Quantized Iterative Hard Thresholding:
Bridging 1-bit and High Resolution Quantized Compressed Sensing

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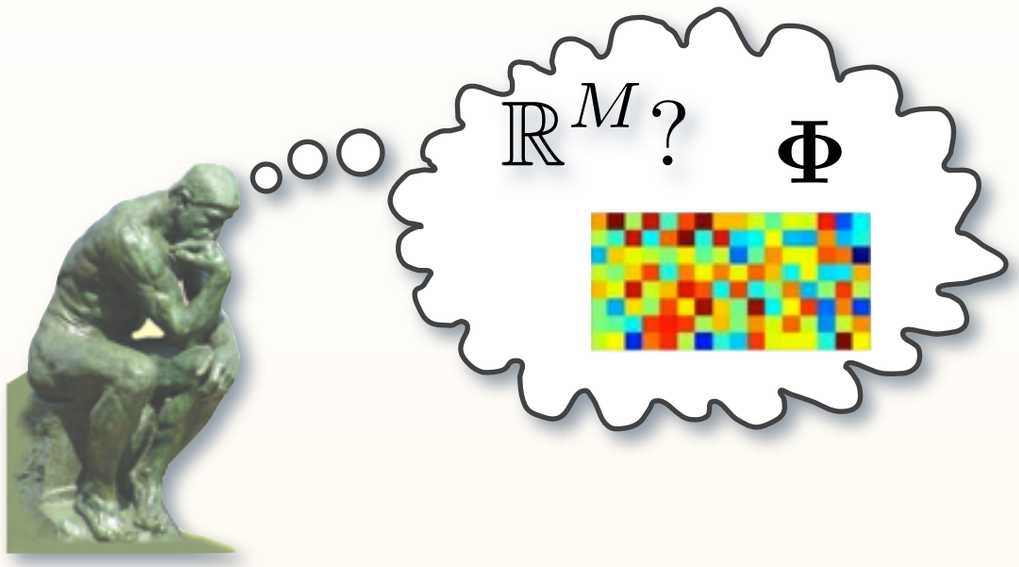
Compressive Sampling and Quantization

Compressed sensing theory says:

"Linearly sample a signal

at a rate function of

its intrinsic dimensionality"

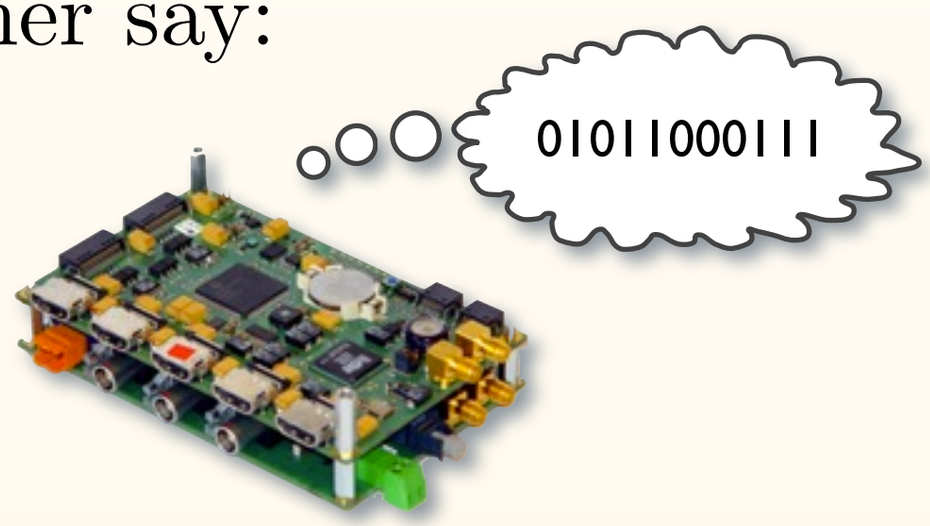


Information theory and sensor designer say:

"Okay, but I need to

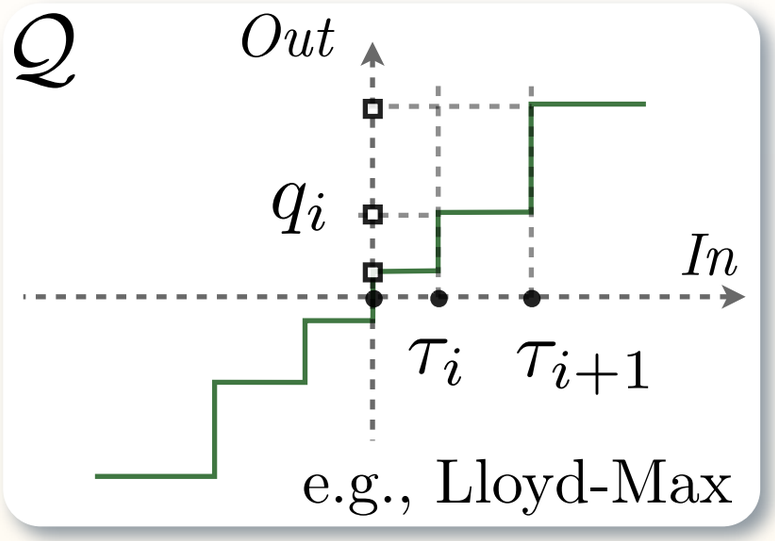
quantize/digitize my measurements!"

(e.g., in ADC)



(this talk) Focus on scalar quantization

Turning measurements into bits → scalar quantization



$$q_i = \mathcal{Q}[(\mathbf{\Phi}\boldsymbol{x})_i] = \mathcal{Q}[\langle \boldsymbol{\phi}_i, \boldsymbol{x}\rangle] \in \Omega \subset \mathbb{R}$$

$$\boldsymbol{q} = \mathcal{Q}\big[\mathbf{\Phi}\boldsymbol{x}\big] \in \mathbf{\Omega} = \Omega^M,$$

$$\Omega = \{q_i \in \mathbb{R} : 1 \leqslant i \leqslant 2^B\}, \quad \text{(levels)}$$

$$\mathcal{T} = \{\tau_i \in \overline{\mathbb{R}} : 1 \leqslant i \leqslant 2^B + 1, \tau_i \leqslant \tau_{i+1}\} \quad \text{(thresholds)}$$

Important conventions:

- Definition of $\mathbf{\Phi}$ independent of M (e.g., $\Phi_{ij} \sim_{\text{iid}} \mathcal{N}(0,1)$)
  → preserves measurement dynamic!
- B bits per measurement
- Total bit budget: $R = BM$
- No further encoding (e.g., entropic)

1. First extreme: CS and high-resolution scalar quantization

Former solutions

- Quantization is like a noise

$$\boldsymbol{q} = \mathcal{Q}\big[\boldsymbol{\Phi x}\big] = \boldsymbol{\Phi x} + \boldsymbol{n}$$

quantization distortion

Former solutions

- Quantization is like a noise

$$\boldsymbol{q} \;=\; \mathcal{Q}\big[\boldsymbol{\Phi x}\big] = \boldsymbol{\Phi x} + \boldsymbol{n}$$

- CS is robust, *e.g.*, with ...
  - Basis Pursuit DeNoise (BPDN):

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{u}\in\mathbb{R}^N}{\operatorname{argmin}} \|\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{\Phi u} - \boldsymbol{q}\| \leqslant \epsilon$$

  - Iterative Hard Thresholding (IHT):

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{u}\in\mathbb{R}^N}{\operatorname{argmin}} \|\boldsymbol{\Phi u} - \boldsymbol{y}\|^2 \text{ s.t. } \|\boldsymbol{u}\|_0 \leqslant K$$

$$\approx \text{iterating: } \boldsymbol{x}^{(n+1)} = \overbrace{\mathcal{H}_K}^{\text{thresholding}}\Big(\boldsymbol{x}^{(n)} + \mu\boldsymbol{\Phi}^*\overbrace{(\boldsymbol{y} - \boldsymbol{\Phi x}^{(n)})}^{\text{gradient descent}}\Big)$$

Former solutions

- Quantization is like a noise

$$\boldsymbol{q} \;=\; \mathcal{Q}\big[\boldsymbol{\Phi x}\big] = \boldsymbol{\Phi x} + \boldsymbol{n}$$

- CS is robust (e.g., with BPDN or IHT)

If $\|\boldsymbol{n}\| \leq \epsilon$ and $\frac{1}{\sqrt{M}}\boldsymbol{\Phi}$ is RIP(δ, aK) with $\delta \leq \delta_0$, then

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \;\leq\; C\frac{\epsilon}{\sqrt{M}} + D\, e(K),$$

for some $C, D > 0$ and $e(K) =$ deviation to K-sparsity.

$$e(K) = \|\boldsymbol{x} - \boldsymbol{x}_K\|_1/\sqrt{K} \quad \text{or} \quad \|\boldsymbol{x} - \boldsymbol{x}_K\|_2 + \|\boldsymbol{x} - \boldsymbol{x}_K\|_1/\sqrt{K}$$

[Candès, 08] $(a = 2,\ \delta_0 < \sqrt{2} - 1)$
[Blumensath, Davies, 09] $(a = 3, \delta_0 < 1/32)$

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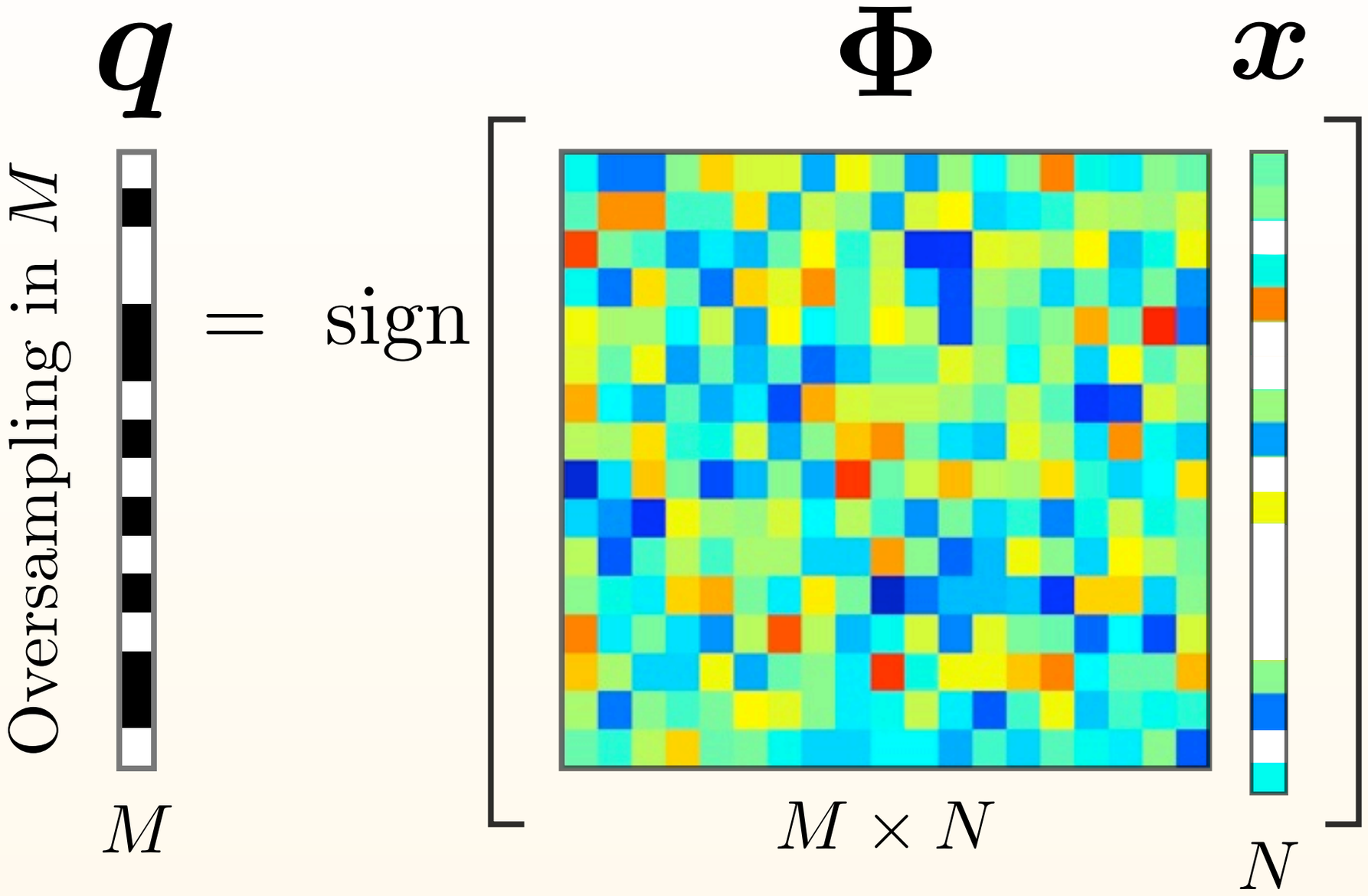
- Question: finding a ϵ for QCS? High-Resolution Bounds!

  e.g., $\frac{\epsilon}{\sqrt{M}} = O(\alpha)$ (unif. $\mathcal{Q}$ bin width α) or $O(2^{-B})$ for B-bit $\mathcal{Q}$

2. Second extreme: CS and 1-bit scalar quantization

1-bit Compressed Sensing

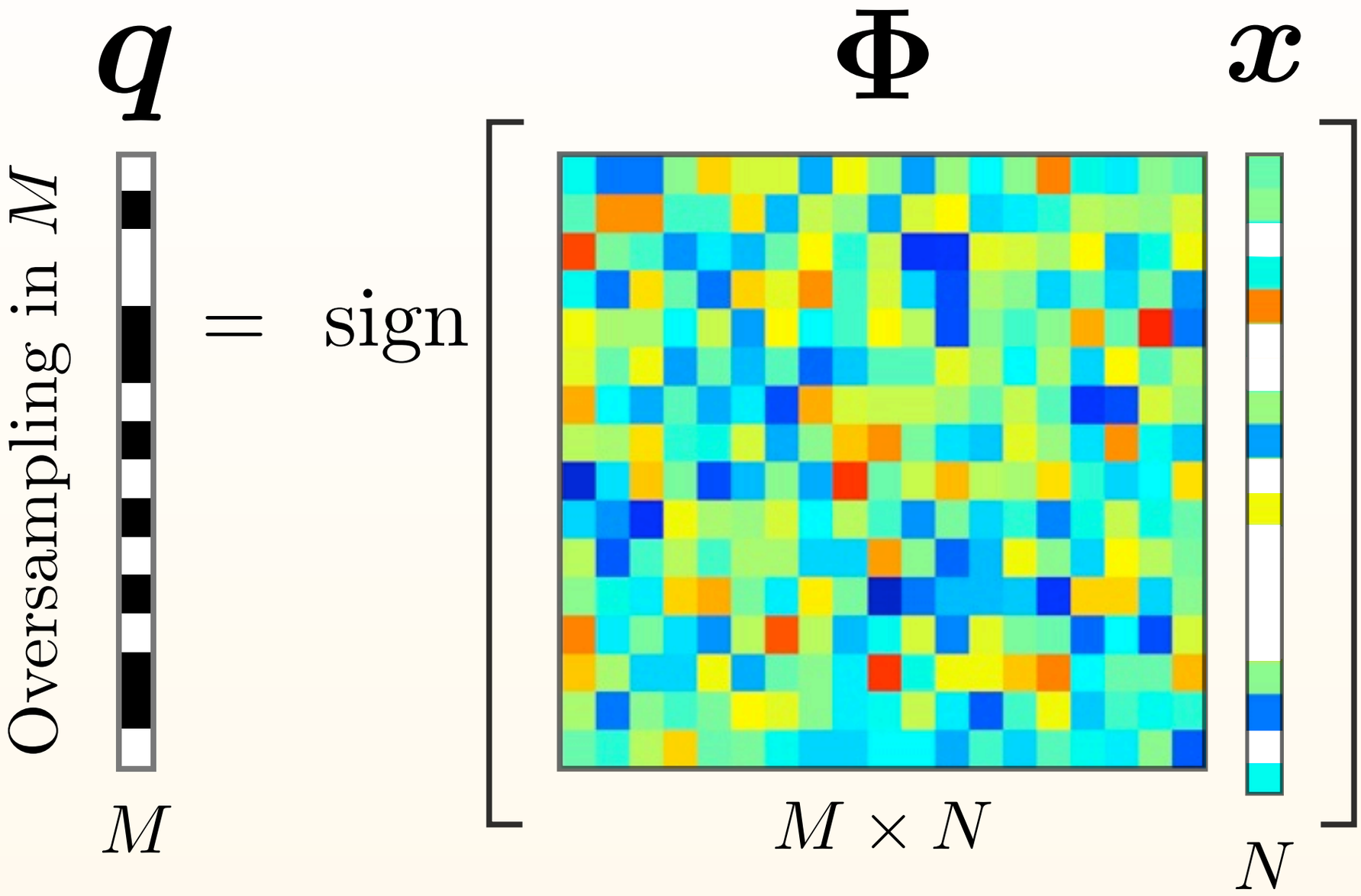


[Boufounos, Baraniuk, 08]

M-bits! But, which information inside $\boldsymbol{q}$?

1-bit Computational ~~-pressed~~ Sensing

bits matter!



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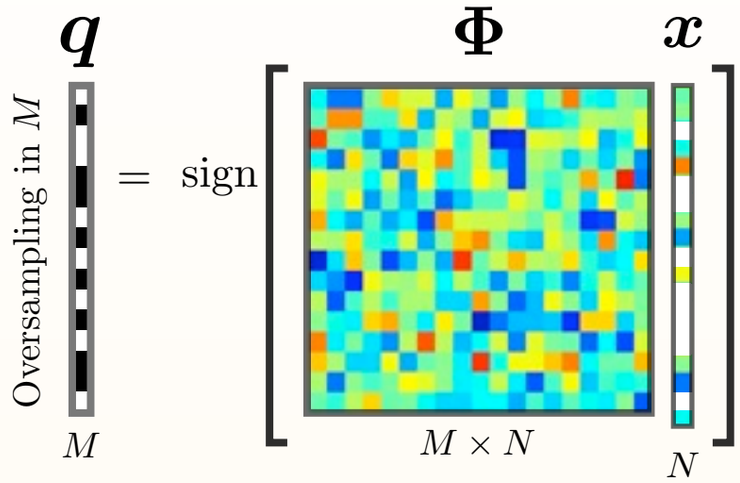


Warning 1: signal amplitude is lost!

$\Rightarrow$ Amplitude is arbitrarily fixed

Examples : $\|\boldsymbol{x}\| = 1$ or $\|\boldsymbol{\Phi x}\|_1 = 1$

Warning 2: $\exists$ forbidden sensing!

(e.g. Bernoulli + canonical sparsity)

[Plan, Vershynin, 11]

Why does it work?

$\boldsymbol{x}$ on S^2

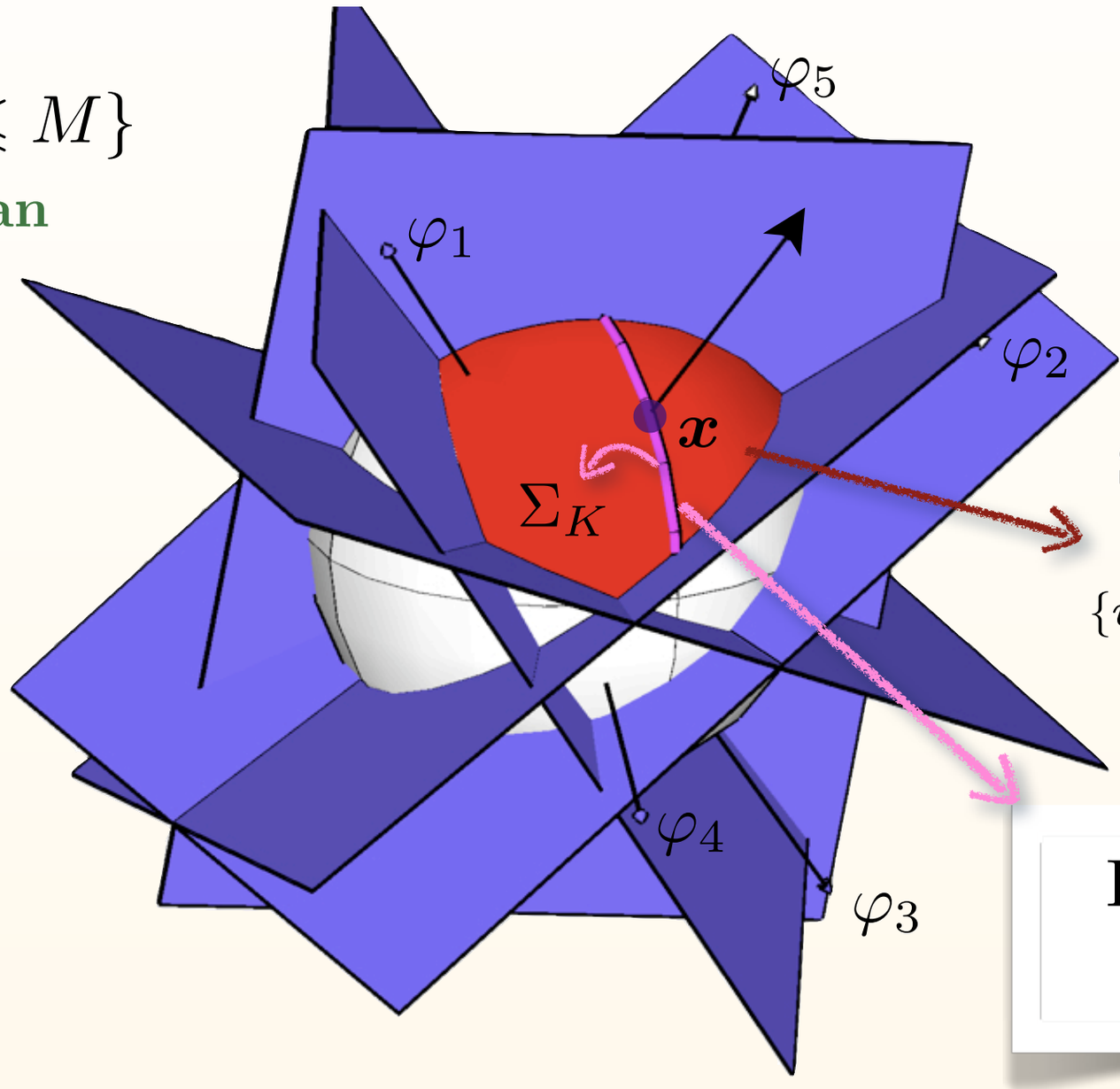
M vectors:

$$\{\boldsymbol{\varphi}_i : 1 \leqslant i \leqslant M\}$$

iid Gaussian

1-bit Measurements

$$\begin{aligned}
&\langle \boldsymbol{\varphi}_1, \boldsymbol{x} \rangle > 0 \\
&\langle \boldsymbol{\varphi}_2, \boldsymbol{x} \rangle > 0 \\
&\langle \boldsymbol{\varphi}_3, \boldsymbol{x} \rangle \leqslant 0 \\
&\langle \boldsymbol{\varphi}_4, \boldsymbol{x} \rangle > 0 \\
&\langle \boldsymbol{\varphi}_5, \boldsymbol{x} \rangle > 0 \\
&\vdots
\end{aligned}$$



Smaller and smaller when M increases

$$\{\boldsymbol{u} : \operatorname{sign}(\boldsymbol{\Phi u}) = \operatorname{sign}(\boldsymbol{\Phi x})\}$$

Lower bound on this width?

1-bit CS bounds

Let $A(\cdot) := \operatorname{sign}(\boldsymbol{\Phi}\,\cdot)$ with $\boldsymbol{\Phi} \sim \mathcal{N}^{M\times N}(0,1)$.

If $M = O(\epsilon^{-1} K \log N)$, then, w.h.p,

for any two unit K-sparse vectors $\boldsymbol{x}$ and $\boldsymbol{s}$,

if only b bits are different between $A(\boldsymbol{x})$ and $A(\boldsymbol{s})$ $\quad \Rightarrow \|\boldsymbol{x} - \boldsymbol{s}\| \leqslant \frac{K+b}{K}\,\epsilon$

[LJ, Laska, Boufounos, Baraniuk, 13] $(b = 0)$
[LJ, Degraux, De Vleeschouwer, 13] $(b \neq 0)$

If $M = O(\epsilon^{-2} K \log N) \rightarrow$ Binary ϵ Stable Embedding (BϵSE):

$$d_{\text{ang}}(\boldsymbol{x}, \boldsymbol{s}) - \epsilon \leq d_H(A(\boldsymbol{x}), A(\boldsymbol{s})) \leq d_{\text{ang}}(\boldsymbol{x}, \boldsymbol{s}) + \epsilon$$

$\neq$ RIP like

[LJ, Laska, Boufounos, Baraniuk, 13] **[Plan, Vershynin, 11, 12]**

Lesson: possible recovery should be as consistent as possible.

Easiest 1-bit reconstruction

Fact: If $M = O(\epsilon^{-2} K \log N/K)$ (for $\boldsymbol{x} \in \Sigma_K$ fixed, $\forall\, \boldsymbol{s} \in \Sigma_K$)

$$\left|\tfrac{\sqrt{\pi}/2}{M} \langle \mathrm{sign}\,(\boldsymbol{\Phi x}), \boldsymbol{\Phi s}\rangle - \langle \boldsymbol{x}, \boldsymbol{s}\rangle\right| \leq \epsilon \quad \text{(w.h.p)}$$

[Plan, Vershynin, 12]

$\Downarrow$

Let $\boldsymbol{x} \in \Sigma_K \cap S^{N-1}$ and $\boldsymbol{q} = \mathrm{sign}(\boldsymbol{\Phi x})$.
Compute

$$\hat{\boldsymbol{x}} = \tfrac{\pi}{2M}\, \mathcal{H}_K(\boldsymbol{\Phi}^* \boldsymbol{q})$$

Then, if previous property holds, $\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \leq 2\epsilon$.

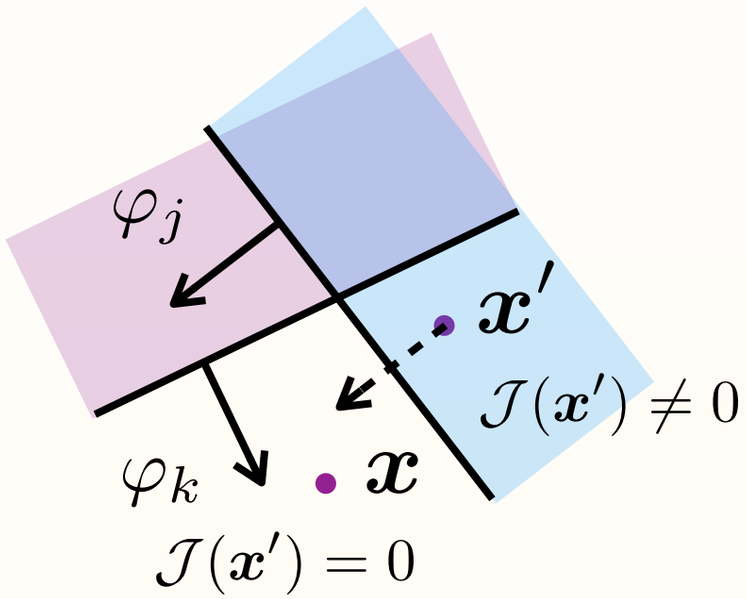
[LJ, Degraux, De Vleeschouwer, 13]

$$\propto \quad \arg\max_{\boldsymbol{u} \in \mathbb{R}^N} \ \boldsymbol{q}^T \boldsymbol{\Phi u} \quad \text{s.t.} \quad \|\boldsymbol{u}\|_0 \leq K,\ \|\boldsymbol{u}\| = 1$$

$$\equiv \ \text{“PV-L0 problem”}$$

[Bahmani, Boufounos, Raj, 13]

Binary Iterative Hard Thresholding (BIHT)



Idea: "greedily"

- forcing consistency
- forcing sparsity

$$\mathcal{J}(\boldsymbol{x}') = \sum_{j=1}^{M} \left| \left(\overbrace{\operatorname{sign}\left(\langle \boldsymbol{\varphi}_j, \boldsymbol{x} \rangle\right)}^{q_j} \langle \boldsymbol{\varphi}_j, \boldsymbol{x}' \rangle \right)_- \right|$$

$$\text{with } (\lambda)_- = (\lambda - |\lambda|)/2$$

or $$\boxed{\mathcal{J}(\boldsymbol{x}') = \| [\operatorname{diag}(\boldsymbol{q})(\boldsymbol{\Phi}\boldsymbol{x}')]_- \|_1}$$

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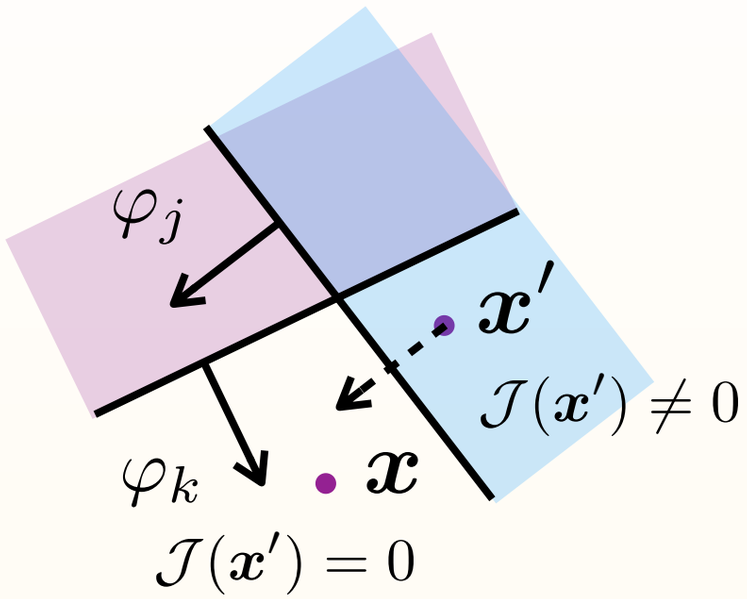

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with $(\lambda)_- = (\lambda - |\lambda|)/2$

or $\boxed{\mathcal{J}(\boldsymbol{x}') = \| [\operatorname{diag}(\boldsymbol{q})(\boldsymbol{\Phi}\boldsymbol{x}')]_- \|_1}$

Objective: $\operatorname{argmin}_{\boldsymbol{u}} \mathcal{J}(\boldsymbol{u})$ s.t. $\|\boldsymbol{u}\|_0 \leqslant K$

Given $\boldsymbol{q} = A(\boldsymbol{x})$ and K, set $l = 0$, $\boldsymbol{x}^0 = 0$:

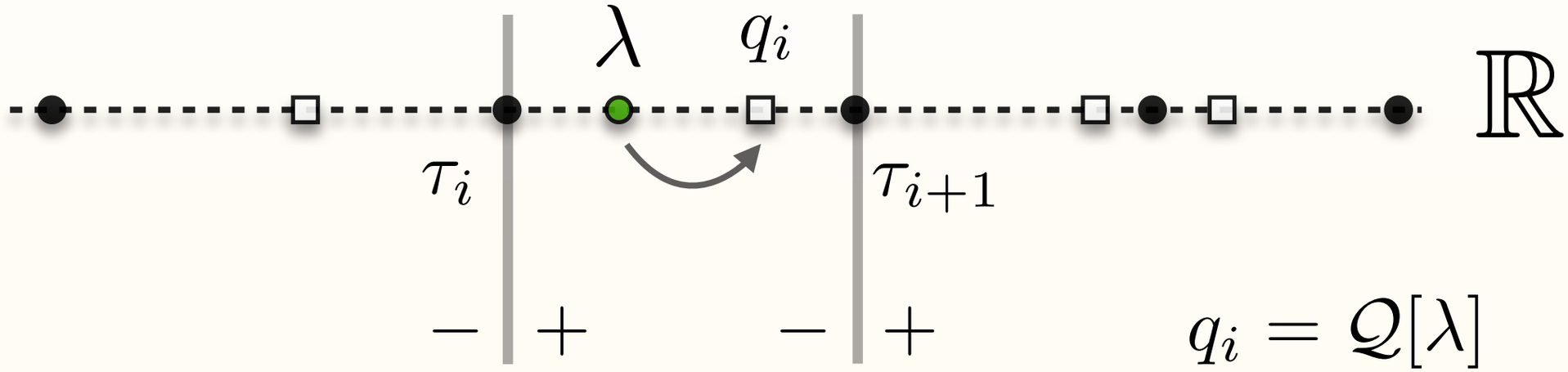
$$\boxed{\begin{aligned} \boldsymbol{a}^{l+1} &= \boldsymbol{x}^l + \tfrac{\tau}{2} \boldsymbol{\Phi}^T \left(\boldsymbol{q} - A(\boldsymbol{x}^l)\right), \\ \boldsymbol{x}^{l+1} &= \mathcal{H}_K(\boldsymbol{a}^{l+1}), \quad l \leftarrow l+1 \end{aligned}}$$

("gradient" towards consistency)
($\tau > 0$ controls gradient descent)
(proj. K-sparse signal set)

3. Bridging 1-bit & B -bit CS?

Bridging 1-bit & B-bit CS?

- B-bit quantizer defined with thresholds:



$$\lambda \in \mathcal{R}_i = [\tau_i, \tau_{i+1}) \Leftrightarrow \left(\mathrm{sign}\,(\lambda - \tau_i) = +1 \;\&\; \mathrm{sign}\,(\lambda - \tau_{i+1}) = -1 \right)$$

- Can we combine multiple thresholds in 1-bit CS?

Bridging 1-bit & B-bit CS?

Given $\mathcal{T} = \{\tau_j\}$ and $\Omega = \{q_j\}$ ($|\mathcal{T}| = 2^B + 1 = |\Omega| + 1$), let's define

$$J(\nu, \lambda) = \sum_{j=2}^{2^B} w_j \left| \left(\operatorname{sign}(\lambda - \tau_j)(\nu - \tau_j) \right)_- \right|,$$

with $w_j = q_j - q_{j-1}$.

Remark: for symmetric $\mathcal{Q}$, $\mathcal{Q}(\lambda) = \sum_{j=2}^{2^B} w_j \operatorname{sign}(\lambda - \tau_j)$

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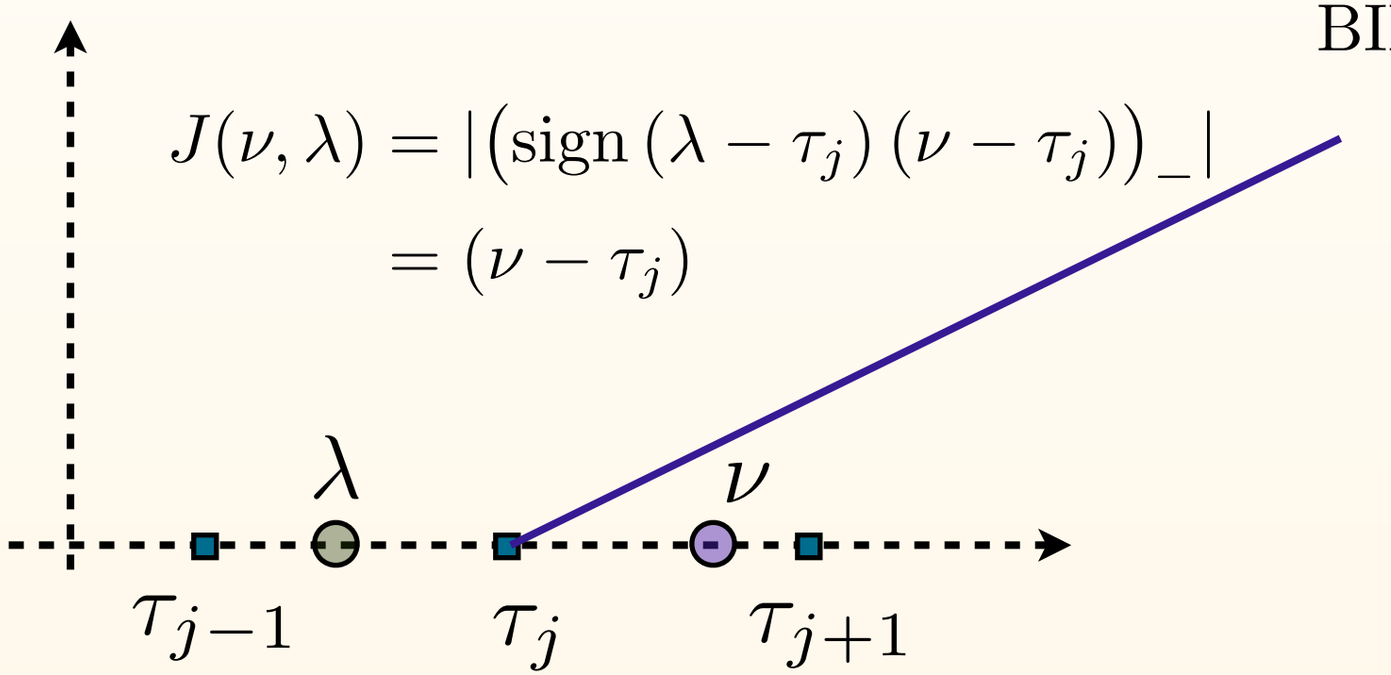
Illustration: $\lambda \in [\tau_{j-1}, \tau_j)$, $\nu \in [\tau_j, \tau_{j+1})$ "delocalized"

BIHT ℓ_1-sided norm

$$J(\nu, \lambda) = \left| \left(\operatorname{sign}\left(\lambda - \tau_j\right) \left(\nu - \tau_j\right) \right)_- \right|$$
$$= \left(\nu - \tau_j\right)$$



(for $w_j = 1$)

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Illustration: $\lambda \in [\tau_{j-1}, \tau_j)$, $\nu \in [\tau_{j+1}, \tau_{j+2})$

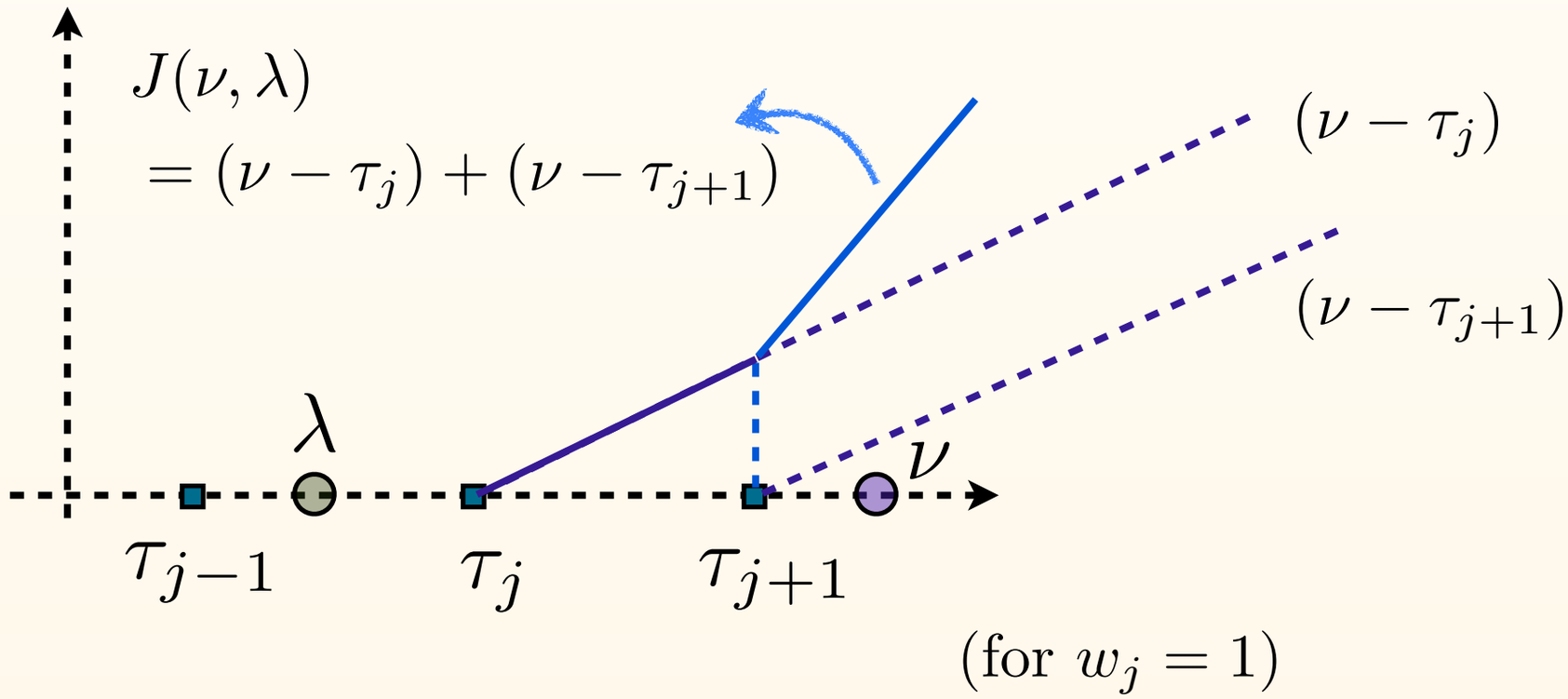

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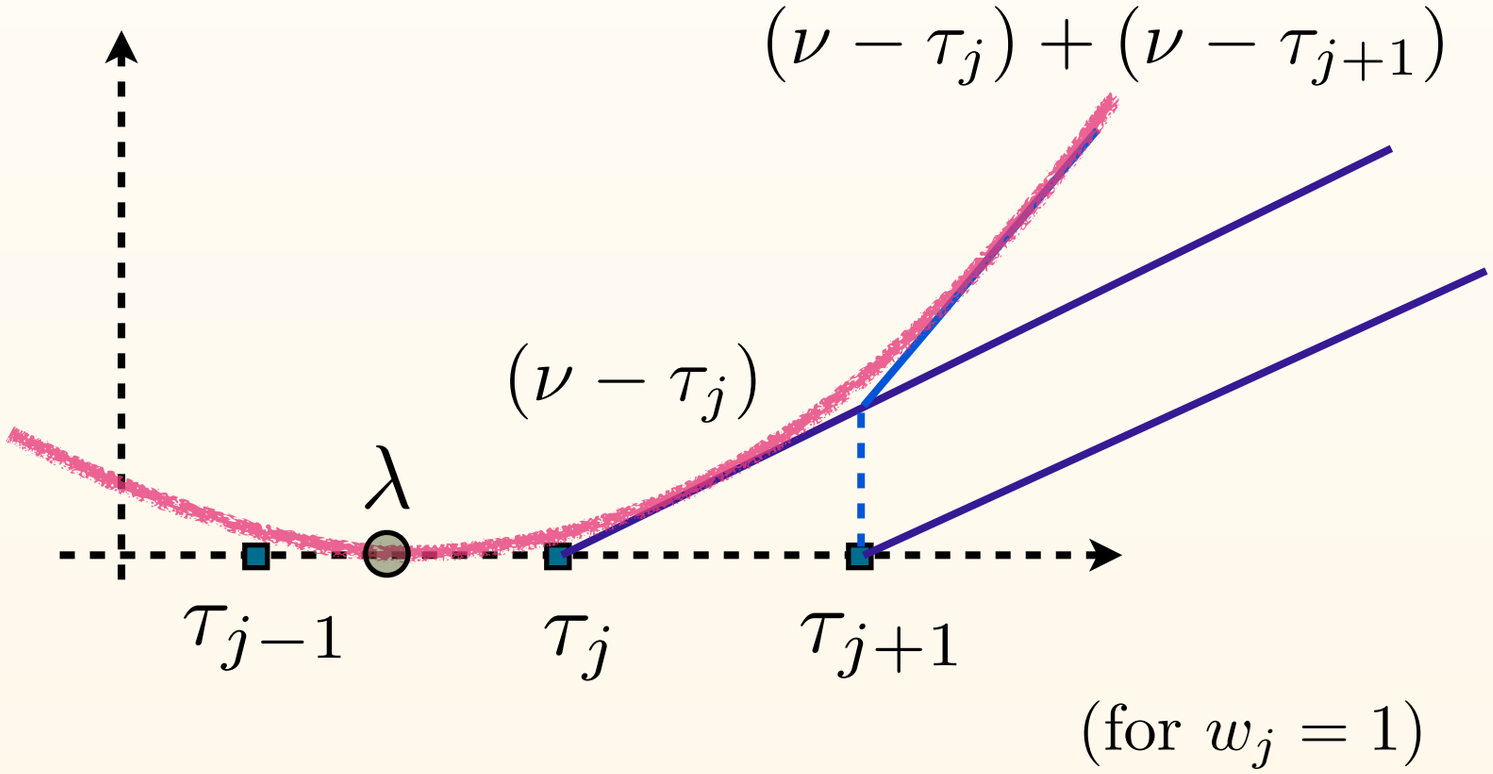
Illustration:



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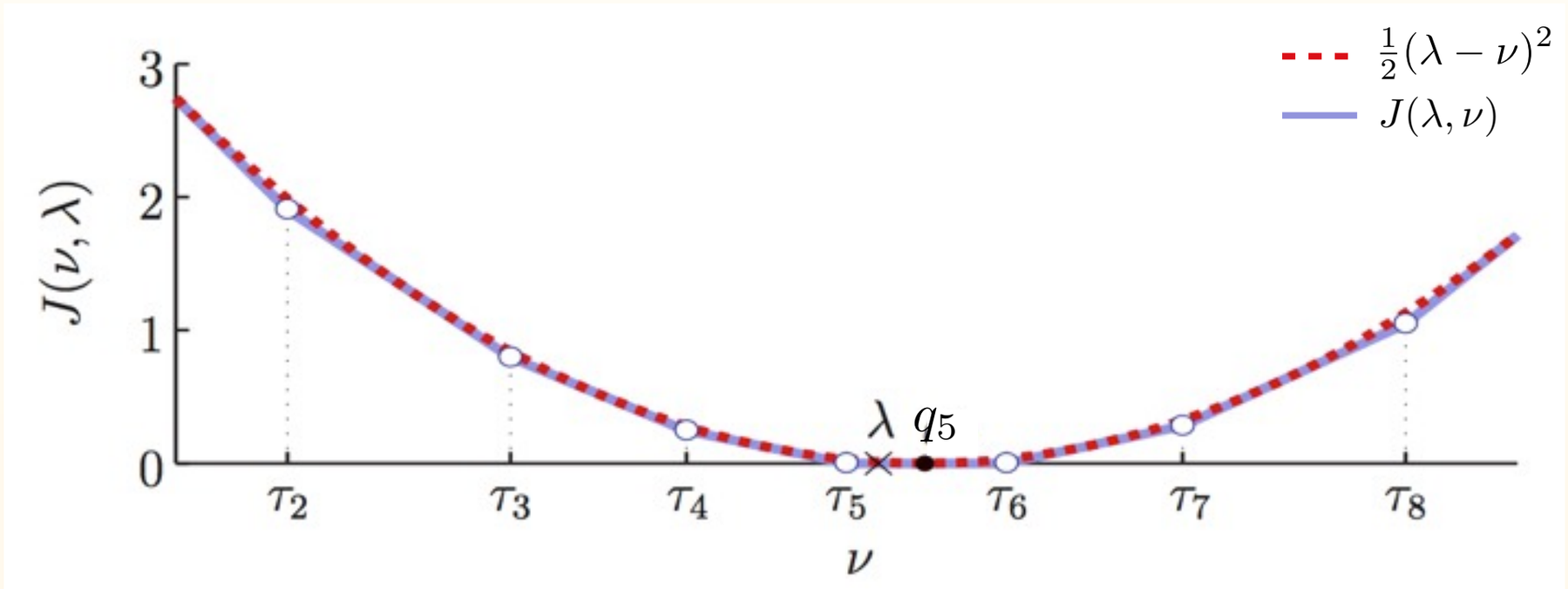
with $w_j = q_j - q_{j-1}$.

Illustration: *more bins,* $\lambda \in \mathcal{R}_5$



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with $w_j = q_j - q_{j-1}$.

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^M$: $\mathcal{J}(\boldsymbol{u}, \boldsymbol{v}) := \sum_{k=1}^{M} J(u_k, v_k)$ (component wise)

Remarks:

- J is convex in ν
- For $B = 1$ ($j = 2$ only):

  $\mathcal{J}(\boldsymbol{u}, \boldsymbol{v}) \propto \|(\operatorname{sign}(\boldsymbol{v}) \odot \boldsymbol{u})_-\|_1 \rightarrow \ell_1$-sided 1-bit energy
- For $B \gg 1$:

  $J(\nu, \lambda) \rightarrow \frac{1}{2}(\nu - \lambda)^2$ and $\mathcal{J}(\boldsymbol{u}, \boldsymbol{v}) \rightarrow \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{v}\|^2$ (quadratic energy)

Bridging 1-bit & B-bit CS?

- Let's define an *inconsistency* energy:

$$\mathcal{E}_B(\boldsymbol{u}) := \mathcal{J}(\boldsymbol{\Phi u}, \boldsymbol{q}) \text{ with } \boldsymbol{q} = \mathcal{Q}_B[\boldsymbol{\Phi x}] \text{ and } \mathcal{E}_B(\boldsymbol{x}) = 0$$

- Idea: Minimize it in Σ_K (as for Iterative Hard Thresholding)

[Blumensath, Davies, 08]

$$\min_{\boldsymbol{u} \in \mathbb{R}^N} \mathcal{E}_B(\boldsymbol{u}) \text{ s.t. } \|\boldsymbol{u}\|_0 \leqslant K,$$

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$$\min_{\boldsymbol{u}\in\mathbb{R}^N} \mathcal{E}_B(\boldsymbol{u}) \text{ s.t. } \|\boldsymbol{u}\|_0 \leqslant K,$$

- combinatorial but greedy solution (as for IHT):

$$\boldsymbol{x}^{(n+1)} = \mathcal{H}_K[\boldsymbol{x}^{(n)} - \mu\, \partial\, \mathcal{E}_B(\boldsymbol{x}^{(n)})] \text{ and } \boldsymbol{x}^{(0)} = 0. \qquad (\mu > 0, \text{ see after})$$

(sub) gradient

$$\partial\, \mathcal{E}_B(\boldsymbol{u}) = \boldsymbol{\Phi}^*(\mathcal{Q}_B(\boldsymbol{\Phi u}) - \boldsymbol{q})$$

$B = 1$: $\boldsymbol{\Phi}^*(\text{sign}\,(\boldsymbol{\Phi u}) - \text{sign}\,(\boldsymbol{\Phi x}))$ — BIHT!

$B \gg 1$: $\boldsymbol{\Phi}^*(\boldsymbol{\Phi u} - \boldsymbol{q})$ — IHT!

Quantized IHT (QIHT)

"all that.. for this!?" ;-)

Bridging 1-bit & B-bit CS?

- So, QIHT reads

$$\boldsymbol{x}^{(n+1)} = \mathcal{H}_K\left[\boldsymbol{x}^{(n)} + \mu\,\boldsymbol{\Phi}^*\left(\boldsymbol{q} - \mathcal{Q}_B(\boldsymbol{\Phi}\boldsymbol{x}^{(n)})\right)\right] \text{ and } \boldsymbol{x}^{(0)} = 0.$$

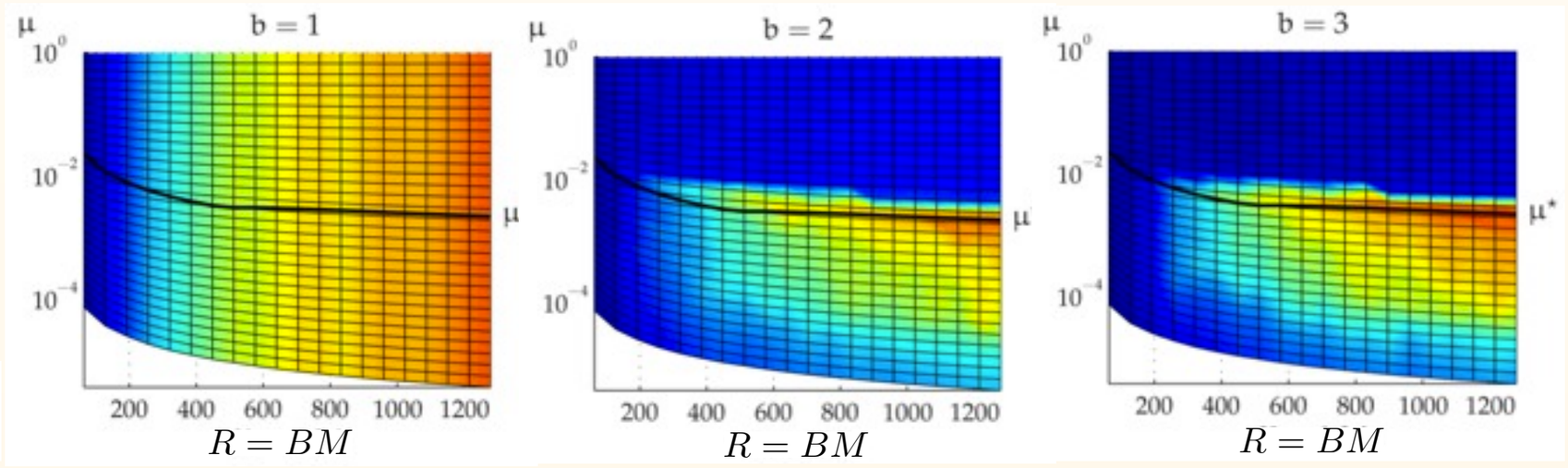
- Setting μ?
  - for $B = 1$, μ has no impact
  - for high B, IHT solution: **[Blumensath, Davies, 08]**

$$\text{If } \boldsymbol{\Phi}/\sqrt{M} \text{ is RIP, } \mu < \frac{1}{(1+\delta_{2K})M}$$

  - for any B, heuristic: $\mu = \frac{1}{M}(1 - \sqrt{cK/M})$ for some $c > 0$.

Validated by extensive simulations ($c = 3$)

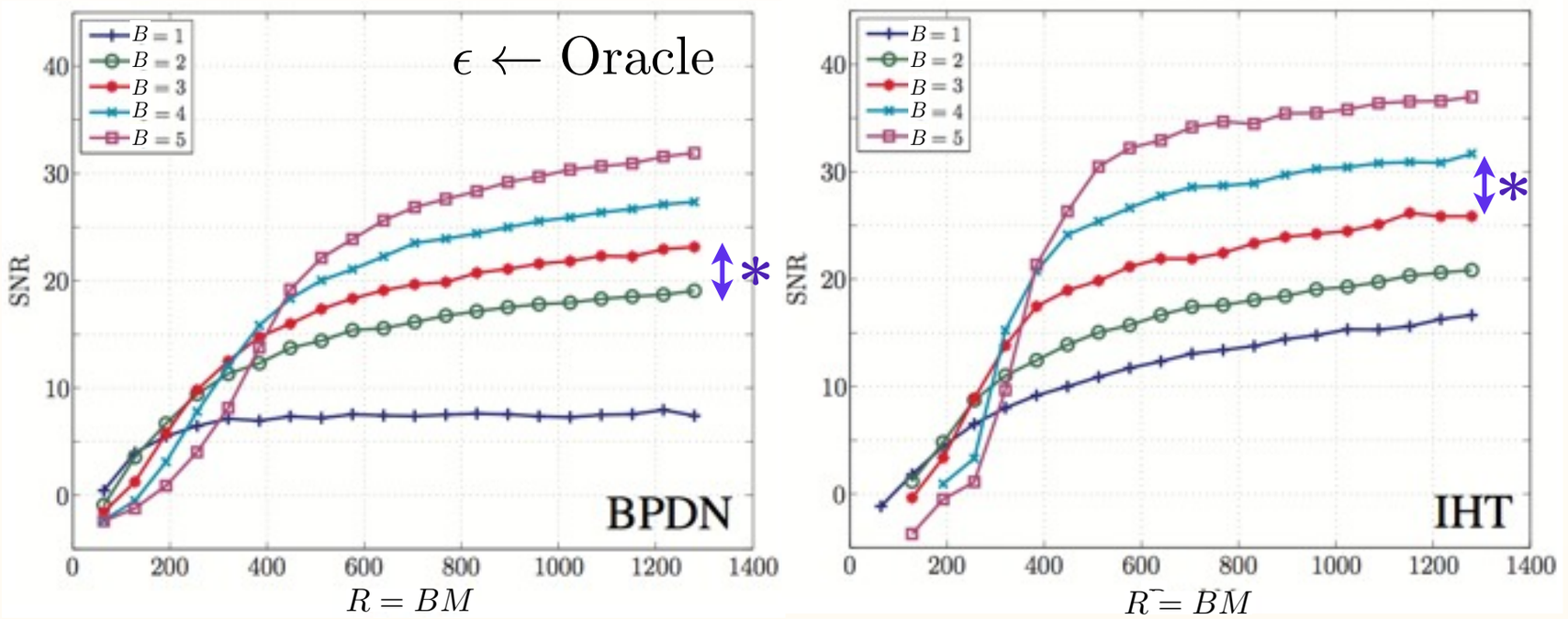
SNR: 10 20 30 40 50



QIHT simulations

$N = 1024,\ K = 16,\ R = BM \in \{64, 128, \cdots, 1280\}, 100$ trials (+ Lloyd-Max Gauss. Q.)

Note: entropy could be computed instead of B (*e.g.*, for further efficient coding)



*: almost "6dB per bit" gain

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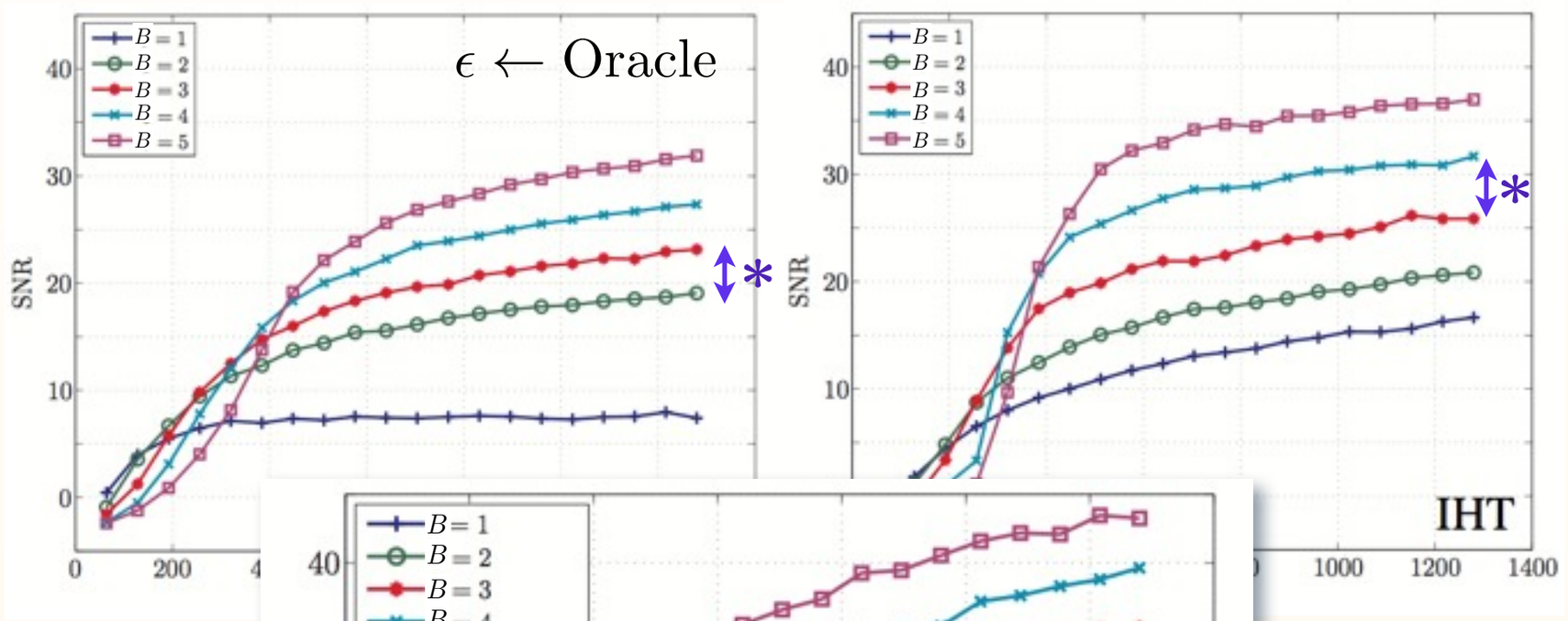

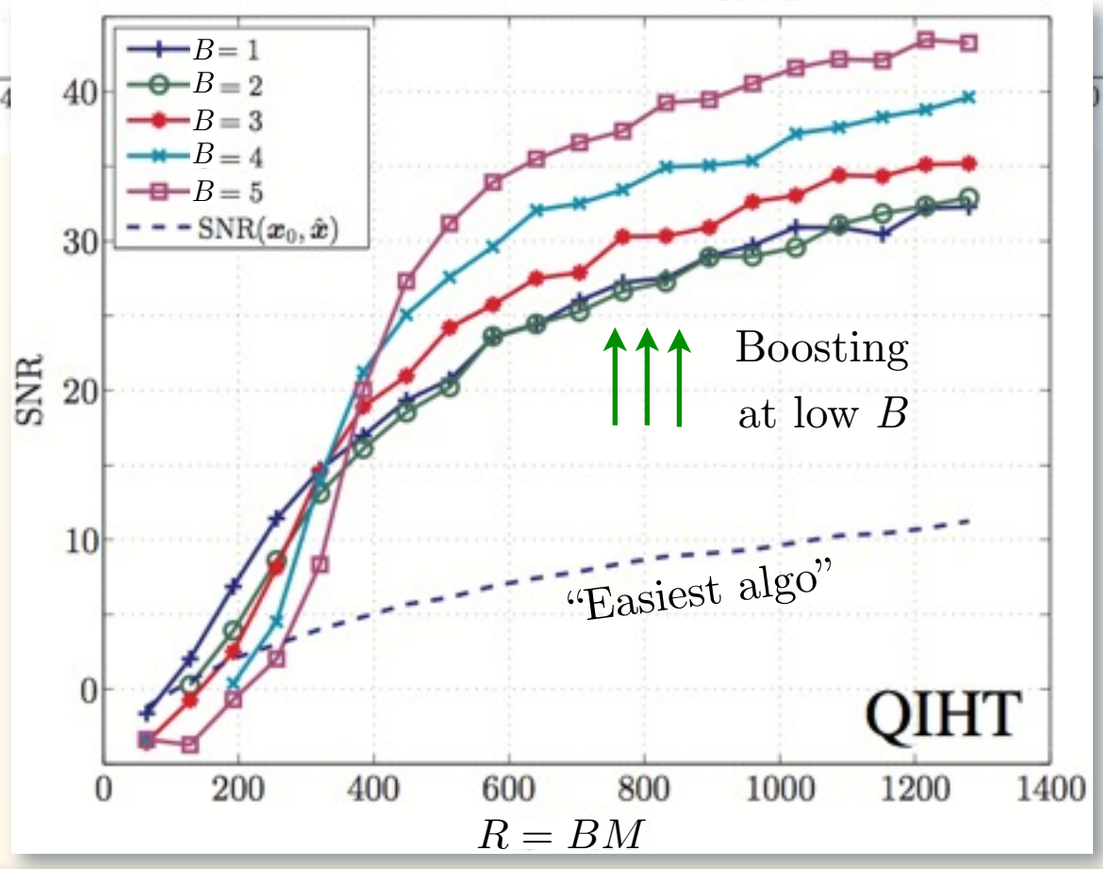

QIHT stops if

$$\frac{\|\boldsymbol{x}^{(n+1)} - \boldsymbol{x}^{(n)}\|}{\|\boldsymbol{x}^{(n)}\|} < 10^{-4}$$

or $n = 500$.

*: almost "6dB per bit" gain

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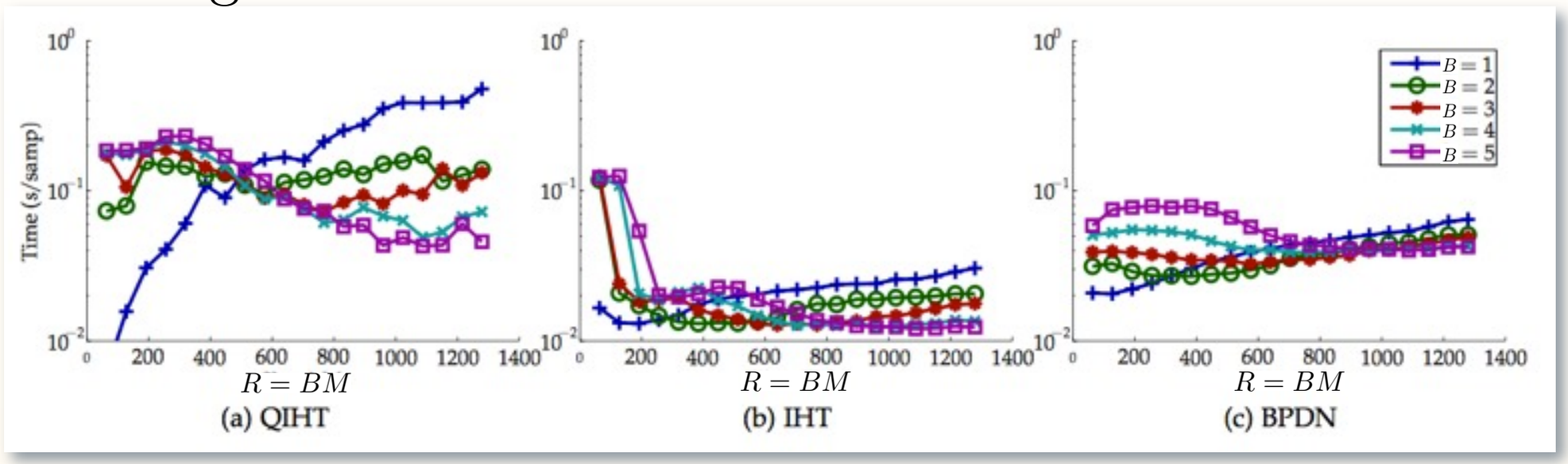
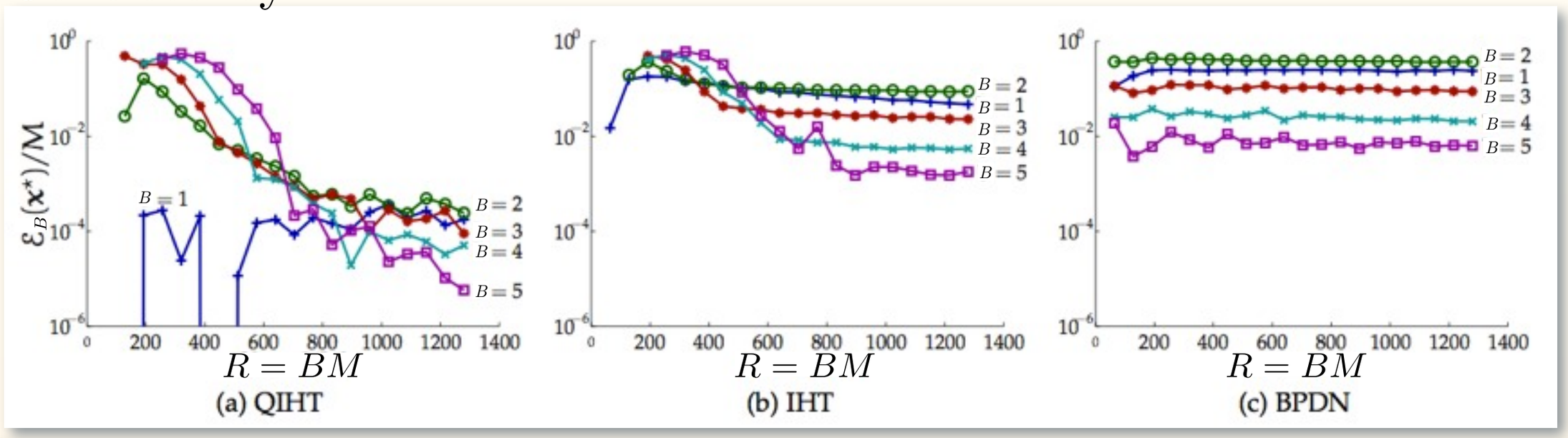
Running time:



Consistency?



Conclusions

- IHT framework can integrate quantization
- Lot of theoretical works to do/come...
  - BIHT/QIHT convergence/optimality guarantees?
  - New embedding results?

    considering $\mathcal{Q}(\lambda) = \sum_{j=2}^{2^B} w_j \operatorname{sign}(\lambda - \tau_j)$ = sum of 1-bit costs + thresholds?

  - Blend of RIP/BεSE? as in

$$\big(1-\delta\big)\|\boldsymbol{x}-\boldsymbol{x}'\| - \epsilon \;\leqslant\; \operatorname{dist}\big(\mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x}), \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x}')\big) \;\leqslant\; \big(1+\delta\big)\|\boldsymbol{x}-\boldsymbol{x}'\| + \epsilon \;?$$

    for sparse $\boldsymbol{x}, \boldsymbol{x}'$ and with $\epsilon \to_{B,M} 0$ and $\delta \to_M 0$?
- Coming:

  Application to B-bit compressive sensors (RMPI)

Further Reading

- (this work) L. Jacques, K. Degraux, C. De Vleeschouwer, “Quantized Iterative Hard Thresholding: Bridging 1-bit and High-Resolution Quantized Compressed Sensing", SAMPTA 2013, http://arxiv.org/abs/1305.1786.
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