Keep the phase!
Signal recovery in phase-only compressive sensing


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Oppenheim and Lim, 1981:

"*What's the most important information between the spectral amplitude and phase of signals?*"

Amplitude and Phase of Image Frequencies (1/2)

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A simple experiment: Let $\mathcal{F}$ the (2-D) Fourier transform

Original image $f \in \mathbb{R}^{N_x \times N_y}$

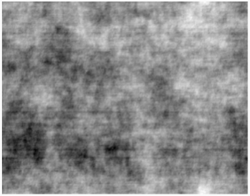


Image reconstructed with spectral amplitude

$f' = \mathcal{F}^{-1}(|\mathcal{F}f|)$

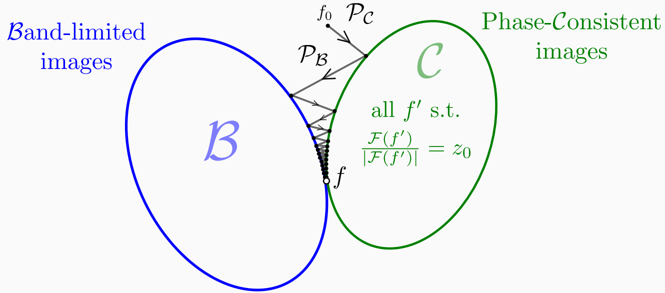
Image reconstructed with spectral phase

$f' = \mathcal{F}^{-1}(\frac{\mathcal{F}f}{|\mathcal{F}f|})$

Fact: $\exists$ algorithm to recover band-limited images from their spectral phase (up to a global amplitude).

$\Rightarrow$ Use *alternate projections onto convex sets*, *i.e.*,

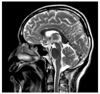
- given $z_0 = \mathcal{F}(f)/|\mathcal{F}(f)|$, the observes spectral phase,
- assuming $f \in \mathcal{B} :=$ set of band-limited images (for some cutoff freq.).

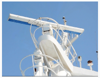


$$f^{(n+1)} = \mathcal{P}_\mathcal{B}\mathcal{P}_\mathcal{C} f^{(n)},\ f^{(0)} = f_0 \in \mathbb{C}^n, \quad \lim_{n\to+\infty} f^{(n)} = c\, f.$$

Why could it be useful?

Numerous Fourier/spectral sensing applications:





- Magnetic resonance imaging (MRI);
- Radar systems;
- Michelson interferometry / Fourier transform imaging;
- Aperture synthesis by radio interferometry.

Challenges:

- Massive data stream imposes new data compression strategies.
- Compress but keep useful information (*e.g.*, for subsequent imaging).
- Large magnitude variations $\Rightarrow$ different compression impact.

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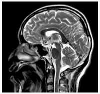
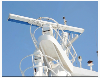


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Questions: Which systems are compatible with phase-only signal estimation?
$\triangleright$ (this talk) Is *complex compressive sensing* compatible?

Why asking?

- If compatible/robust, quantize the spectral phase for compression!
- Robust to large observation amplitudes; easy quantizers (over $[0, 2\pi]$).

(Complex) Compressive Sensing

Let's collect $m < n$ measurements about $\boldsymbol{x}$ from this linear model:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n} \in \mathbb{C}^m, \tag{CS}$$

with:
- a low-complexity vector $\boldsymbol{x} \in \mathcal{L} \subset \mathbb{C}^n$
  (with $\mathcal{L}$ the set of, *e.g.*, sparse signals, low-rank matrices, ...),
- a complex sensing matrix $\boldsymbol{A} \in \mathbb{C}^{m \times n}$,
- a given (additive) noise $\boldsymbol{n} \in \mathbb{C}^m$ and $\|\boldsymbol{n}\|_2 \leqslant \varepsilon$.

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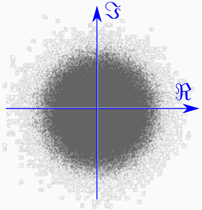
Compressive sensing:

If m larger than $\mathcal{L}$'s "dimension", and $\boldsymbol{A}$ is "random",
the vector $\boldsymbol{x}$ can be exactly recovered, or estimated (if noise).

[Candes and Tao, 2005; Foucart and Rauhut, 2013]

Let's be more specific ... let's focus on the Gaussian case.

Restricted isometry property



For some $0 < \delta < 1$ and $k < m < n$, if

$$m \geqslant C\delta^{-2} k \log(n/k),$$

and $\sqrt{m}\, A_{ij} \sim_{\text{i.i.d.}} \mathbb{C}\mathcal{N}(0,2) \sim \mathcal{N}(0,1) + \mathrm{i}\,\mathcal{N}(0,1)$,

then, *with high probability* (w.h.p.),

$$(1-\delta^2)\|\boldsymbol{v}\|^2 \leqslant \|\boldsymbol{A}\boldsymbol{v}\|_2^2 \leqslant (1+\delta^2)\|\boldsymbol{v}\|^2, \quad \forall k\text{-sparse } \boldsymbol{v}. \qquad (\mathrm{RIP}(k,\delta))$$

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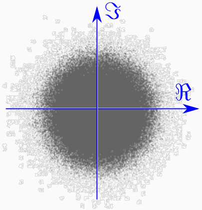

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So, why does CS work?

- $\mathrm{RIP}(2k,\delta) \Rightarrow \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{u}\|_2^2 = \|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{u})\|_2^2 \approx \|\boldsymbol{x}-\boldsymbol{u}\|_2^2$, for all k-sparse $\boldsymbol{x}, \boldsymbol{u}$.
- $\boldsymbol{A}$ is *essentially* invertible over the set of sparse vectors.

The RIP supports (one of) the "fundamental theorem(s) of CS"

Theorem: If $\boldsymbol{A}$ is RIP$(2k, \delta)$ with $0 < \delta < \delta_0$ (*e.g.*, $\delta_0 = 1/\sqrt{2}$), then the *basis pursuit denoise* estimate:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u} \in \mathbb{C}^n} \underbrace{\|\boldsymbol{u}\|_1}_{\text{sparsity promoting}} \quad \text{s.t.} \quad \underbrace{\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{u}\|_2 \leqslant \varepsilon}_{\text{data fidelity}}, \qquad \text{(BPDN)}$$

satisfies the *instance optimality*

$$\underbrace{\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_2}_{\text{Rec. error; } \approx \text{ MSE}} \leqslant C \underbrace{\frac{\|\boldsymbol{x} - \boldsymbol{x}_k\|_1}{\sqrt{k}}}_{\text{deviation to sparsity}} + \underbrace{D\,\varepsilon}_{\text{noise}}.$$

See, *e.g.*, Candès, 2008; Foucart and Rauhut, 2013.

Phase-Only Sensing Model

Inspired by Oppenheim and Lim, 1981; Boufounos, 2013,

in the context of CS, let's consider the phase-only (non-linear) sensing model:

$$\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) + \boldsymbol{\epsilon} \in \mathbb{C}^m, \qquad \text{(PO-CS)}$$

with:
- $\boldsymbol{x}$ is real and k-sparse;
- $\mathrm{sign}_{\mathbb{C}}(re^{\mathrm{i}\theta}) := e^{\mathrm{i}\theta}$ (and 0 if $r = 0$), applied pointwise;
- and $\boldsymbol{\epsilon} \in \mathbb{C}^m$ a bounded noise with $\|\boldsymbol{\epsilon}\|_\infty \leqslant \tau$ for some $\tau \geqslant 0$.

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Key observations:

1. If $\boldsymbol{x} \to C\boldsymbol{x}$ with $C > 0$, $\boldsymbol{z}$ is unchanged (Signal amplitude is lost)

2. If both $\boldsymbol{A}$ and $\boldsymbol{x}$ are real, then $\boldsymbol{z} \in \{\pm 1\}^m$ (Real PO-CS $\to$ 1-bit CS)

   Fact: In noiseless 1-bit CS, best estimate s.t. $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| = \Omega(1/m)$ if $m \uparrow$.

   [Boufounos and Baraniuk, 2008; Jacques et al., 2013; Plan and Vershynin, 2012]

Noiseless Signal Recovery for PO-CS ($\epsilon = 0$) **(1/3)**

Principle: Turn the non-linear PO model into linear one.

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A. Let's normalize x

Since signal amplitude is lost, we still have $\mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}^\star)$ with

$$\boldsymbol{x}^\star := \frac{\kappa\sqrt{m}}{\|\boldsymbol{A}\boldsymbol{x}\|_1}\,\boldsymbol{x}, \quad \text{with } \kappa := \sqrt{\tfrac{\pi}{2}}.$$

$\Rightarrow$ We now focus on the recovery of $\boldsymbol{x}^\star$ ($\rightarrow$ encodes signal direction).

Rationale:

- Useful for our proofs;
- For complex Gaussian $\sqrt{m}\,\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m\times n}(0,2)$ and $g \sim \mathcal{N}(0,1)$,

$$\mathbb{E}|g| = \kappa \quad \Rightarrow \quad \mathbb{E}\|\boldsymbol{A}\boldsymbol{x}\|_1 = \kappa\sqrt{m}\,\|\boldsymbol{x}\|_2 \quad \Rightarrow \quad \|\boldsymbol{x}^\star\|_2 \approx 1.$$

$\Rightarrow$ $\boldsymbol{x}^\star$ is (almost) a unit length vector, a direction

Principle: Turn the non-linear PO model into linear one. Step by step ...

B. Estimate constraints: From the noiseless model

$$\boldsymbol{z} = \operatorname{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}^\star),$$

we see that $\boldsymbol{u} = \boldsymbol{x}^\star \in \mathbb{R}^n$ respects both:

$$\underbrace{\langle \boldsymbol{z}, \boldsymbol{A}\boldsymbol{u} \rangle}_{= \|\boldsymbol{A}\boldsymbol{x}^\star\|_1 \text{ if } \boldsymbol{u} = \boldsymbol{x}^\star} = \kappa\sqrt{m} \quad \Leftrightarrow \langle \underbrace{\tfrac{1}{\kappa\sqrt{m}} \boldsymbol{A}^* \boldsymbol{z}}_{:= \boldsymbol{\alpha}_{\boldsymbol{z}}}, \boldsymbol{u} \rangle = 1 \qquad \text{(normalization)}$$

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Let's relax the phase consistency, *i.e.*, impose $\mathrm{diag}(\boldsymbol{z})^* \boldsymbol{A}\boldsymbol{u} \in \mathbb{R}^m$, that is

$$0 = \Im(\mathrm{diag}(\boldsymbol{z})^* \boldsymbol{A}\boldsymbol{u}) = (\mathrm{diag}(z)^{\Re} \boldsymbol{A}^{\Im} - \mathrm{diag}(z)^{\Im} \boldsymbol{A}^{\Re})\boldsymbol{u} =: \boldsymbol{H}_z \boldsymbol{u}.$$

noting also that

$$\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{u} \rangle = 1 \quad \Leftrightarrow \quad \langle \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{u} \rangle = 1, \ \langle \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{u} \rangle = 0.$$

In summary, $\boldsymbol{u} = \boldsymbol{x}^\star$ respects the relaxed, real $m+2$ constraints ...

$$\boldsymbol{A_z}\boldsymbol{u} = \boldsymbol{e}_1 := (1, 0, \cdots, 0)^\top \quad \Rightarrow$$

This is a linear sensing model!

with

$$\boldsymbol{A_z} := (\boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{H}_{\boldsymbol{z}}^\top)^\top \in \mathbb{R}^{(m+2)\times n}.$$

In other words,

- A good estimate $\hat{x}$ of $\boldsymbol{x}^\star$ should respect the linear model $\boldsymbol{A_z}\hat{\boldsymbol{x}} = \boldsymbol{e}_1$ since $\boldsymbol{x}^\star \in \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{A_z}\hat{\boldsymbol{u}} = \boldsymbol{e}_1\}$.
- We know this estimate should be sparse (as $\boldsymbol{x}^\star$ is)

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- We know this estimate should be sparse (as $\boldsymbol{x}^\star$ is)

$\Rightarrow$ As in linear CS, we can compute $\hat{x}$ from a *basis pursuit* program (BP)

$$\hat{\boldsymbol{x}} = \operatorname*{arg\,min}_{\boldsymbol{u}\in\mathbb{C}^n} \|\boldsymbol{u}\|_1 \ \text{ s.t. } \quad \boldsymbol{A_z}\boldsymbol{u} = \boldsymbol{e}_1, \qquad (\mathrm{BP}(\boldsymbol{A_z}, \boldsymbol{e}_1))$$

Question: How far is $\hat{x}$ from $\boldsymbol{x}^\star$? Well, let's see if $\boldsymbol{A_z}$ respects the RIP!

RIP for A_z?

How could $\boldsymbol{A_z} := (\boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{H}_{\boldsymbol{z}}^{\top})^{\top}$ respect the RIP?

For a sparse $\boldsymbol{v}$, $\|\boldsymbol{A_z v}\|_2^2 := |\langle \boldsymbol{\alpha_z}, \boldsymbol{v}\rangle|^2 + \|\boldsymbol{H}_z \boldsymbol{v}\|_2^2$

you can show that, for complex Gaussian $\boldsymbol{A}$:

- $\langle \boldsymbol{\alpha_z}, \boldsymbol{v}\rangle \approx \langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}, \boldsymbol{v}\rangle \approx$ projection of $\boldsymbol{v}$ onto $\mathcal{X} := \mathbb{R}\,\bar{\boldsymbol{x}}$.
- $\boldsymbol{H}_z \boldsymbol{x} = 0$ & $\boldsymbol{H}_z$ RIP on $\mathcal{X}^{\perp} \cap$ $2k$-sparse signals.

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- $\boldsymbol{H_z x} = 0$ & $\boldsymbol{H_z}$ RIP on $\mathcal{X}^{\perp} \cap$ $2k$-sparse signals.

Theorem: Given $\boldsymbol{x}$ and $0 < \delta < 1$, $\sqrt{m}\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m\times n}(0,2)$, if

$$m \geqslant C\delta^{-2} k \log(n/k),$$

then, w.h.p., $\boldsymbol{A_z}$ satisfies the RIP (k, δ).

Consequences:

- For $\hat{\boldsymbol{x}} = \mathrm{BP}(\boldsymbol{A}_z, \boldsymbol{e}_1)$, if $\boldsymbol{A}_z$ is RIP$(\delta < \delta_0, 2k)$,
  we get exact reconstruction of signal direction, *i.e.*, $\hat{\boldsymbol{x}} = \boldsymbol{x}^{\star}$!
- Instance optimality for the noisy setting (with BPDN) (not covered here)

Let's plot a *phase-transition curve*: we generate $\sqrt{m}\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m \times 256}(0, 2)$ &

- 20-sparse vectors in $\mathbb{R}^{256}$;
- $m \in [1, 256]$ and average over 100 trials;
- Reconstruction successful if SNR $\geqslant$ 60 dB.

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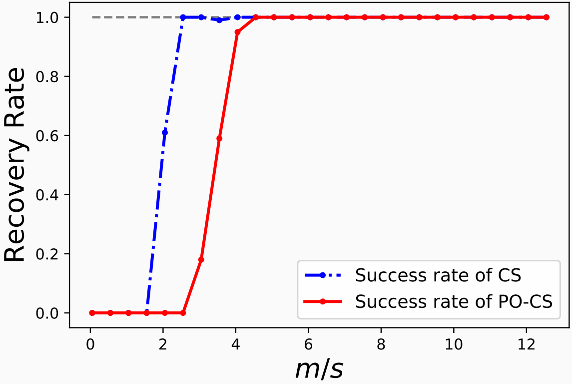

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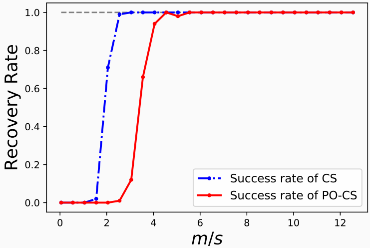
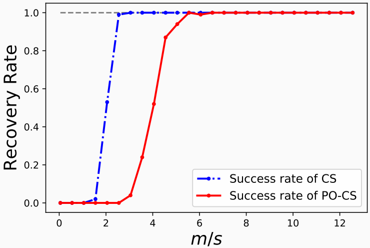
$A_{ij} \sim_{\mathrm{iid}} \{\pm 1\}$



Random partial Fourier
(A = sub-sampled $\mathcal{F}(\boldsymbol{x})$)



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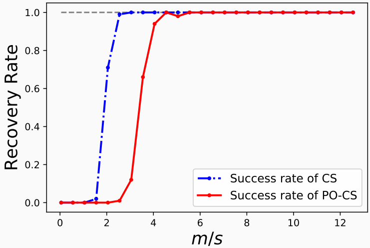

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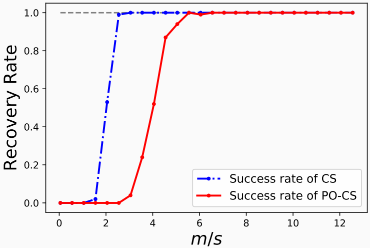

Interestingly:

- These results are not covered by theory.
- Bernoulli random matrices do not work for 1-bit CS.
- Fourier sensing has PO-CS counter-examples (that cannot be recovered)!

$$e.g., \text{ for } \boldsymbol{x}' := \boldsymbol{h} * \boldsymbol{x} \text{ with } \hat{h}_k > 0, \forall k, \quad \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}') = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}).$$

Take-Away Messages

1. In Gauss' world, despite:
   - the non-linearity of its sensing model,
   - and the bad example of 1-bit CS (the "real" PO-CS),

   phase-only compressive sensing works "*as well as*" (linear) CS.

2. What is recovered/estimated is the signal direction (via $\boldsymbol{x}^\star$).

3. Applications: phase-quantization procedures with bounded distortion
   e.g., in radar, MRI, ...

4. Open questions:
   - (minor) Extension to complex signals.
   - (major) Theoretical extension to other random sensing matrices.

. — . — .

Thank you!

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Part I

Extra slides

Extra simulations: noisy case

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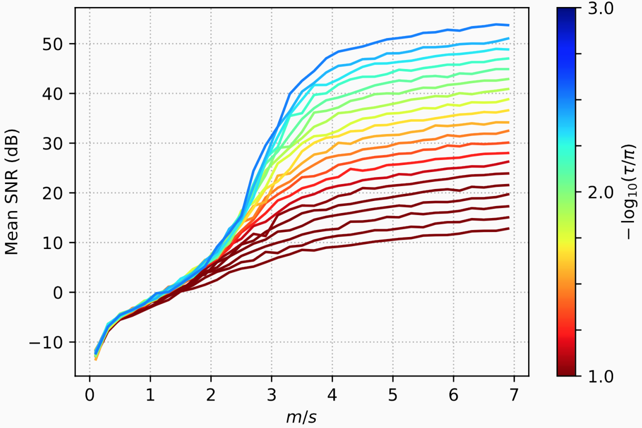

Simplifying hypothesis

Phase-only observation in Compressive Sensing?

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Let's first simplify the context . . .

1. We consider the sensing of real vectors $\boldsymbol{x} \in \mathbb{R}^n$.

Note: If complex signal $\boldsymbol{x}$, we can always rewrite

$$\boldsymbol{A}\,\boldsymbol{x} = (\boldsymbol{A}^{\Re} + \mathrm{i}\boldsymbol{A}^{\Im})(\boldsymbol{x}^{\Re} + \mathrm{i}\boldsymbol{x}^{\Im}) = \big(\boldsymbol{A},\, \mathrm{i}\boldsymbol{A}\big)\begin{pmatrix}\boldsymbol{x}^{\Re}\\ \boldsymbol{x}^{\Im}\end{pmatrix} = \overline{\boldsymbol{A}}\,\bar{\boldsymbol{x}},$$

with $\bar{\boldsymbol{x}} \in \mathbb{R}^{2n}$ and $\overline{\boldsymbol{A}} \in \mathbb{C}^{m \times 2n}$.

Simplifying hypothesis

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2. We focus here on the case of sparse vectors in $\mathbb{R}^n$.

However, extension to any low-complexity signals is possible (with small "dimension", that is *Gaussian mean width*)