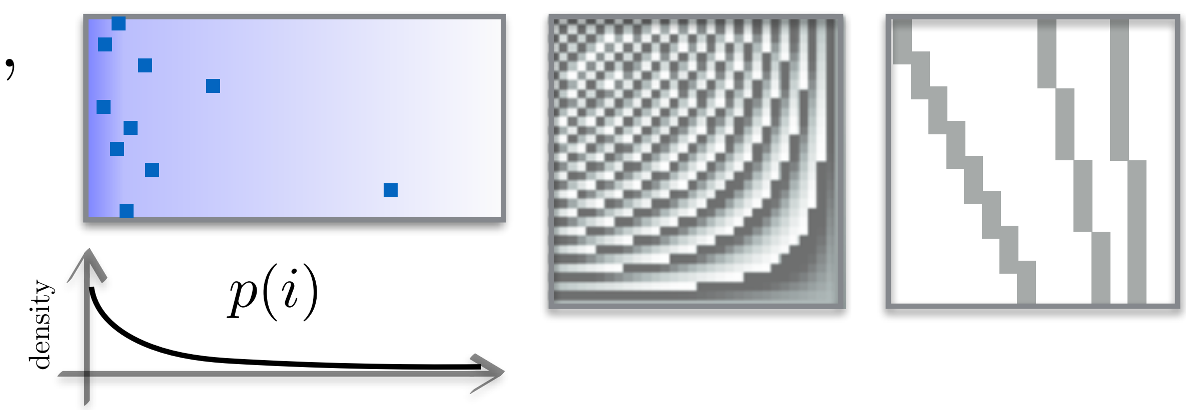
Variable Density Sampling (VDS) in CS

- VDS:

$$\boldsymbol{A} = \boldsymbol{R}_\Omega \, \boldsymbol{U}^* \boldsymbol{\Psi} \in \mathbb{C}^{M \times N}$$

VDS Fourier Wavelet

e.g.,



with $\boldsymbol{U}, \boldsymbol{\Psi}$ ONB, $\Omega = \{\Omega_1, \cdots, \Omega_M\}$, $\Omega_j \sim_{\rm iid} \beta$, with

$$\mathbb{P}(\beta = i) = p(i), \quad i \in \{1, \cdots, N\}).$$

(sampling without replacement)

- RIP? **[Kramer & Ward, 14]**

Given $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_N)^\top$, with $\kappa_i \geqslant \overbrace{\mu_i = \mu_i(\boldsymbol{U}^*\boldsymbol{\Psi})}^{\text{local coherence}} := \sqrt{N} \max_j |(\boldsymbol{U}^*\boldsymbol{\Psi})_{ij}|$,

if $p(i) \propto \kappa_i^2$ (with $\sum_i p(i) = 1$), then

$$\boxed{M \geqslant C\delta^{-2} \|\boldsymbol{\kappa}\|^2 K \ln^3(K) \ln(N)} \stackrel{\text{w.h.p.}}{\Longrightarrow} \tfrac{1}{\sqrt{M}} \boldsymbol{D}\boldsymbol{A} \text{ is } \mathrm{RIP}(\delta, \Sigma_k)$$

with $\boldsymbol{D} = \mathrm{diag}\left(\sqrt{\frac{1}{p(\Omega_1)}}, \cdots, \sqrt{\frac{1}{p(\Omega_M)}}\right)$.

(note: in avg, D accounts for multiplicities)

So, new game: reduce $\|\kappa\|^2$!

Variable Density Sampling (VDS) in CS

- Reconstruction in VDS?

Assuming $\boldsymbol{y} = \overbrace{\boldsymbol{R}_\Omega \boldsymbol{U}}^{\text{sensing}}{}^* \boldsymbol{x} + \boldsymbol{n}$ with $\|\frac{1}{\sqrt{M}}\boldsymbol{D}\boldsymbol{n}\| \leqslant \epsilon$, and

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u}} \|\boldsymbol{\Psi}^* \boldsymbol{u}\|_1 \ \text{s.t.}\ \|\frac{1}{\sqrt{M}}\boldsymbol{D}(\underbrace{\boldsymbol{R}_\Omega \boldsymbol{U}}_{\text{sensing}}{}^* \boldsymbol{u} - \boldsymbol{y})\| \leqslant \epsilon$$

If $\frac{1}{\sqrt{M}}\boldsymbol{D}\boldsymbol{A}$ is $\mathrm{RIP}(\delta < 1/3, \Sigma_{5k})$, then

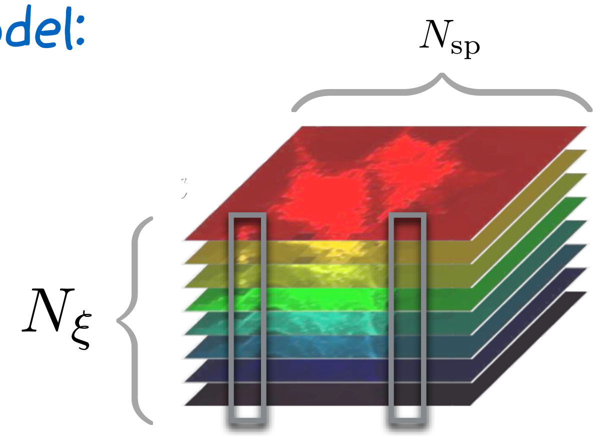
$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(k^{-1/2}\|\boldsymbol{\Psi}^* \boldsymbol{x} - (\boldsymbol{\Psi}^* \boldsymbol{x})_k\|_1 + \epsilon)$$

(& special care for noise power estimation)

Instantiation on Compressive FTI (theory)

Coded Illumination (CI-FTI):

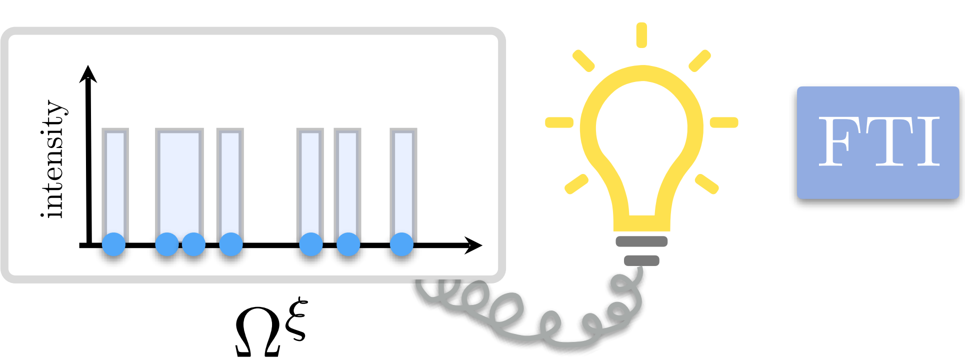
Sparsity model:



(locally) K_ξ-sparse
in $\boldsymbol{\Psi}_{\rm 1D}$ (wavelet)

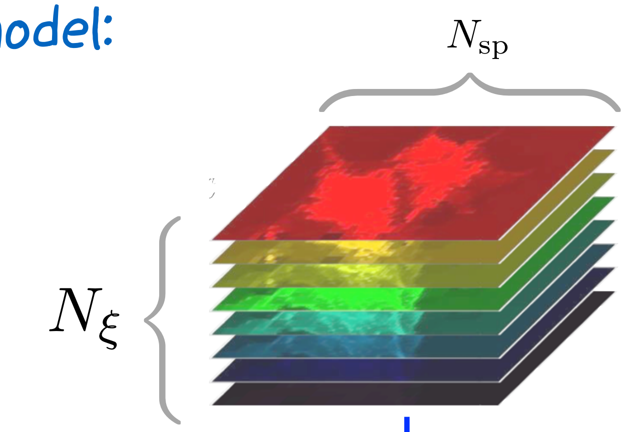
Sensing model:

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{F}^*_{\Omega^\xi} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}^*_{\Omega^\xi} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{F}^*_{\Omega^\xi} \end{bmatrix}$$



Structured Illumination (SI-FTI):

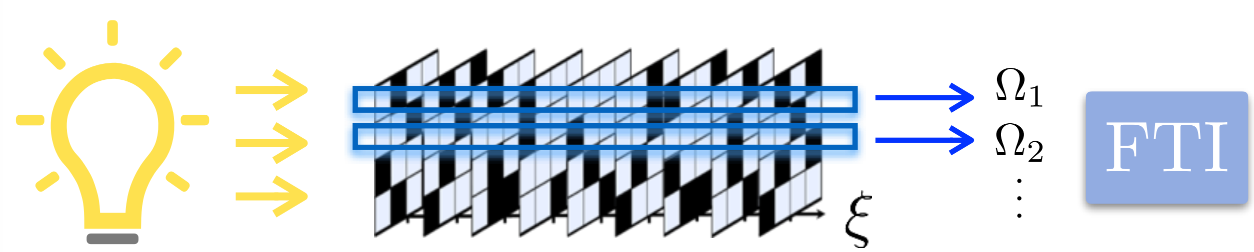
Sparsity model:



(globally) K-sparse
in $\boldsymbol{\Psi}_{\rm 1D} \otimes \boldsymbol{\Psi}_{\rm 2D}$
(wavelet x wavelet)

Sensing model:

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{F}^*_{\Omega_1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}^*_{\Omega_2} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{F}^*_{\Omega_{N_{\rm p}}} \end{bmatrix},$$



Instantiation on Compressive FTI (theory)

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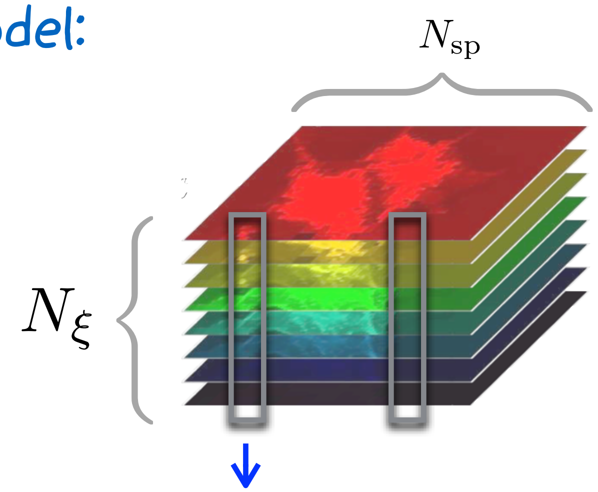

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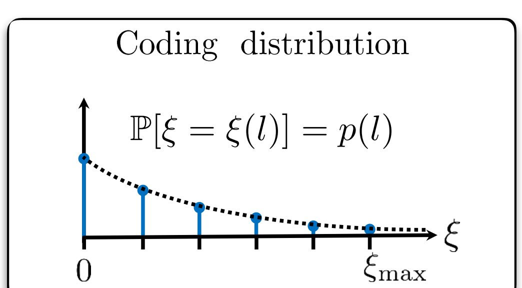
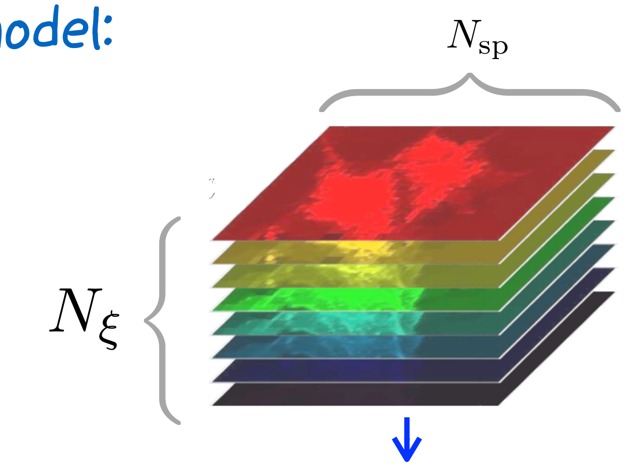
$$p(j) = C \min\big(1, \tfrac{1}{|j - \frac{1}{2} N_\xi|}\big) \quad *$$



Structured Illumination (SI-FTI):

Sparsity model:



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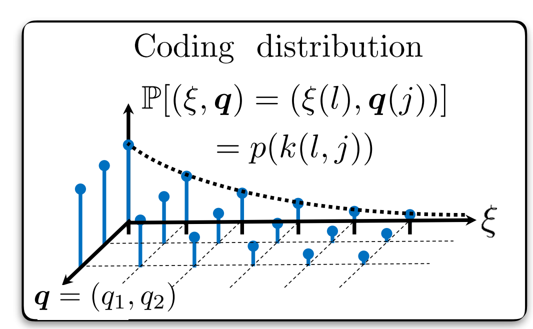
$$p(j_\xi, j_x, j_y) = C \min\Big(1, \tfrac{1}{N_{\rm sp} |j_\xi - \frac{1}{2} N_\xi|}\Big) \quad *$$

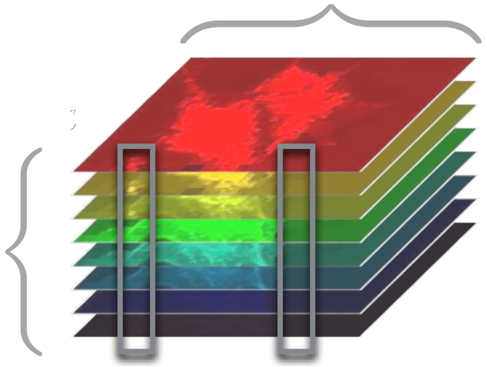


(*) By bounding the local coherence (with Fourier/Haar)

Instantiation on Compressive FTI (theory)

Coded Illumination (CI-FTI):

Sparsity model:



(locally) K_ξ-sparse
in $\boldsymbol{\Psi}_{\mathrm{1D}}$ (wavelet)

$$\boldsymbol{X} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{N_{\mathrm{sp}}})$$
$$\boldsymbol{x} = \mathrm{vec}(\boldsymbol{X})$$

(vectorized model)

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{n}$$

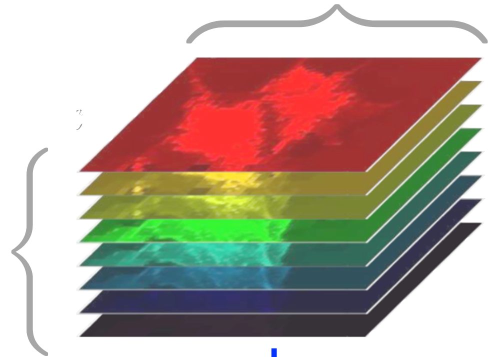
Reconstruction:

On each pixel $j \in [N_{\mathrm{sp}}]$, solve:

$$\hat{\boldsymbol{x}}_j = \mathrm{argmin}_{\boldsymbol{u} \in \mathbb{R}^{N_\xi}} \|\boldsymbol{\Psi}_{\mathrm{1D}}^\top \boldsymbol{u}\| \quad \text{s.t.} \quad \tfrac{1}{\sqrt{M}}\|\boldsymbol{D}(\boldsymbol{y}_j - \boldsymbol{F}^*_{\Omega^\xi}\boldsymbol{u})\| \leqslant \overbrace{\tfrac{\epsilon}{\sqrt{N_{\mathrm{sp}}}}}^{\text{noise per pixel}}$$

Structured Illumination (SI-FTI):

Sparsity model:



(globally) K-sparse
in $\boldsymbol{\Psi}_{\mathrm{1D}} \otimes \boldsymbol{\Psi}_{\mathrm{2D}}$
(wavelet x wavelet)

Reconstruction:

$$\hat{\boldsymbol{x}} = \mathrm{argmin}_{\boldsymbol{u} \in \mathbb{R}^{N}} \|(\overbrace{\boldsymbol{\Psi}_{\mathrm{2D}} \otimes \boldsymbol{\Psi}_{\mathrm{1D}}}^{\boldsymbol{\Psi}})^\top \boldsymbol{u}\| \quad \text{s.t.} \quad \tfrac{1}{\sqrt{M}}\|\boldsymbol{D}(\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{u})\| \leqslant \epsilon$$

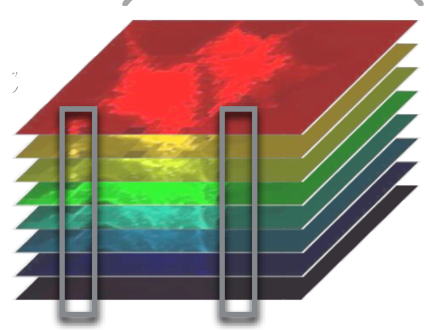
Instantiation on Compressive FTI (theory)

Coded Illumination (CI-FTI):

Sparsity model:

N_{sp}

N_ξ



(locally) K_ξ-sparse
in $\boldsymbol{\Psi}_{1\text{D}}$ (wavelet)

$$\boldsymbol{X} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{N_{\text{sp}}})$$
$$\boldsymbol{x} = \operatorname{vec}(\boldsymbol{X})$$

(vectorized model)

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Error: 🙂

If $M_\xi \gtrsim K_\xi \ln^3(K_\xi)\ln^2(N_\xi)$, then, w.h.p.,

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O\Big(K_\xi^{-1/2}\big(\sum_{j=1}^{N_{\text{sp}}} \|\boldsymbol{\Psi}_{1\text{D}}^\top \boldsymbol{x}_j - (\boldsymbol{\Psi}_{1\text{D}}^\top \boldsymbol{x}_j)_{K_\xi}\|_1^2\big)^{1/2} + \epsilon\Big)$$

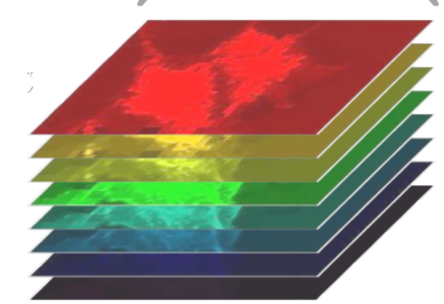
😕

Structured Illumination (SI-FTI):

Sparsity model:

N_{sp}

N_ξ



(globally) K-sparse
in $\boldsymbol{\Psi}_{1\text{D}} \otimes \boldsymbol{\Psi}_{2\text{D}}$
(wavelet x wavelet)

Reconstruction:

$$\hat{\boldsymbol{x}} = \operatorname{argmin}_{\boldsymbol{u} \in \mathbb{R}^N} \|(\overbrace{\boldsymbol{\Psi}_{2\text{D}} \otimes \boldsymbol{\Psi}_{1\text{D}}}^{\boldsymbol{\Psi}})^\top \boldsymbol{u}\| \quad \text{s.t.} \quad \tfrac{1}{\sqrt{M}} \|\boldsymbol{D}(\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{u})\| \leqslant \epsilon$$

Error: 😕

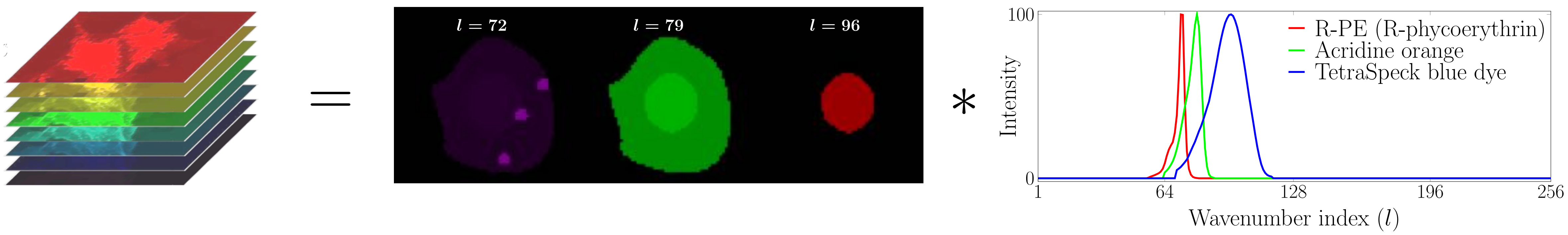
If $M \gtrsim N_{\text{sp}} K \ln^3(K)\ln(N)^2$, then, w.h.p.,

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O\big(K^{-1/2}\|\boldsymbol{\Psi}^\top \boldsymbol{x} - (\boldsymbol{\Psi}^\top \boldsymbol{x})_K\|_1 + \epsilon\big)$$

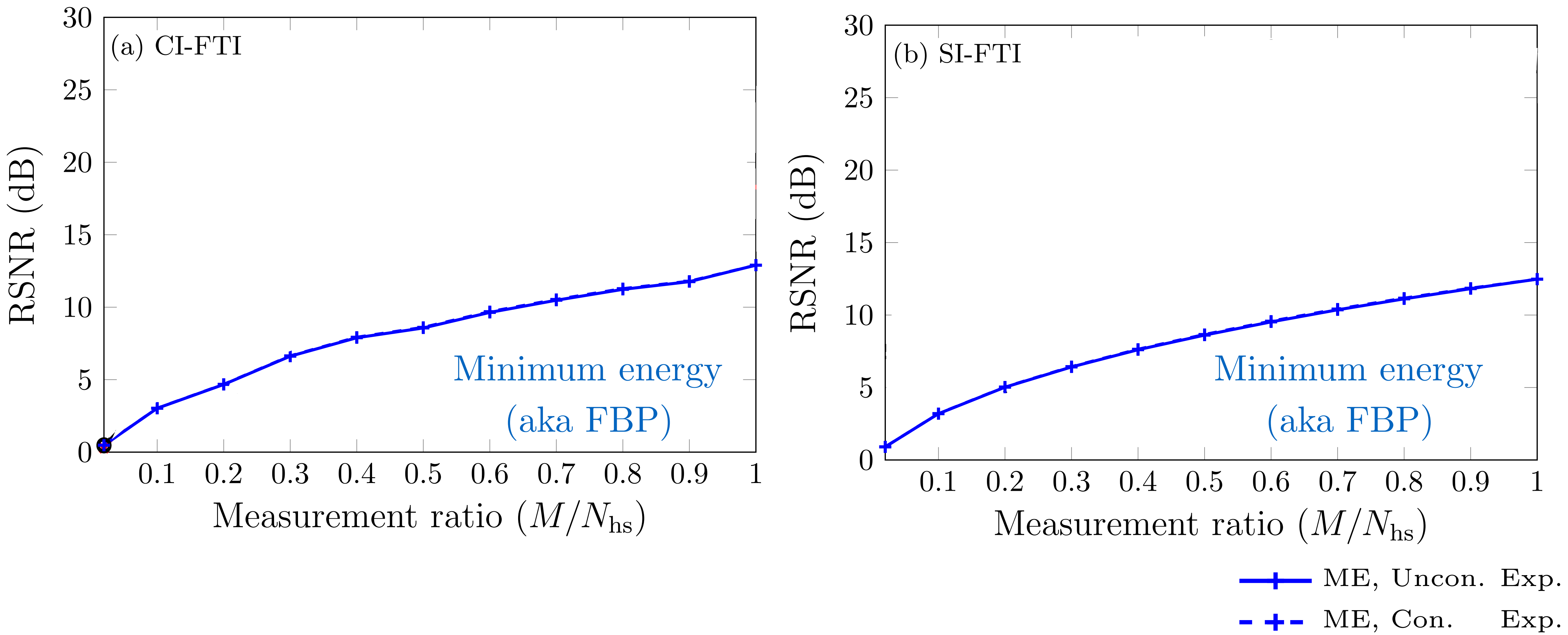
🙂

Numerical Simulations

Synthetic Data:



Reconstruction quality:



Numerical Simulations

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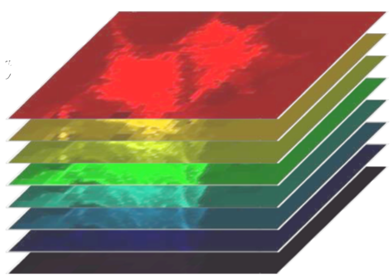
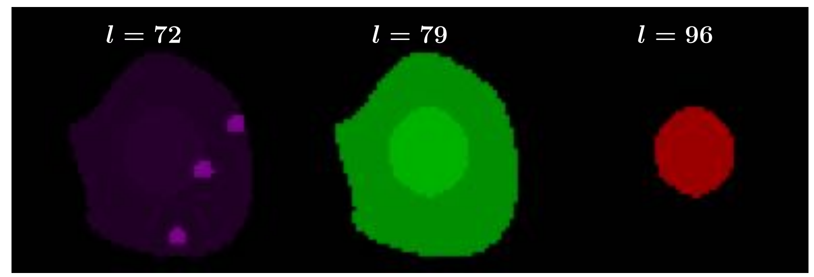
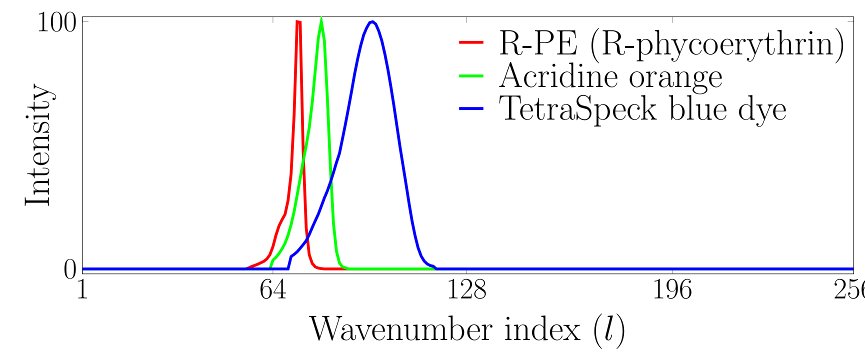


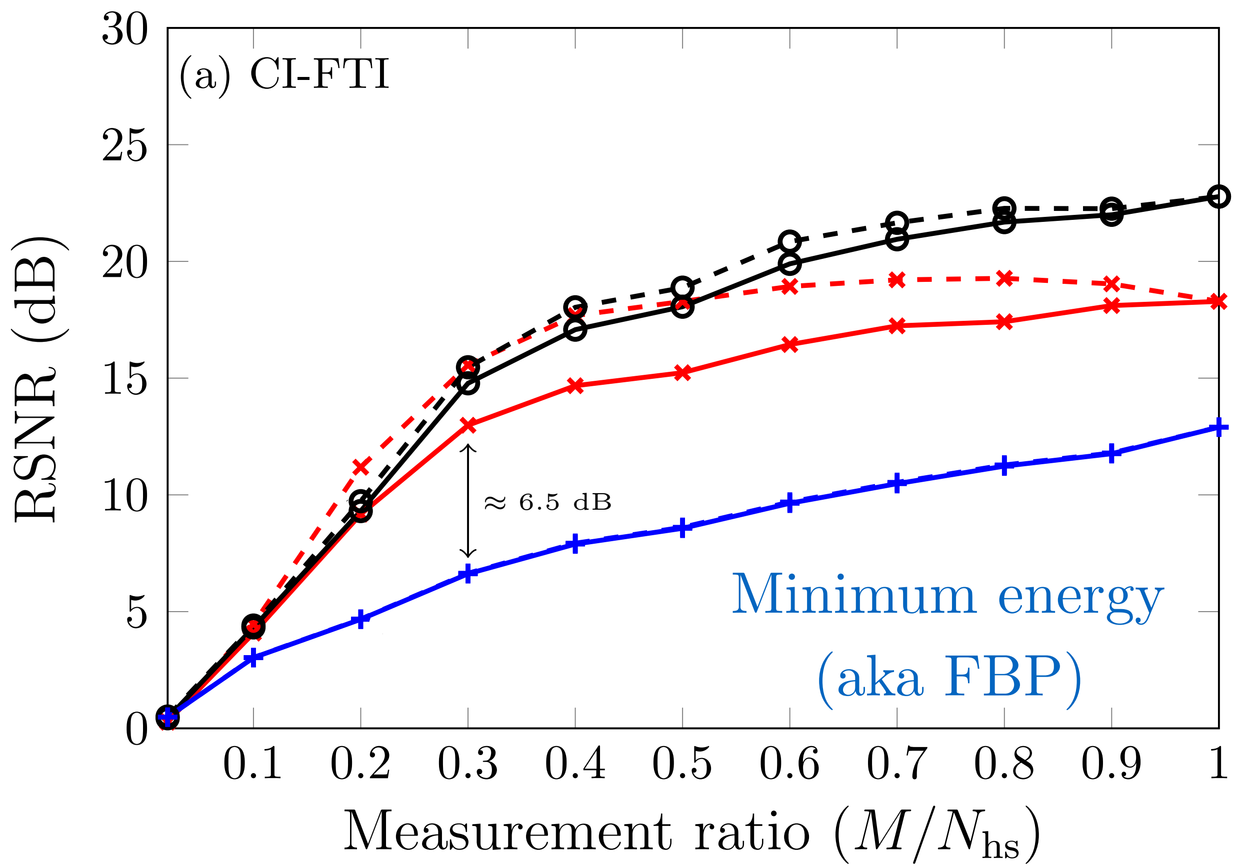
=

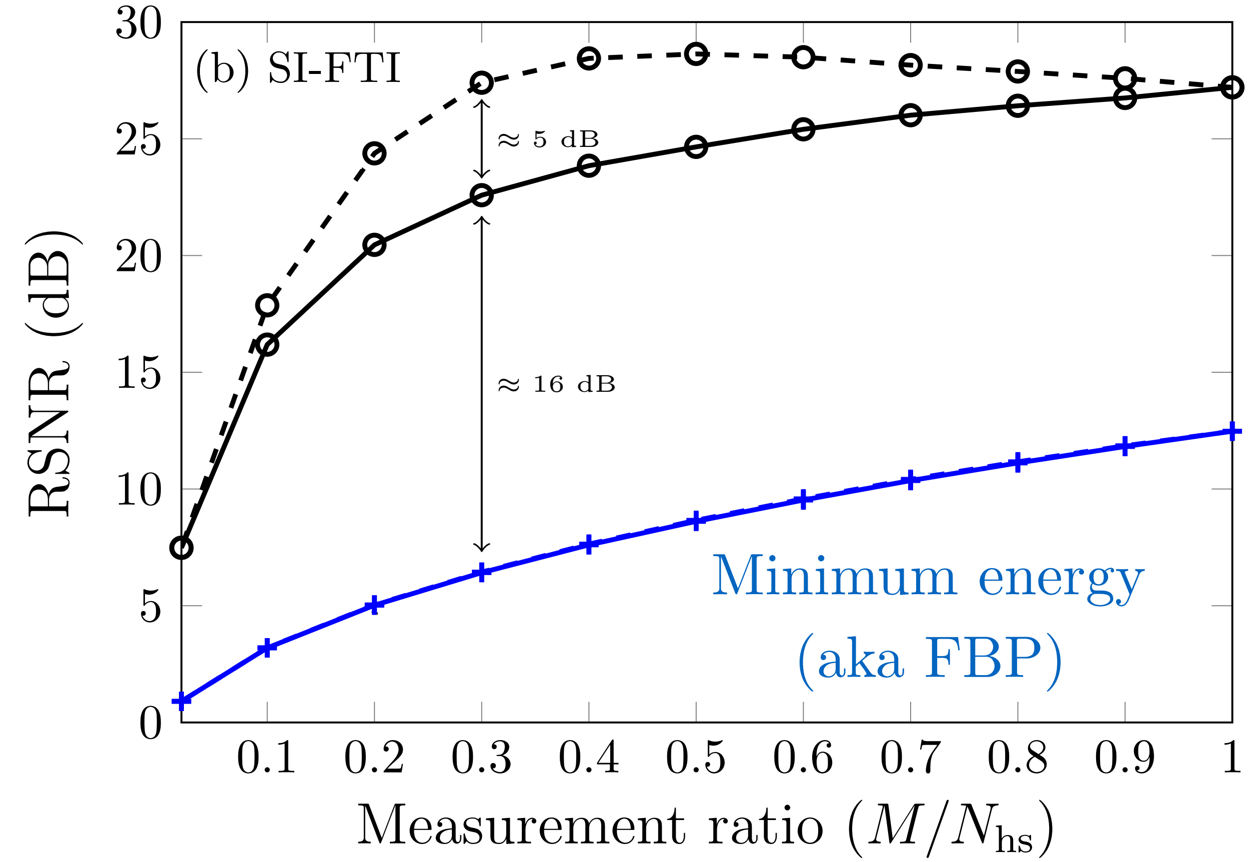


$*$



Reconstruction quality:





CS, Uncon. Exp., $\boldsymbol{\Psi} = \boldsymbol{I}_{N_{\mathrm{p}}} \otimes \boldsymbol{\Psi}_{\mathrm{1D}}$ — CS, Uncon. Exp., $\boldsymbol{\Psi} = \boldsymbol{\Psi}_{\mathrm{2D}} \otimes \boldsymbol{\Psi}_{\mathrm{1D}}$ — ME, Uncon. Exp.

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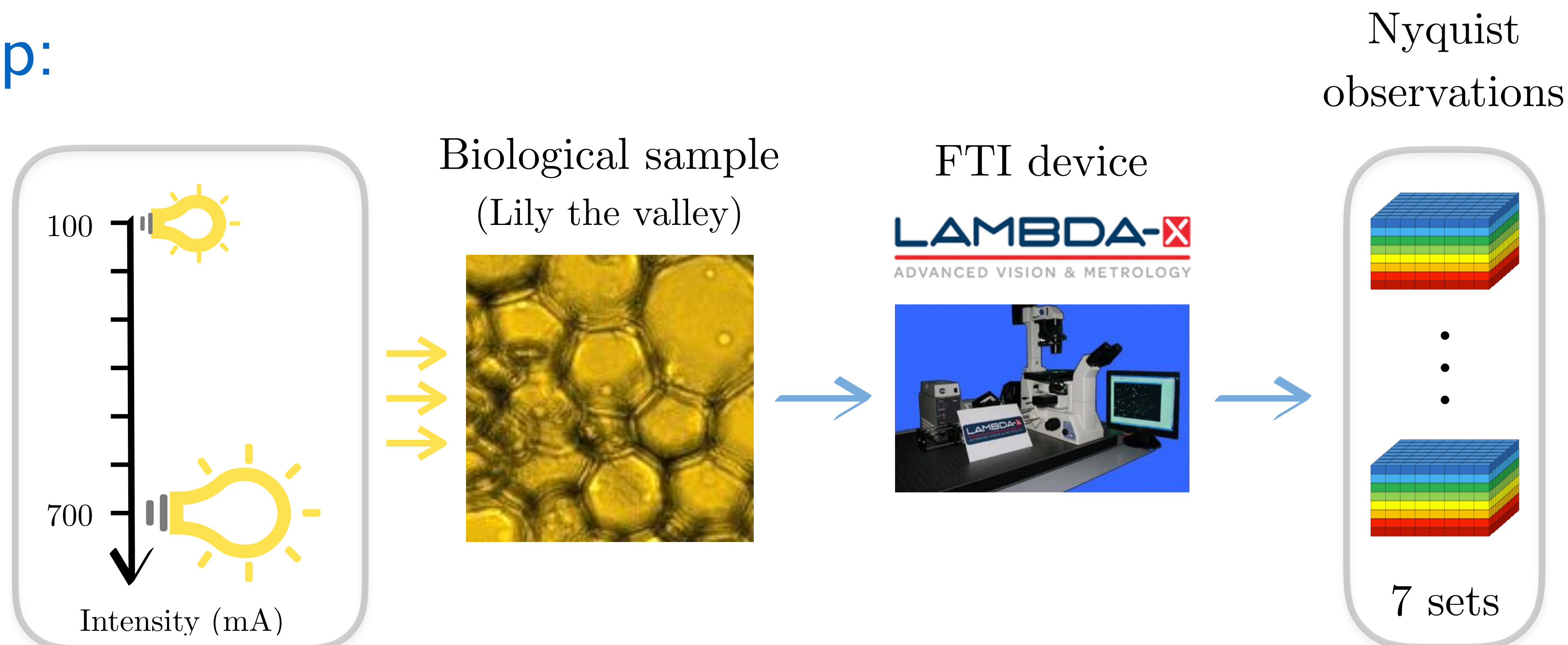
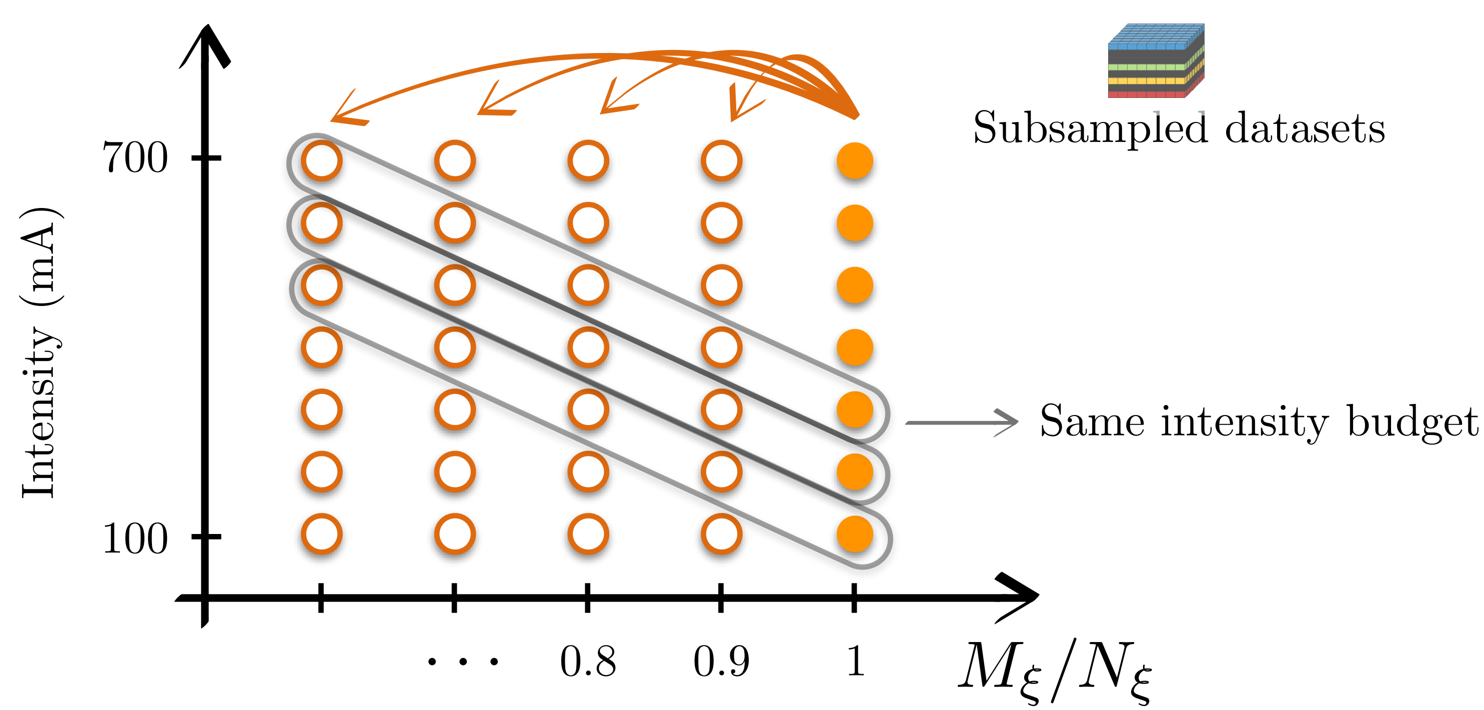
Real experiments with constrained intensity

Setup:

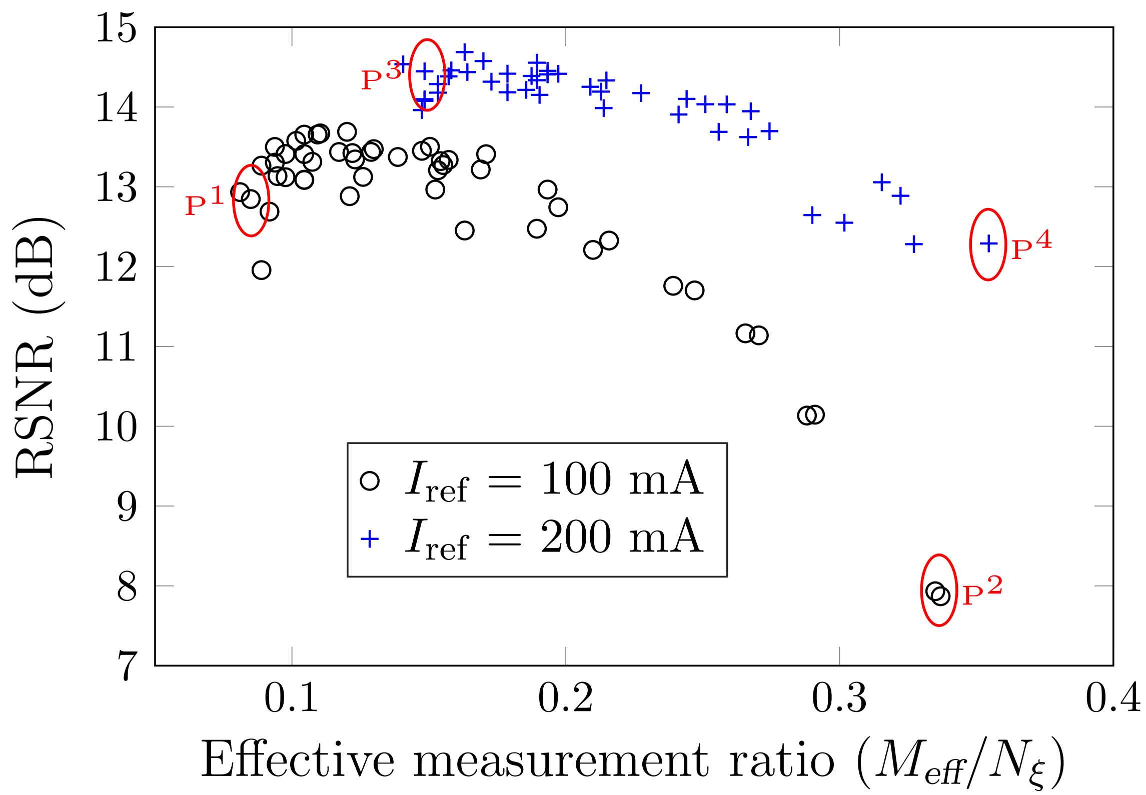


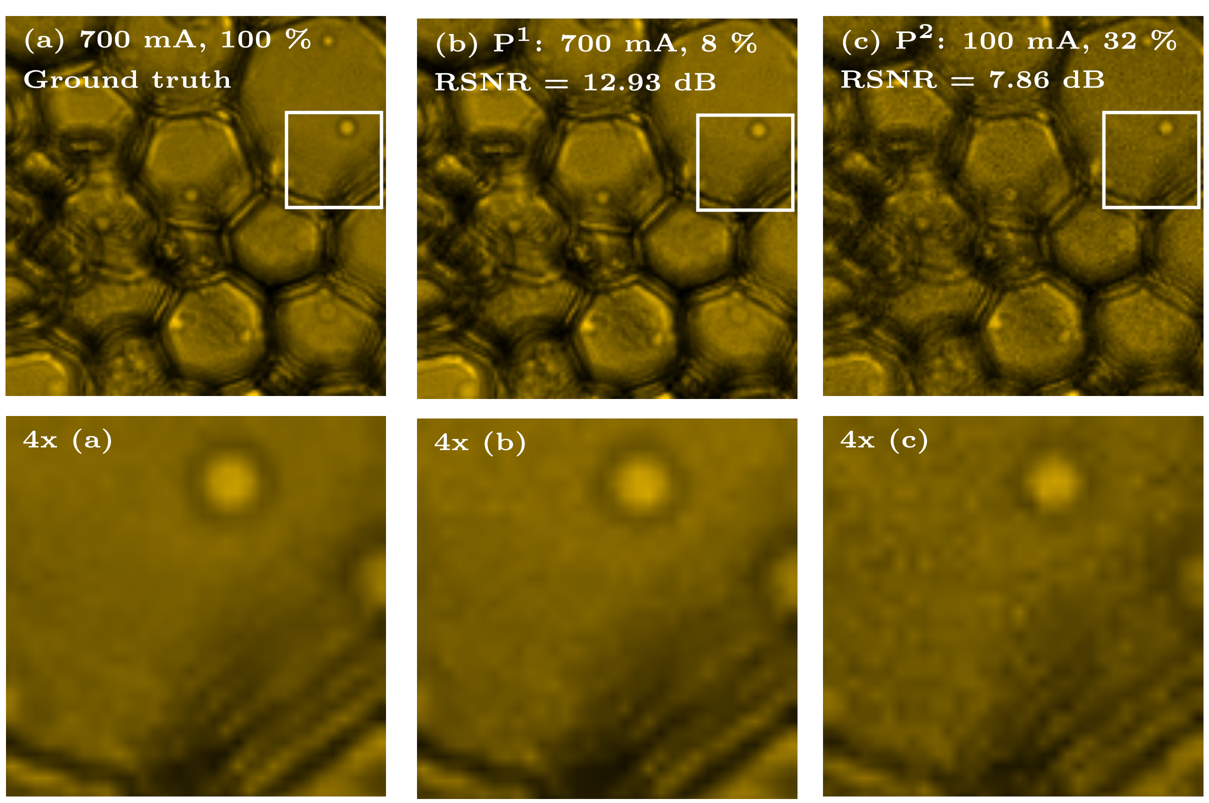
+ subsampling to simulate CI-FTI

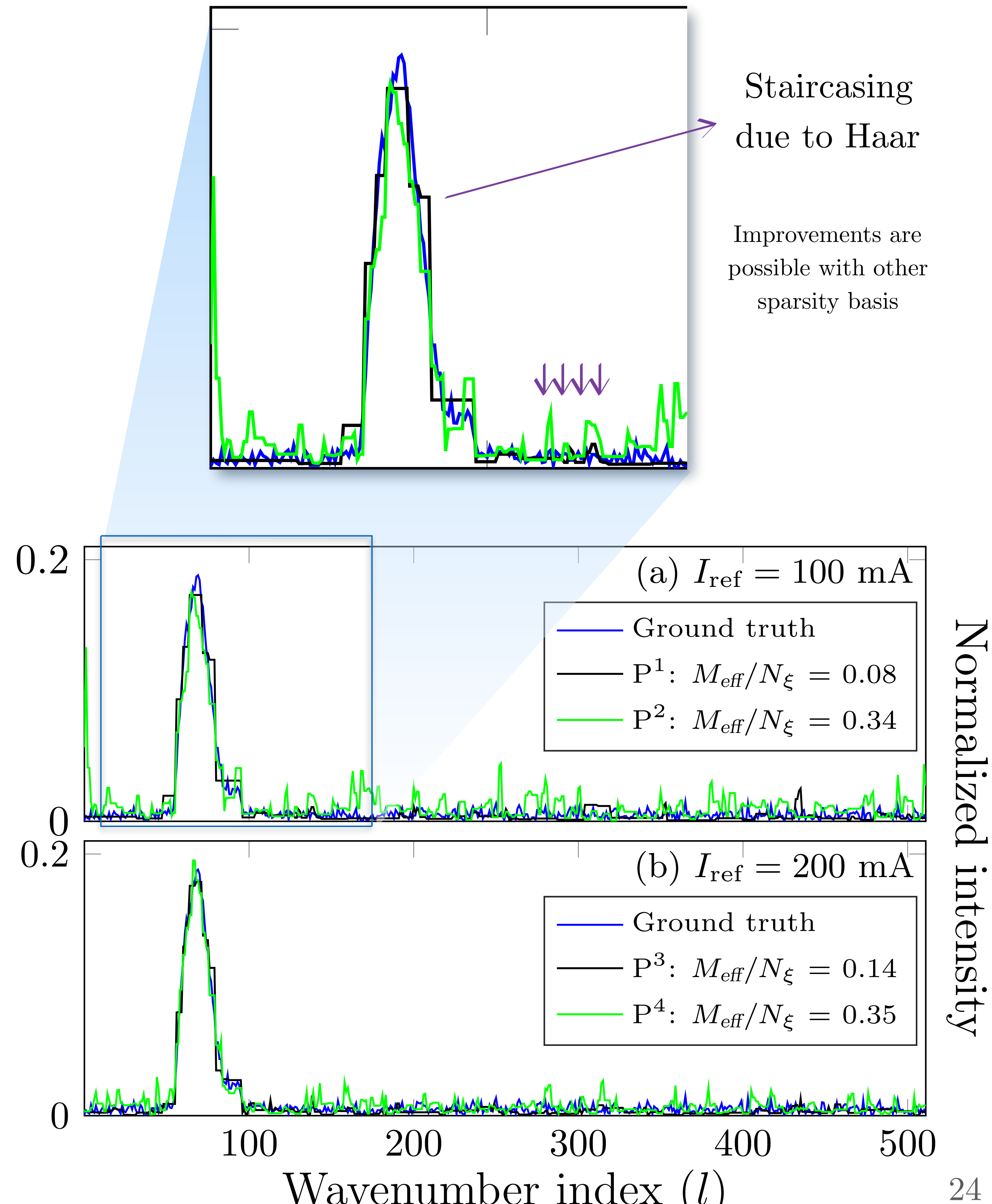
(cheating a bit)



Real experiments with constrained intensity



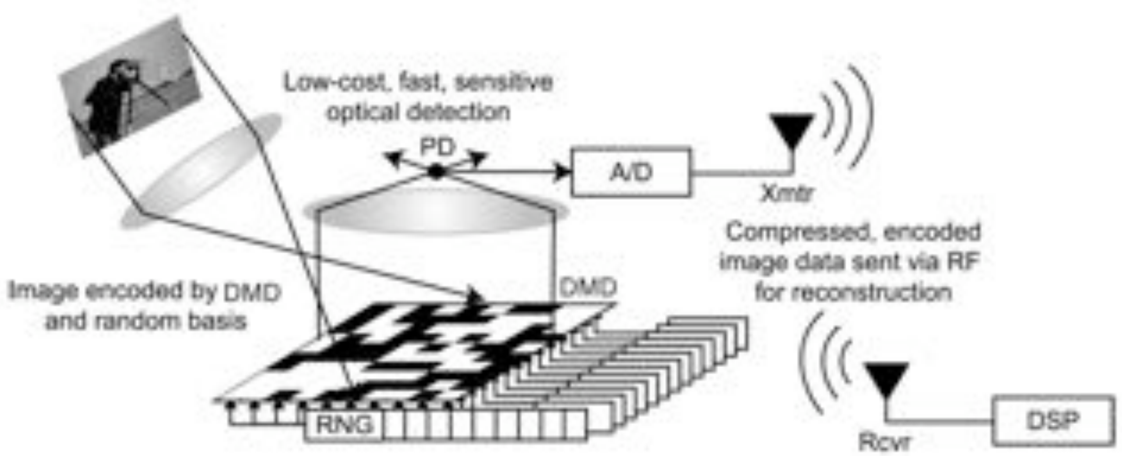


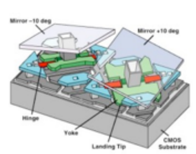


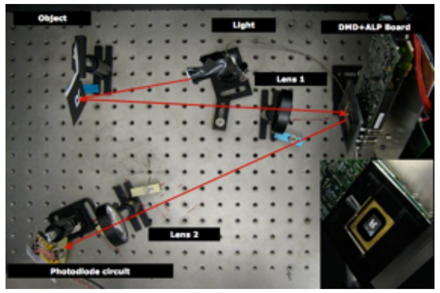
"Single-pixelization" of FTI

Context:

Single Pixel Camera

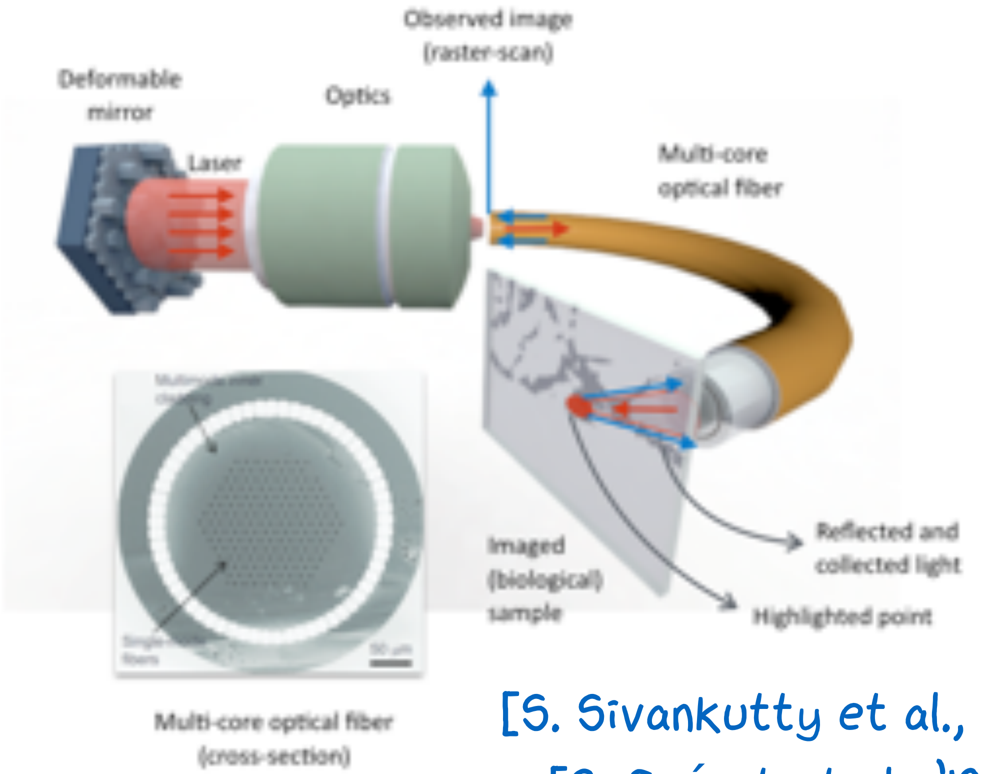






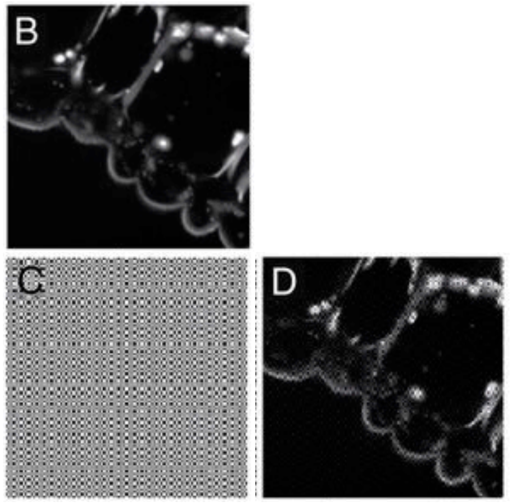
[Duarte et al, '08]

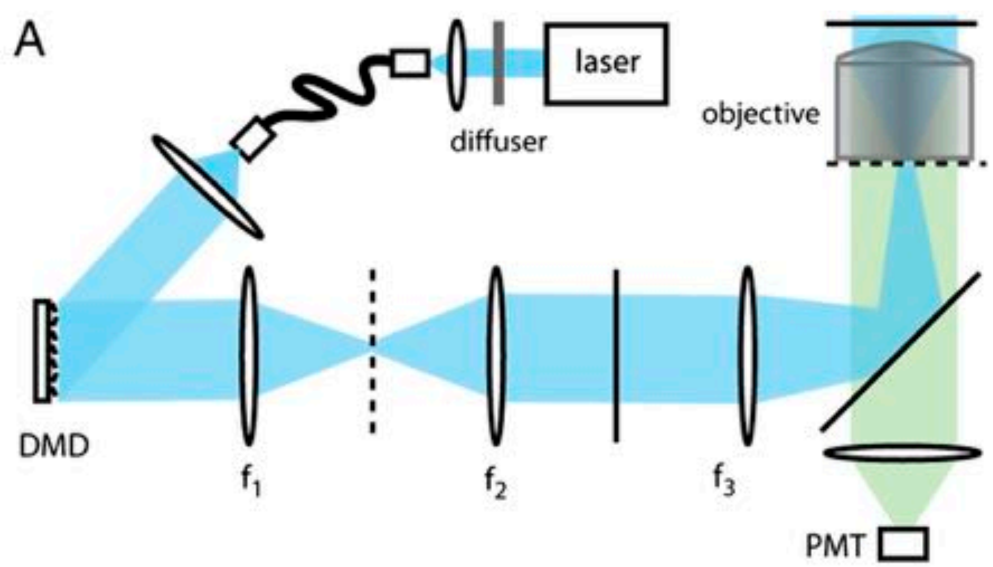
Lensless Endoscope



[S. Sivankutty et al., '18]
[S. Guérit et al., '18]

Compressive fluorescence microscopy

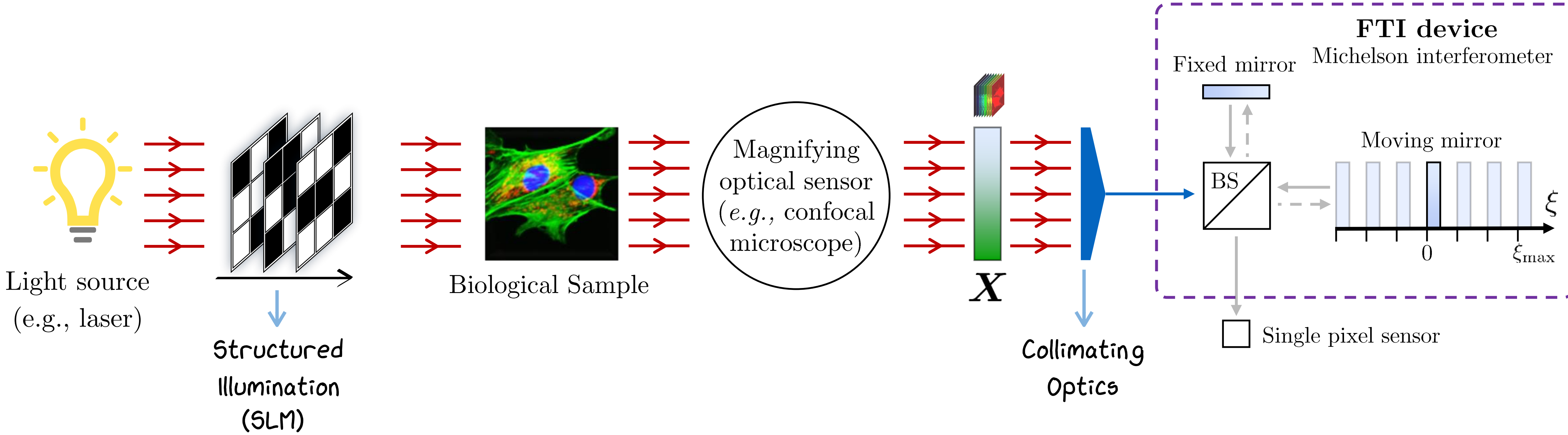




[V. Studer et al, '12]

This part

- "single-pixelization" of FTI !



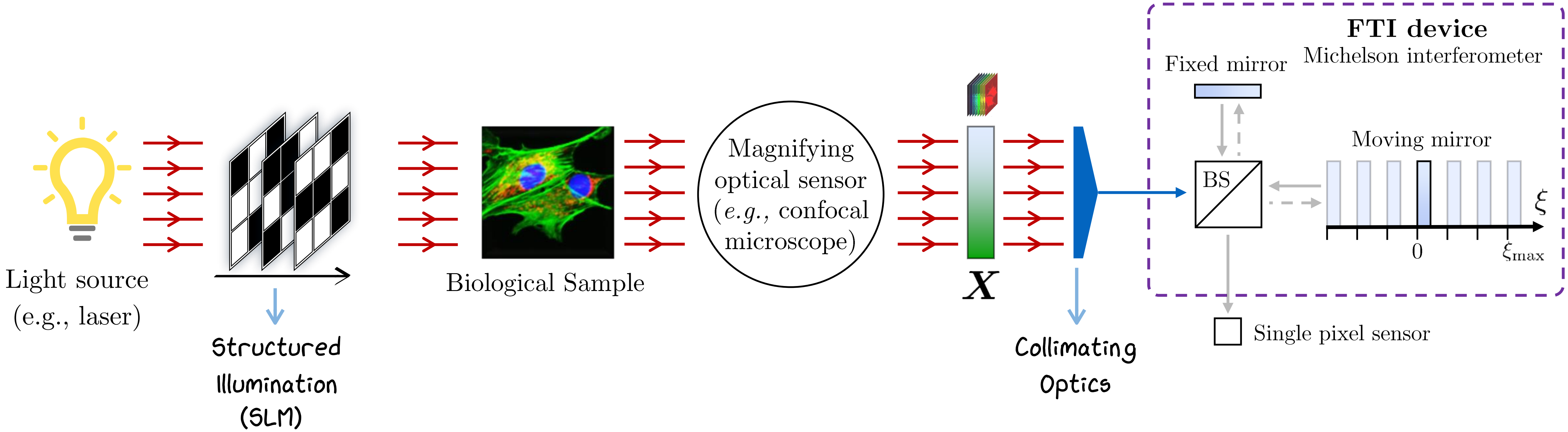
Interest? Bring HS imaging to single-pixel imaging!

<u>Contributions:</u>

- Focus on variable density sampling (VDS) strategies & reconstruction guarantees of such schemes [Krahmer, Ward, 14]
- Analysis of Hadamard-Haar sensing/sparsity systems
- Numerical confirmations of boosted image qualities

This part

- "single-pixelization" of FTI!



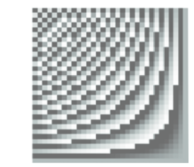
Sensing & sparsity models?

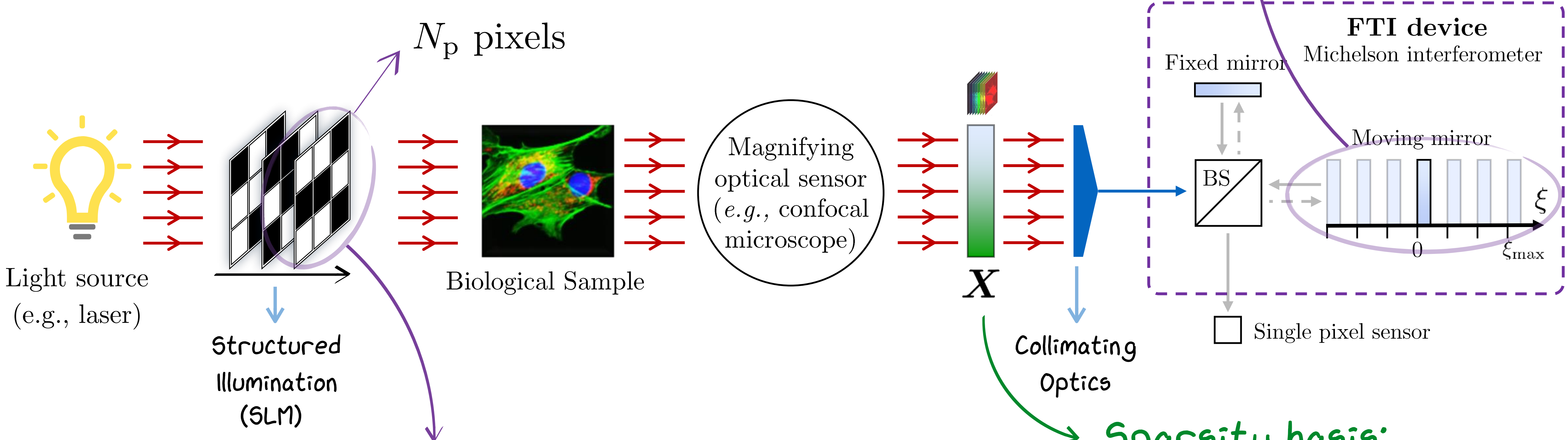
This part

- "single-pixelization" of FTI !

OPD sensing basis:

1-D DFT (imposed by Optics) $\boldsymbol{\Phi}_{\rm dft} \in \mathbb{C}^{N_\xi \times N_\xi}$

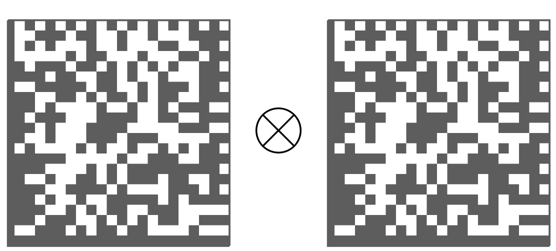
N_ξ OPDs $=$ 



Spatial sensing basis:

2-D Hadamard $\boldsymbol{\Phi}_{\rm had} \in \left\{ \pm \frac{1}{\sqrt{N_{\rm p}}} \right\}^{N_{\rm p} \times N_{\rm p}}$

with $\boldsymbol{\Phi}_{\rm had} = \boldsymbol{\Phi}^{\rm 1d}_{\rm had} \otimes \boldsymbol{\Phi}^{\rm 1d}_{\rm had}$

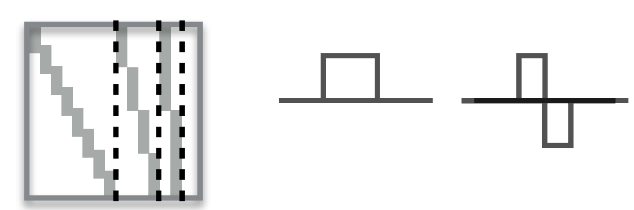
$=$ 

binary patterns
(up to a normalization)

Sparsity basis:

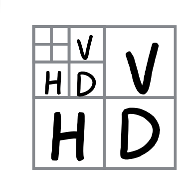
2-D isotropic x 1-D Haar wavelets

$i.e.,$, $\boldsymbol{\Psi}^*_{\rm dhw}\, \boldsymbol{X}\, \boldsymbol{\Psi}^{\rm 1d}_{\rm dhw}$ is sparse

1-D: 

2-D: 

scale by scale product
(with 3 flavors at each scale)

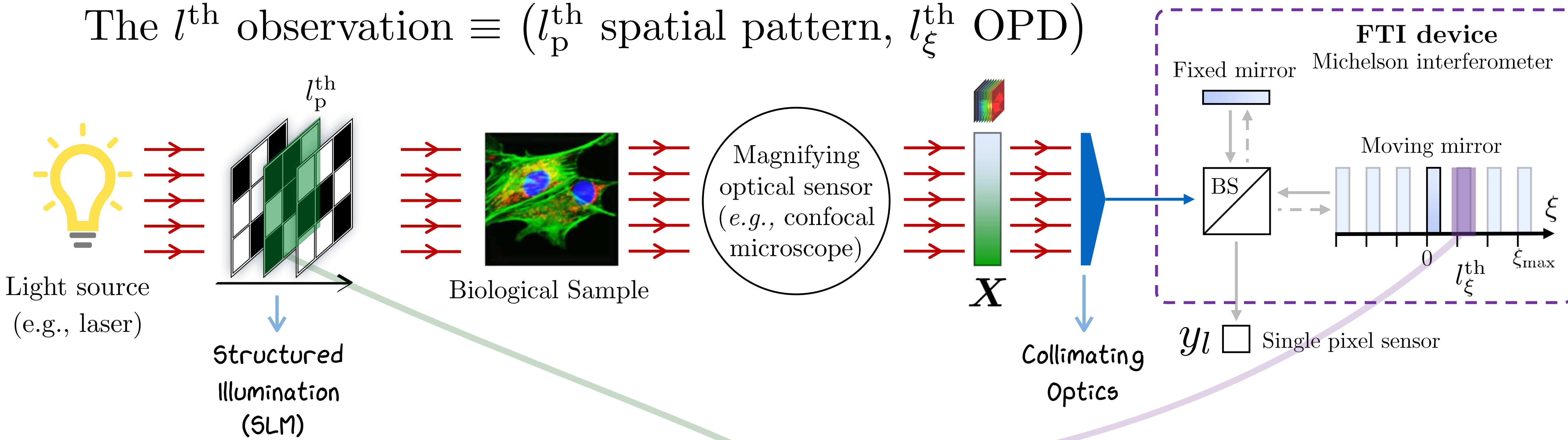


This part

- "single-pixelization" of FTI !

The l^{th} observation $\equiv$ (l_{p}^{th} spatial pattern, l_{ξ}^{th} OPD)

Fourier



Hadamard

Projections due to collimating optics & mirror position

$$y_l = \underbrace{\boldsymbol{R}_{\{l_\xi\}}\,\boldsymbol{\Phi}^*_{\text{dft}}}_{\text{proj. on the } l_\xi^{\text{th}} \text{ column of } \boldsymbol{\Phi}_{\text{dft}}}\ \boldsymbol{X}\ \underbrace{\boldsymbol{\Phi}_{\text{had}}\,\boldsymbol{R}^\top_{\{l_{\text{p}}\}}}_{\text{proj. on the } l_{\text{p}}^{\text{th}} \text{ column of } \boldsymbol{\Phi}_{\text{had}}} + \underbrace{n_l}_{\text{noise}}$$

Restriction operator

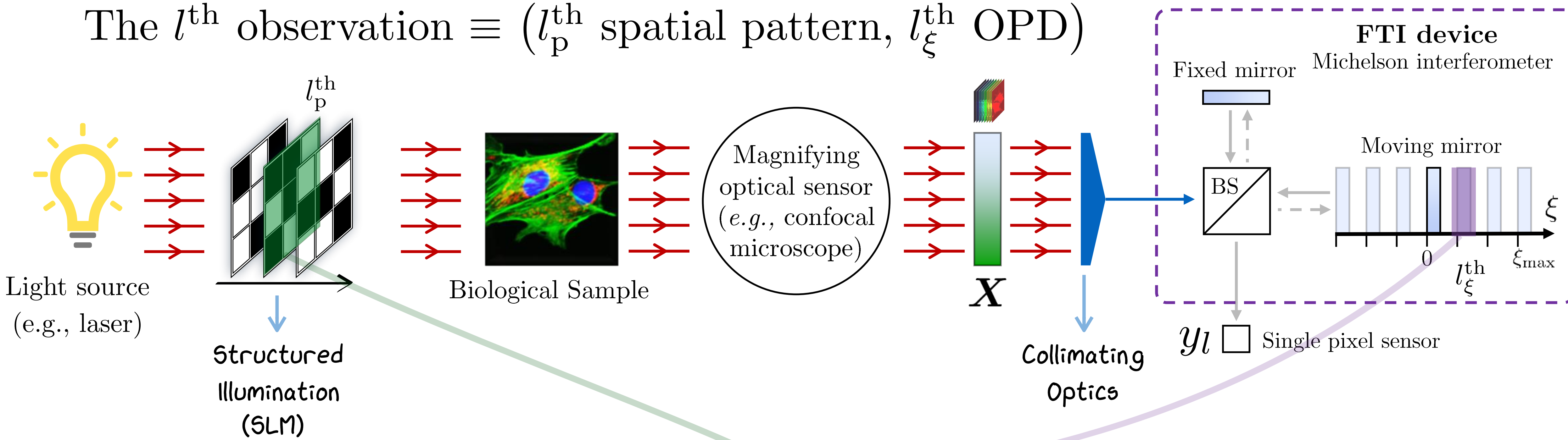
$$\Leftrightarrow \quad \boldsymbol{y} = \boldsymbol{R}_\Omega \boldsymbol{\Phi}^* \boldsymbol{x} + \boldsymbol{n}$$

This part

- "single-pixelization" of FTI !

The l^{th} observation $\equiv$ (l_{p}^{th} spatial pattern, l_{ξ}^{th} OPD)

Fourier



Hadamard

Projections due to collimating optics & mirror position

$$y_l = \underbrace{\boldsymbol{R}_{\{l_\xi\}}\, \boldsymbol{\Phi}_{\text{dft}}^*}_{\substack{\text{proj. on the } l_\xi^{\text{th}} \\ \text{column of } \boldsymbol{\Phi}_{\text{dft}}}}\, \boldsymbol{X}\, \underbrace{\boldsymbol{\Phi}_{\text{had}}\, \boldsymbol{R}_{\{l_{\text{p}}\}}^\top}_{\substack{\text{proj. on the } l_{\text{p}}^{\text{th}} \\ \text{column of } \boldsymbol{\Phi}_{\text{had}}}} + \underbrace{n_l}_{\text{noise}}$$

Restriction operator

Question: minimal number of measurements (i.e., light intensity)?

$$\Leftrightarrow \quad \boldsymbol{y} = \boldsymbol{R}_\Omega \boldsymbol{\Phi}^* \boldsymbol{x} + \boldsymbol{n}$$

$\boldsymbol{\Phi}_{\text{had}} \otimes \boldsymbol{\Phi}_{\text{dft}}$

Variable Density Sampling (VDS) in CS

(from the first part, we know that ...)

- Crucial question: estimate the local coherence of

$\mu_l\Big($  ,  $\Big)$??

Hadamard

Haar

Close encounters of the binary kind ....

- Exploit recurrent H&H design in 1-D, i.e.,

for Hadamard: $$\boldsymbol{H}_N = \tfrac{1}{\sqrt{2}}\Big[\boldsymbol{H}_{\frac{N}{2}} \otimes \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right],\ \boldsymbol{H}_{\frac{N}{2}} \otimes \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\Big], \quad \boldsymbol{H}_1 = [1]$$

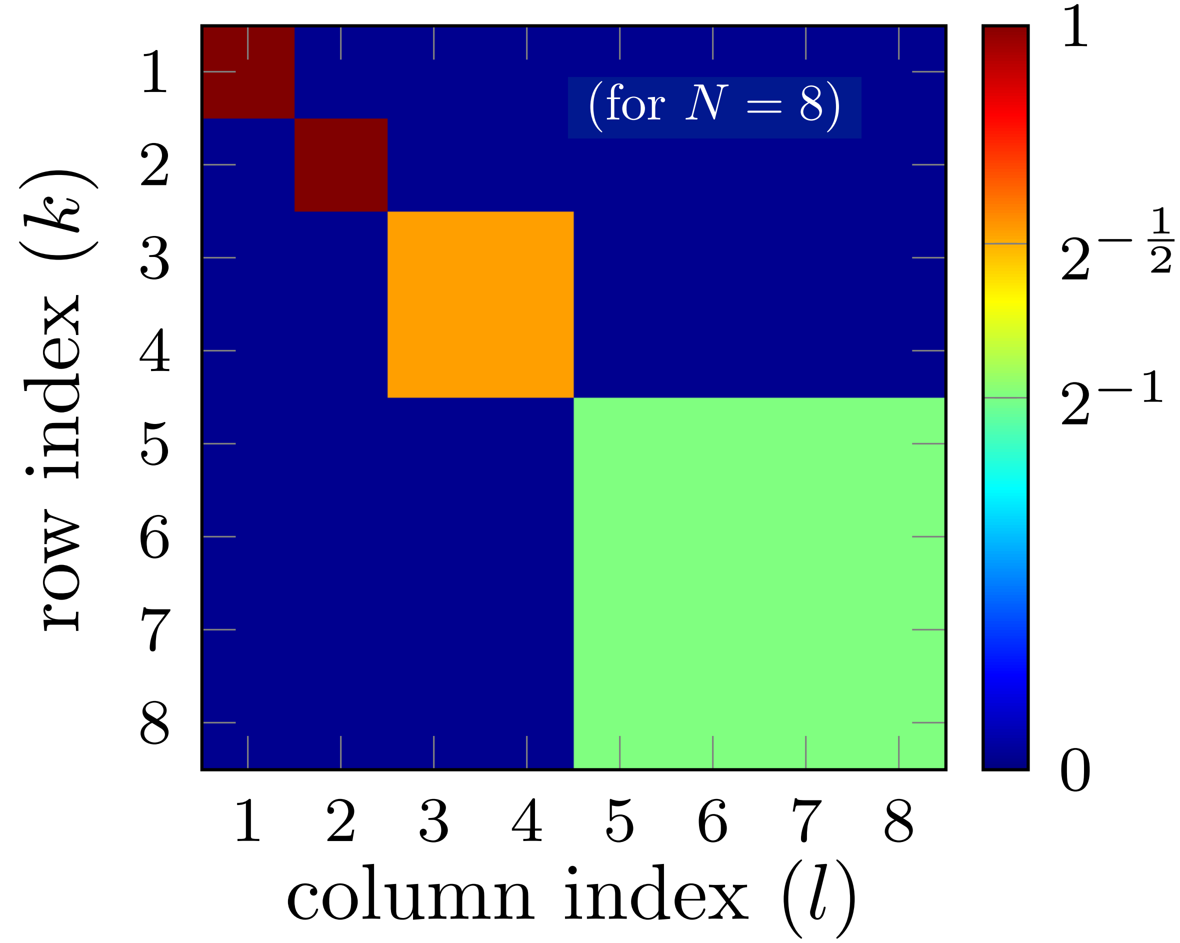
for Haar: (W: wavelet, A: approx.)

$$\begin{cases} \boldsymbol{W}_N = \tfrac{1}{\sqrt{2}}\Big[\boldsymbol{W}_{\frac{N}{2}} \otimes \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right],\ \boldsymbol{I}_{\frac{N}{2}} \otimes \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\Big] \\ \boldsymbol{A}_N = \tfrac{1}{\sqrt{2}}\Big[\boldsymbol{W}_{\frac{N}{2}} \otimes \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right],\ \boldsymbol{I}_{\frac{N}{2}} \otimes \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\Big] \end{cases}, \quad \boldsymbol{W}_1 = \boldsymbol{A}_1 = [1]$$

By ordering i & j by their levels,

$$\left(\boldsymbol{H}_N^\top \boldsymbol{W}_N\right)_{i,j}$$

gets a fractal structure!



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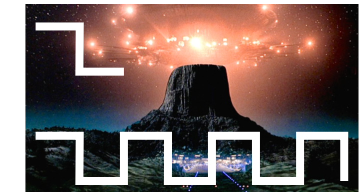

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- Coherence factors with Kronecker product, i.e.,

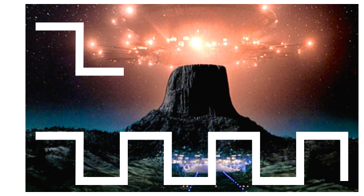
$$\mu_l(\boldsymbol{\Phi}^*\boldsymbol{\Psi}) = \mu_{l_\xi}(\boldsymbol{\Phi}^*_{\mathrm{dft}}\boldsymbol{\Psi}^{\mathrm{1d}}_{\mathrm{dhw}}) \cdot \mu_{l_{\mathrm{p}}}(\boldsymbol{\Phi}^*_{\mathrm{had}}\boldsymbol{\Psi}_{\mathrm{dhw}})$$

$\boldsymbol{\Phi}_{\mathrm{had}} \otimes \boldsymbol{\Phi}_{\mathrm{dft}}$ $\quad$ $\boldsymbol{\Psi}_{\mathrm{dhw}} \otimes \boldsymbol{\Psi}^{\mathrm{1d}}_{\mathrm{dhw}}$ $\qquad \leqslant \sqrt{\tfrac{2}{|l_\xi|}}$ [Krahmer, Ward, 14]

and $\mu_{l_{\mathrm{p}}}(\boldsymbol{\Phi}^*_{\mathrm{had}}\boldsymbol{\Psi}_{\mathrm{dhw}}) \leqslant 2^{-\lfloor \log_2(\max(l_x,l_y)-1)\rfloor}$ [Moshtaghpour et al., 19]

with: $l \equiv (l_\xi, l_{\mathrm{p}}) \equiv (l_\xi, \underbrace{(l_x, l_y)}_{\text{2-D Hadamard “frequencies”}})$

Close encounters of the binary kind ....



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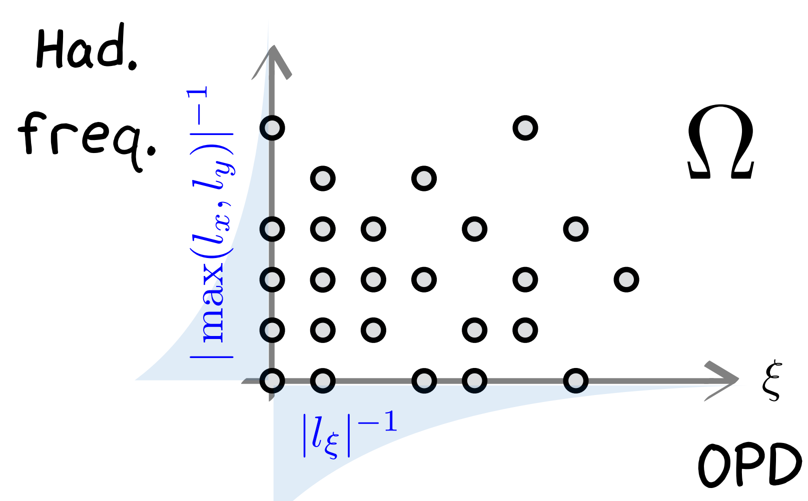
- Coherence factors with Kronecker product
- Combining everything: select patterns with pmf

$$p(l) = p(l_\xi, l_x, l_y) \ \propto \ |l_\xi|^{-1}\,|\max(l_x, l_y)|^{-1}$$



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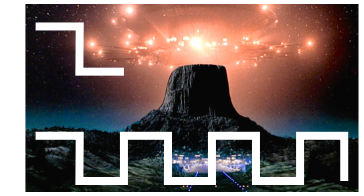

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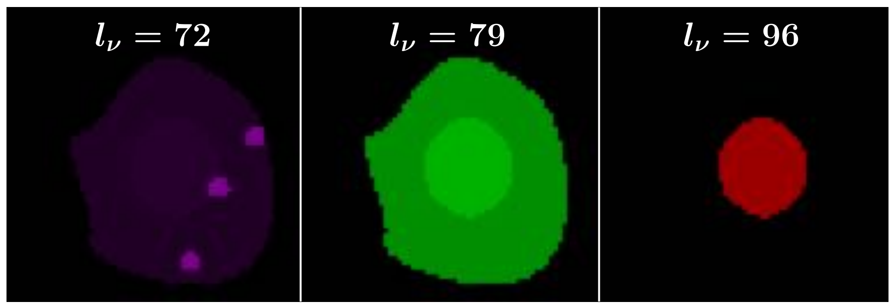
- Sample complexity for K-sparse HS volume:

$$M \geqslant c\,K \log(N_\xi)\log(N_\mathrm{p})\log(\eta)$$

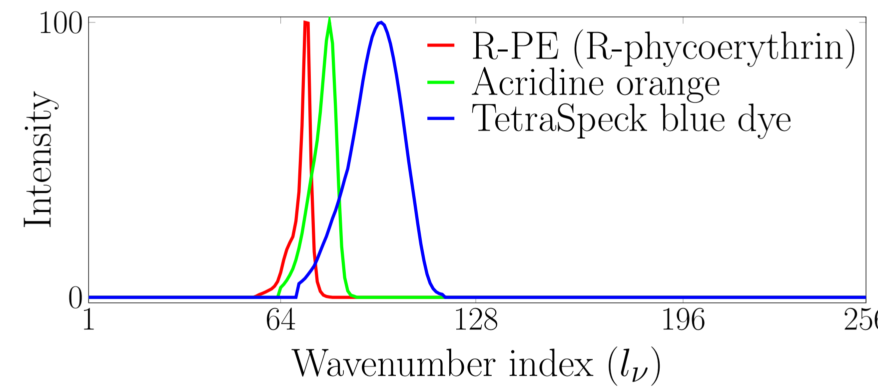
Numerical Simulations

- Synthetic HS volume: $(N_\xi, N_\mathrm{p}) = (512, 64^2)$

3 maps from database [Ruusuvuori et al, 08]



known spectra of 3 fluorochromes



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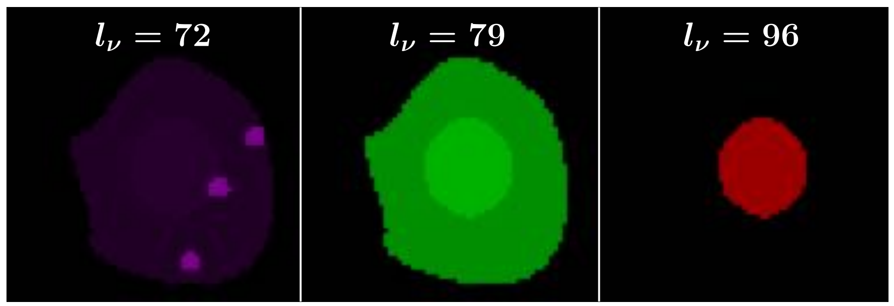

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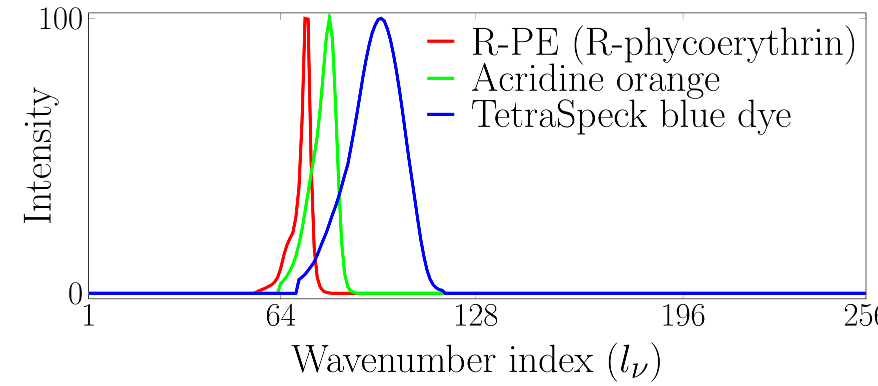
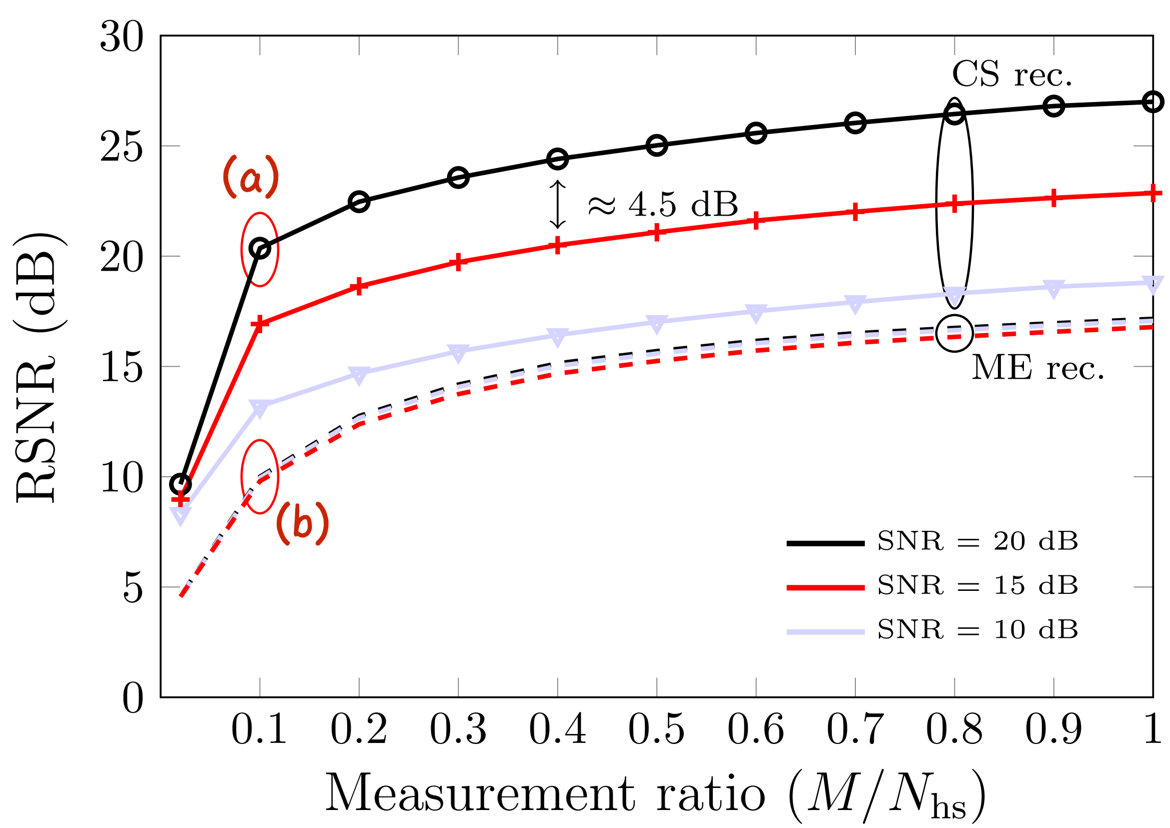
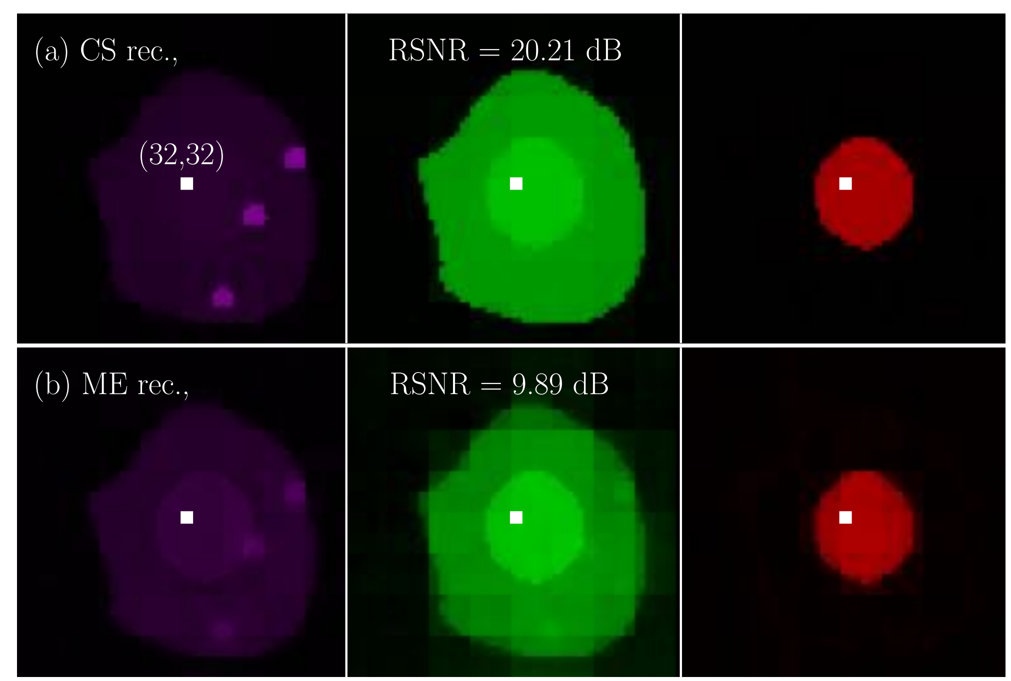


- Measurements obtained with 1-pix FTI model + noise





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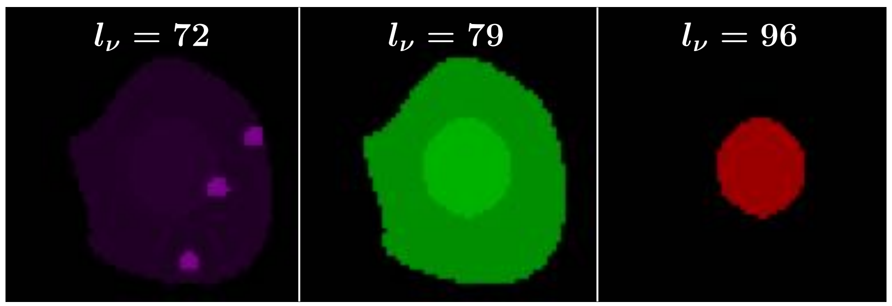

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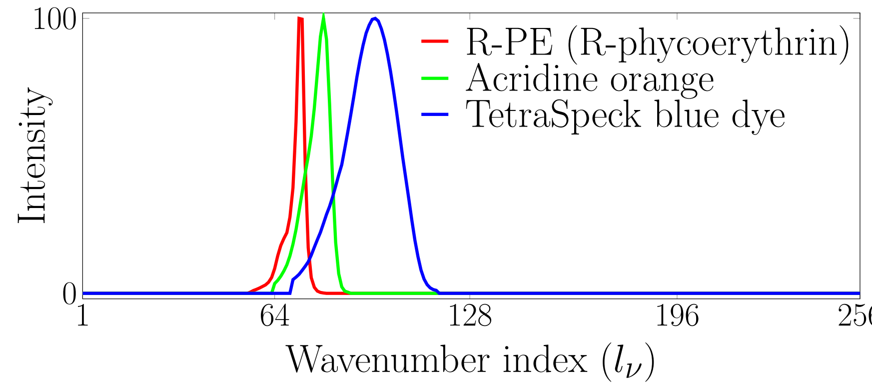

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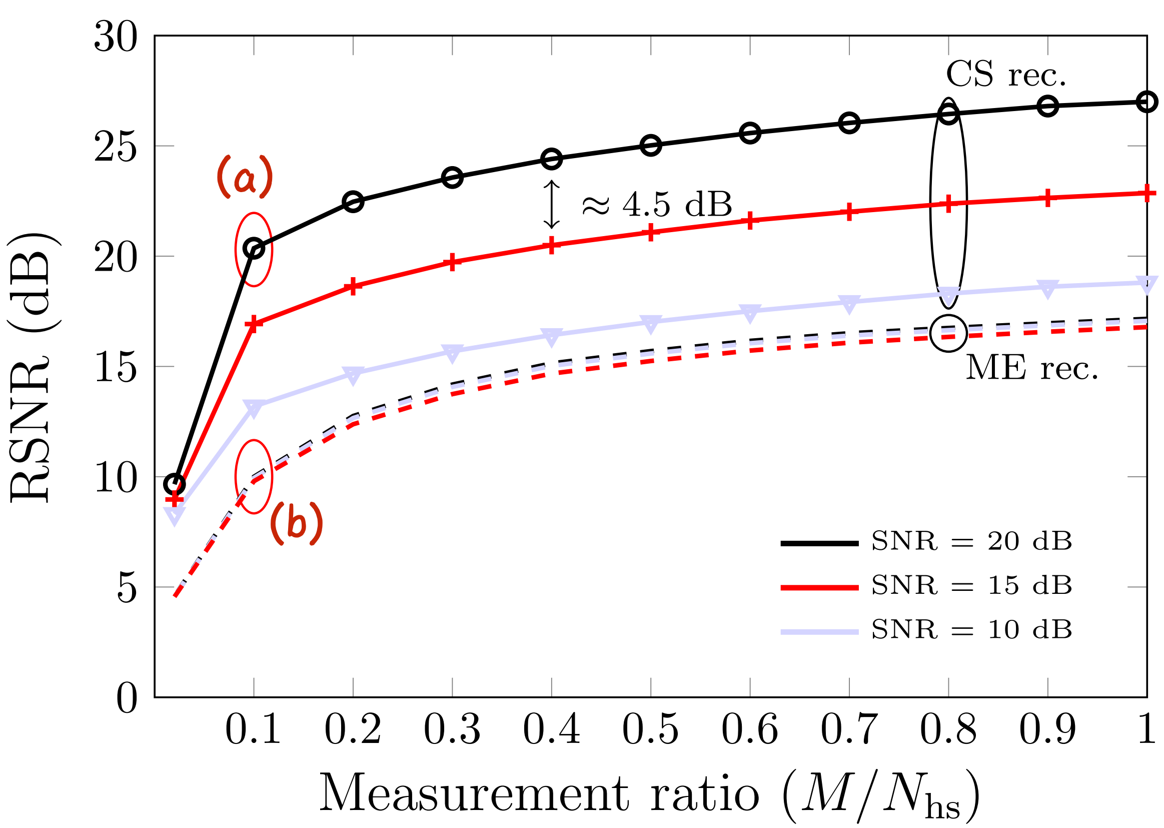

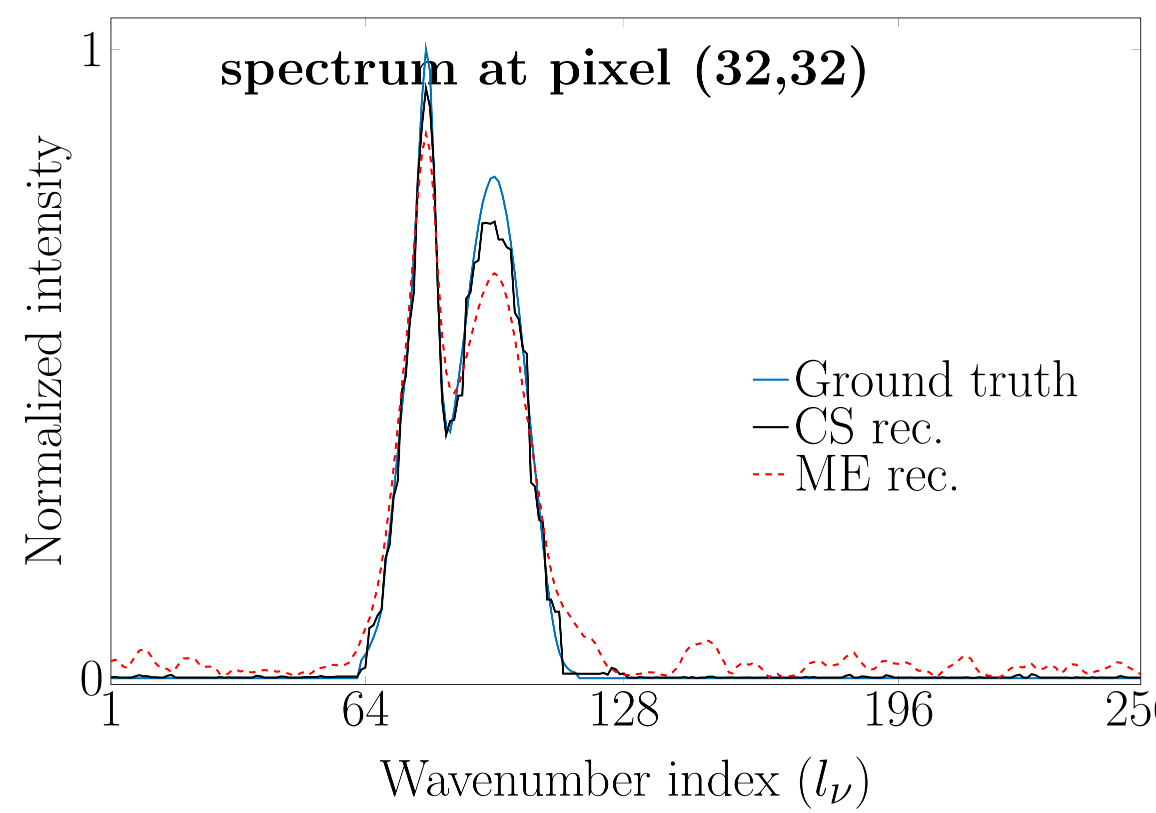

Conclusions and perspectives

- FTI is another modality where VDS/CS helps
- New answers to constrained intensity sensing
  and new possibility to limit, e.g., photobleaching.
- Promising usefulness of single-pixel FTI
  with Hadamard illumination patterns + Haar sparsity basis
- Run this on an actual setup? (for multi-pixel FTI [Moshtaghpour et al., 19])
  Gain for biological factor of merit (e.g., light exposure) ?
- Hadamard with other wavelet bases?
  Is there generalized recurrent relations?
  (connection with bases constructions in quantum physics)
- Extension to Multilevel Density Sampling [Adcock et al., 17]
  (very recent work on this in [Adcock et al., 19])

Thank you for your attention!

(subset of) References

- A. Moshtaghpour, L. Jacques, V. Cambareri, P. Antoine, and M. Roblin, "A variable density sampling scheme for compressive Fourier transform interferometry," SIAM Journal on Imaging Sciences, vol. 12, no. 2, pp. 671–715, 2019.
- V. Studer, J. Bobin, M. Chahid, H. S. Mousavi, E. Candès, and M. Dahan, "Compressive fluorescence microscopy for biological and hyperspectral imaging," Proceedings of the National Academy of Sciences, vol. 109, no. 26, pp. E1679–E1687, 2012.
- A. Michelson, "ART. XXI.--The relative motion of the Earth and the Luminiferous ether." American Journal of Science (1880–1910) 22.128 (1881): 120.
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- P. Ruusuvuori, A. Lehmussola, J. Selinummi, T. Rajala, H. Huttunen, and O. Yli-Harja, "Benchmark set of synthetic images for validating cell image analysis algorithms," in 16th European signal processing conference, 2008, pp. 1–5.
- Ben Adcock, Vegard Antun, Anders C. Hansen, "Uniform recovery in infinite-dimensional compressed sensing and applications to structured binary sampling", arXiv: 1905.00126

- … and the XKCD, font style ;-) (https://github.com/ipython/xkcd-font)

Extra slides

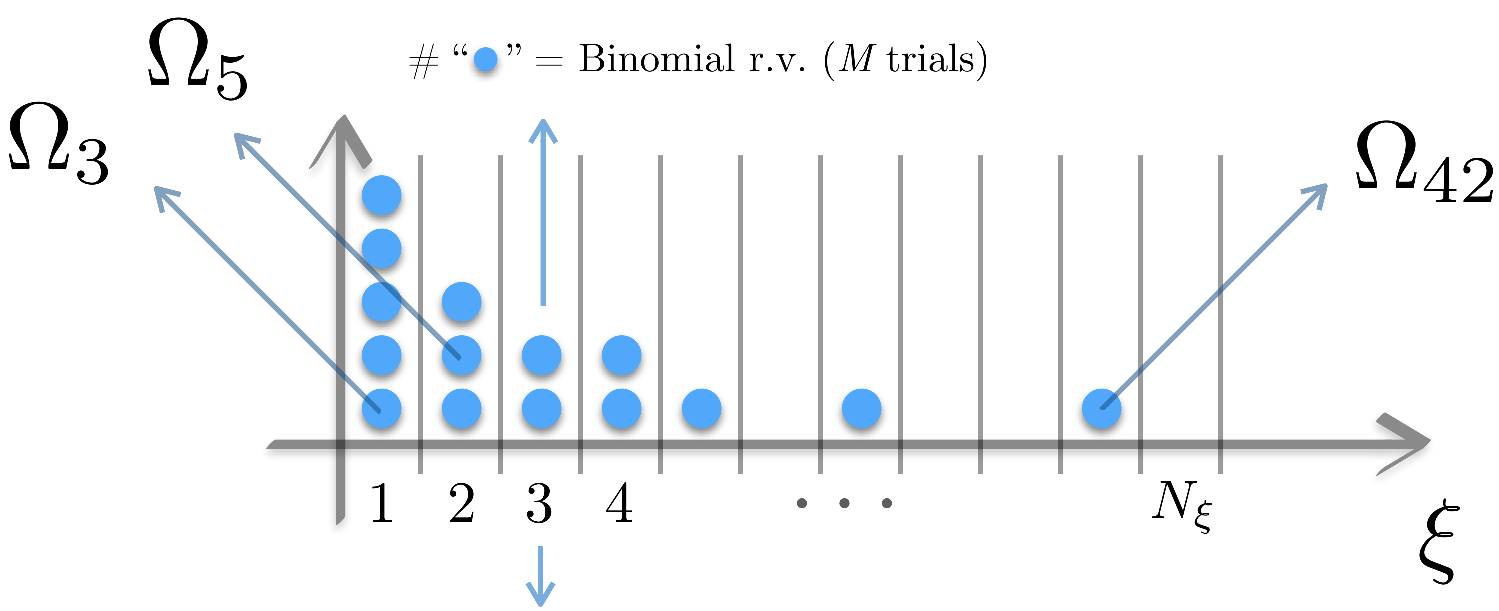
Variable Density Sampling (VDS) in CS

- Reconstruction in VDS?

Assuming $\boldsymbol{y} = \overbrace{\boldsymbol{R}_\Omega \boldsymbol{U}^*}^{\text{sensing}} \boldsymbol{x} + \boldsymbol{n}$ with $\|\frac{1}{\sqrt{M}} \boldsymbol{D} \boldsymbol{n}\| \leqslant \epsilon$, and

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u}} \|\boldsymbol{\Psi}^* \boldsymbol{u}\|_1 \text{ s.t. } \|\tfrac{1}{\sqrt{M}} \boldsymbol{D}(\underbrace{\boldsymbol{R}_\Omega \boldsymbol{U}^*}_{\text{sensing}} \boldsymbol{u} - \boldsymbol{y})\| \leqslant \epsilon$$

Interpretation of $\boldsymbol{D}$:



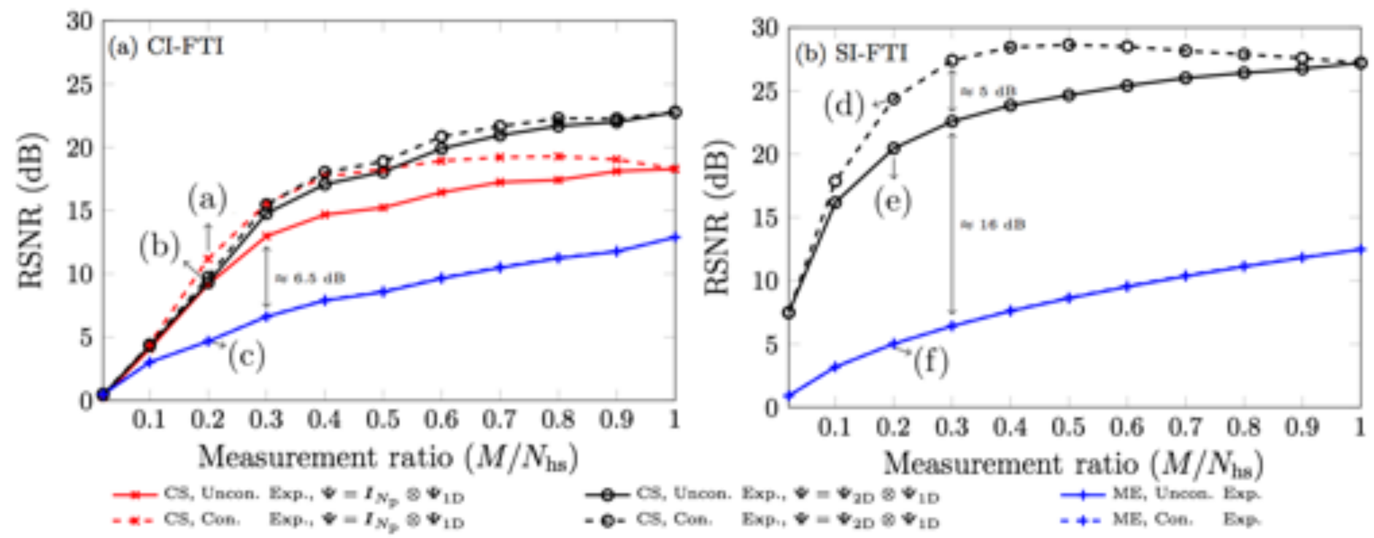
(sampling without replacement)

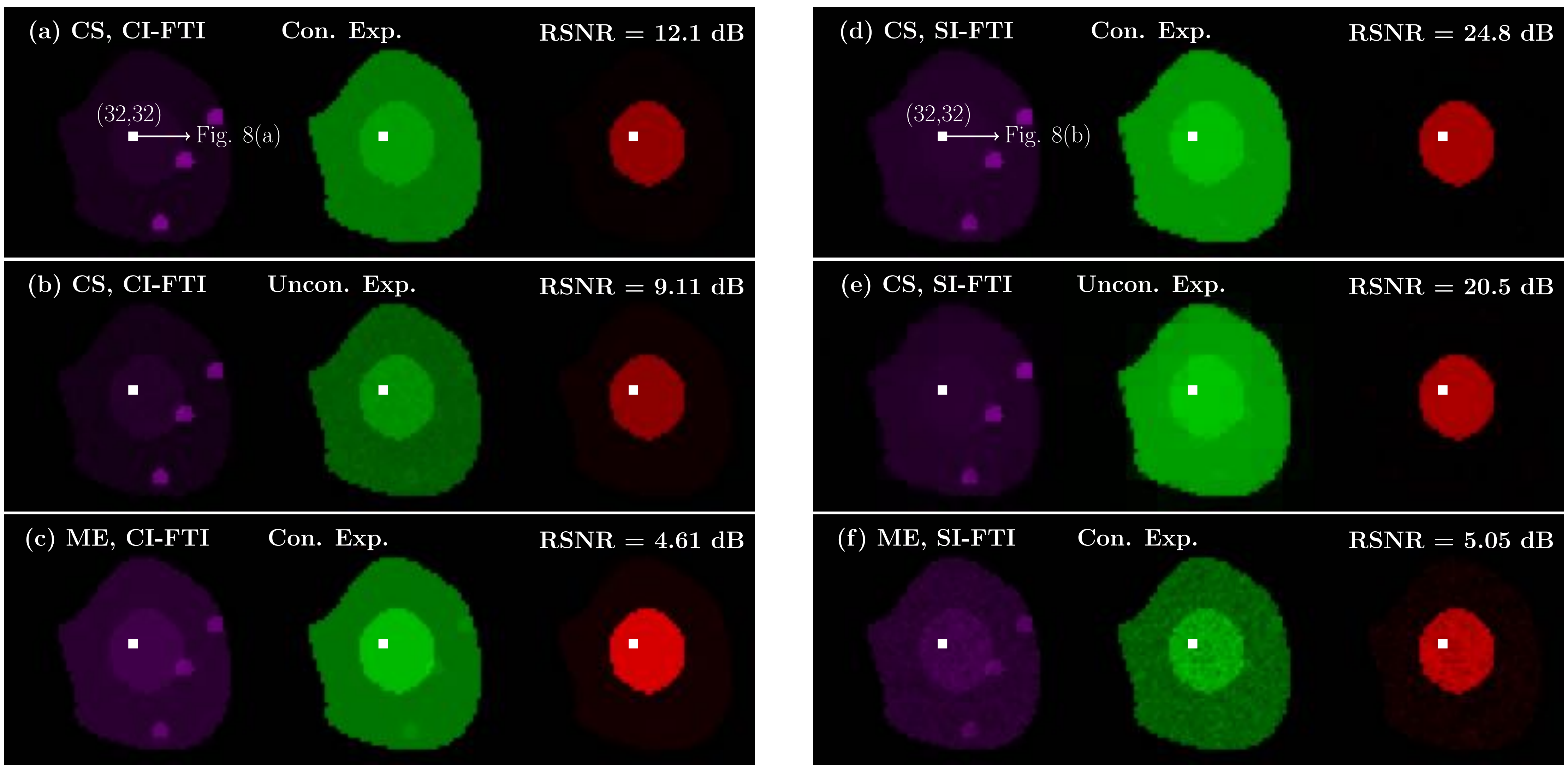
$$\mathbb{E}|\{\Omega_i = 3 : 1 \leqslant i \leqslant M\}| = Mp(3)$$

$$\Rightarrow \tfrac{1}{M} D_{jj}^2 = (\mathbb{E}|\{\Omega_i = j : 1 \leqslant i \leqslant M\}|)^{-1}$$

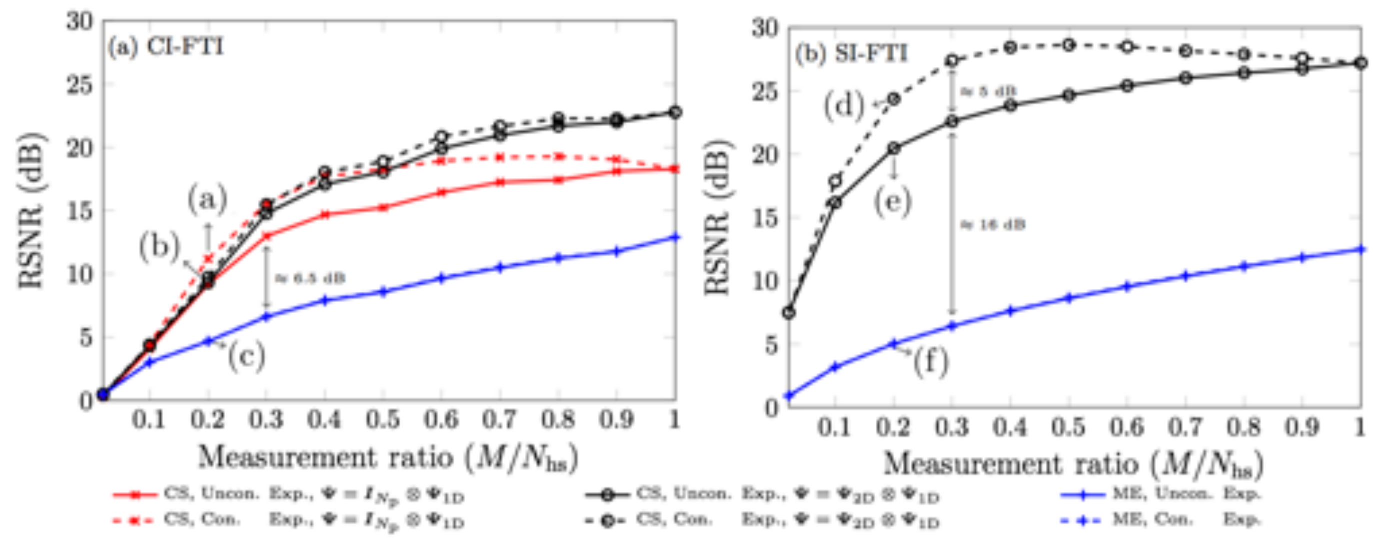
$\Rightarrow$ $\boldsymbol{D}$ accounts for the multiplicities of selected OPDs!

Numerical Simulations

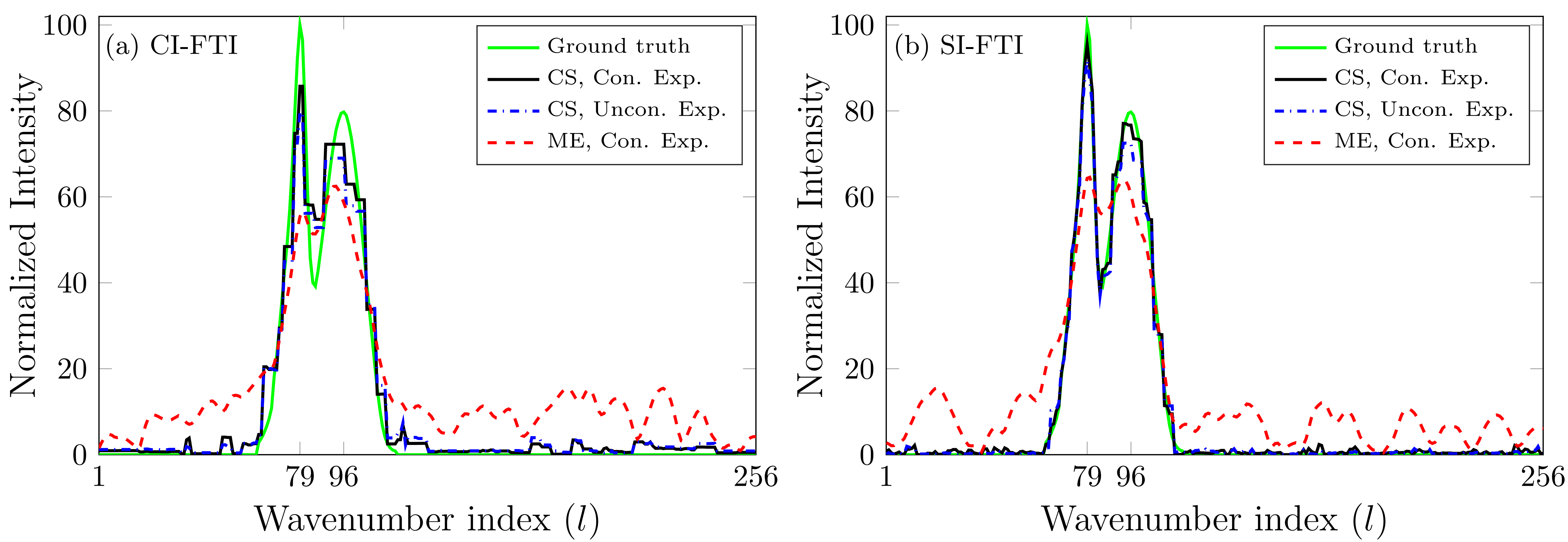




Numerical Simulations



Spectra on a single point



Variable Density Sampling (VDS) in CS

- Reconstruction in VDS?

Assuming $\boldsymbol{y} = \overbrace{\boldsymbol{R}_\Omega \boldsymbol{U}^*}^{\text{sensing}} \boldsymbol{x} + \boldsymbol{n}$ with $\|\frac{1}{\sqrt{M}} \boldsymbol{D}\boldsymbol{n}\| \leqslant \epsilon$, and

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u}} \|\boldsymbol{\Psi}^* \boldsymbol{u}\|_1 \text{ s.t. } \|\frac{1}{\sqrt{M}} \boldsymbol{D}(\underbrace{\boldsymbol{R}_\Omega \boldsymbol{U}^*}_{\text{sensing}} \boldsymbol{u} - \boldsymbol{y})\| \leqslant \epsilon$$

If $\frac{1}{\sqrt{M}} \boldsymbol{D}\boldsymbol{A}$ is $\mathrm{RIP}(\delta < 1/3, \Sigma_{5k})$, then

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(k^{-1/2} \|\boldsymbol{\Psi}^* \boldsymbol{x} - (\boldsymbol{\Psi}^* \boldsymbol{x})_k\|_1 + \epsilon)$$

- Noise estimation? $(e.g.,$ assuming $n_i \sim_{\text{i.i.d.}} \mathcal{N}(0, \sigma^2))$
  - Standard (e.g., UDS):

    Estimator of $\eta^2 \geqslant \|\boldsymbol{n}\|^2 = \sum_i n_i^2$, *e.g.*, $\eta^2 = O\big(\sigma^2 M + c\sigma^2 \sqrt{M}\big)$ w.h.p.
  - VDS, beware of $\boldsymbol{D}$:

$$\tfrac{M}{N} \|\boldsymbol{D}\boldsymbol{n}\|^2 \leqslant \epsilon^2 = O\big(\sigma^2 M + c\,\sigma^2 \sqrt{M}(1 + \rho \ln M)\big)$$

with $\rho = \rho(p)$, *i.e.*, $\rho = 0$ for UDS, $\rho = O(\ln N)$ for $p(i) \propto 1/|i|$.

e.g., for Fourier/Haar