Quantizing compressed sensing: From high resolution to 1-bit quantization scheme

Laurent Jacques, UCL, Belgium

Coherent state transforms, time-frequency
and time-scale analysis, applications

Compressive Sampling

H. Rauhut's tutorial

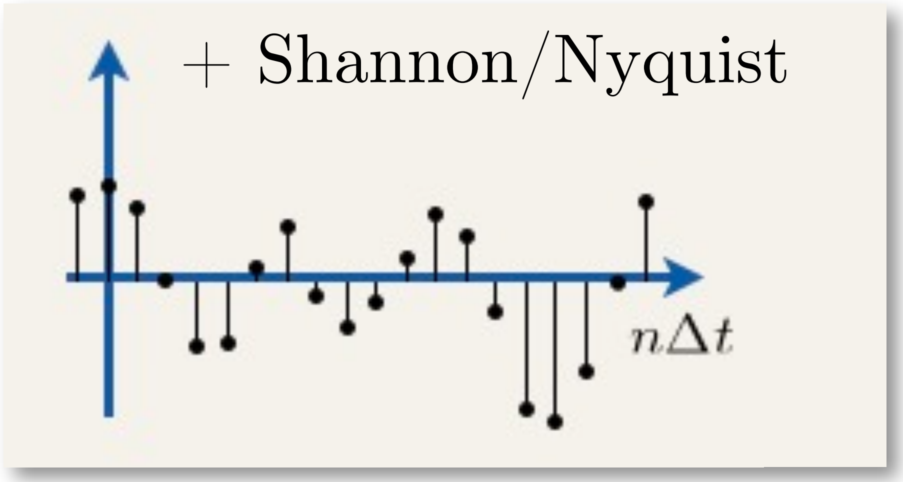
Compressed Sensing

Highly compressed recap of what is ...

Compressive Sensing

Compressed Sampling

Generally, sampling is ...



Human readable signal!

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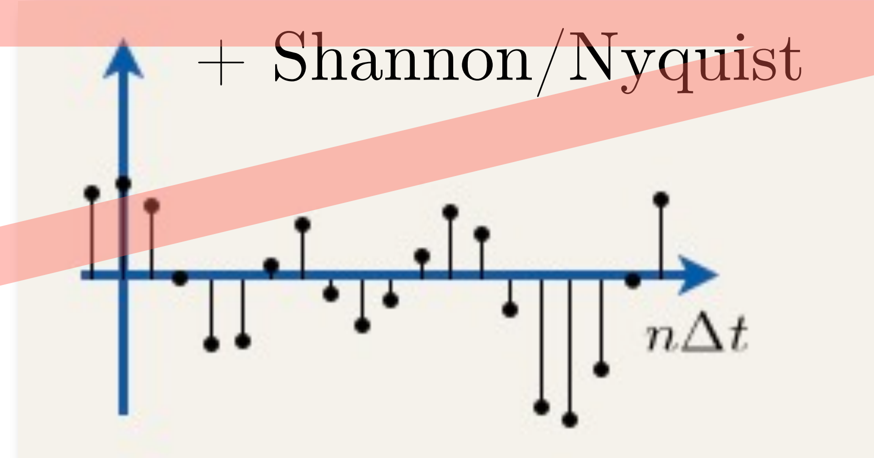


Human readable signal

New ways to sample signals ....

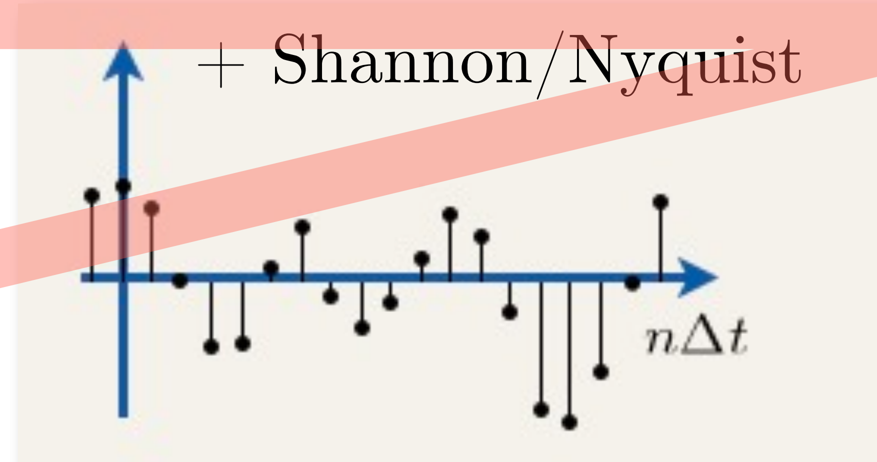
"*Computer readable*" sensing + prior information

World → Signal → Sensing Device → Sensing →

→ Signal → Human

Optimized setup: sampling rate $\propto$ information

Generally, sampling is ...



Human readable signal

New ways to sample signals ....

structures, sparsity, low-rank, ...

"*Computer readable*" sensing + prior information

World → Signal → Sensing Device → Sensing →  → Signal → Human

Optimized setup: sampling rate $\propto$ information

Compressed Sensing

... in a nutshell:

> "Forget" Dirac, forget Nyquist,
> ask *few* (**linear**) *questions*
> about your informative (**sparse**) signal,
> and recover it *differently* (**non-linearly**)"

Compressed Sensing

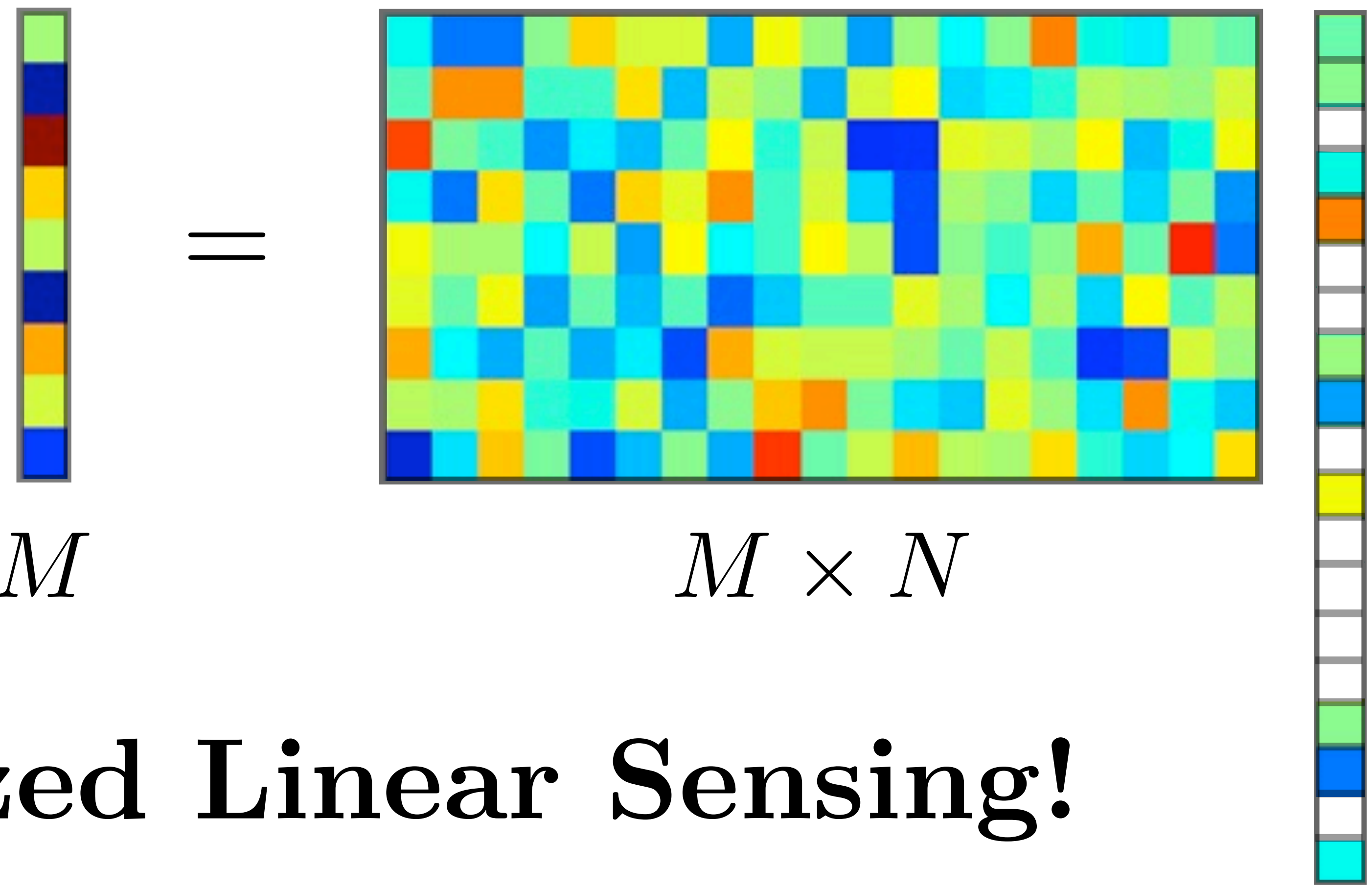
Assumption: **the probability that our world is totally discrete is very high ...**

Compressed Sensing

Then ...

M questions — Sensing method — Signal

$\boldsymbol{y}$ $\quad$ $\boldsymbol{\Phi}$ $\quad$ $\boldsymbol{x}$



M $\quad$ $M \times N$ $\quad$ N

Sparsity Prior ($\Psi = \mathrm{Id}$)

A signal in this discrete world

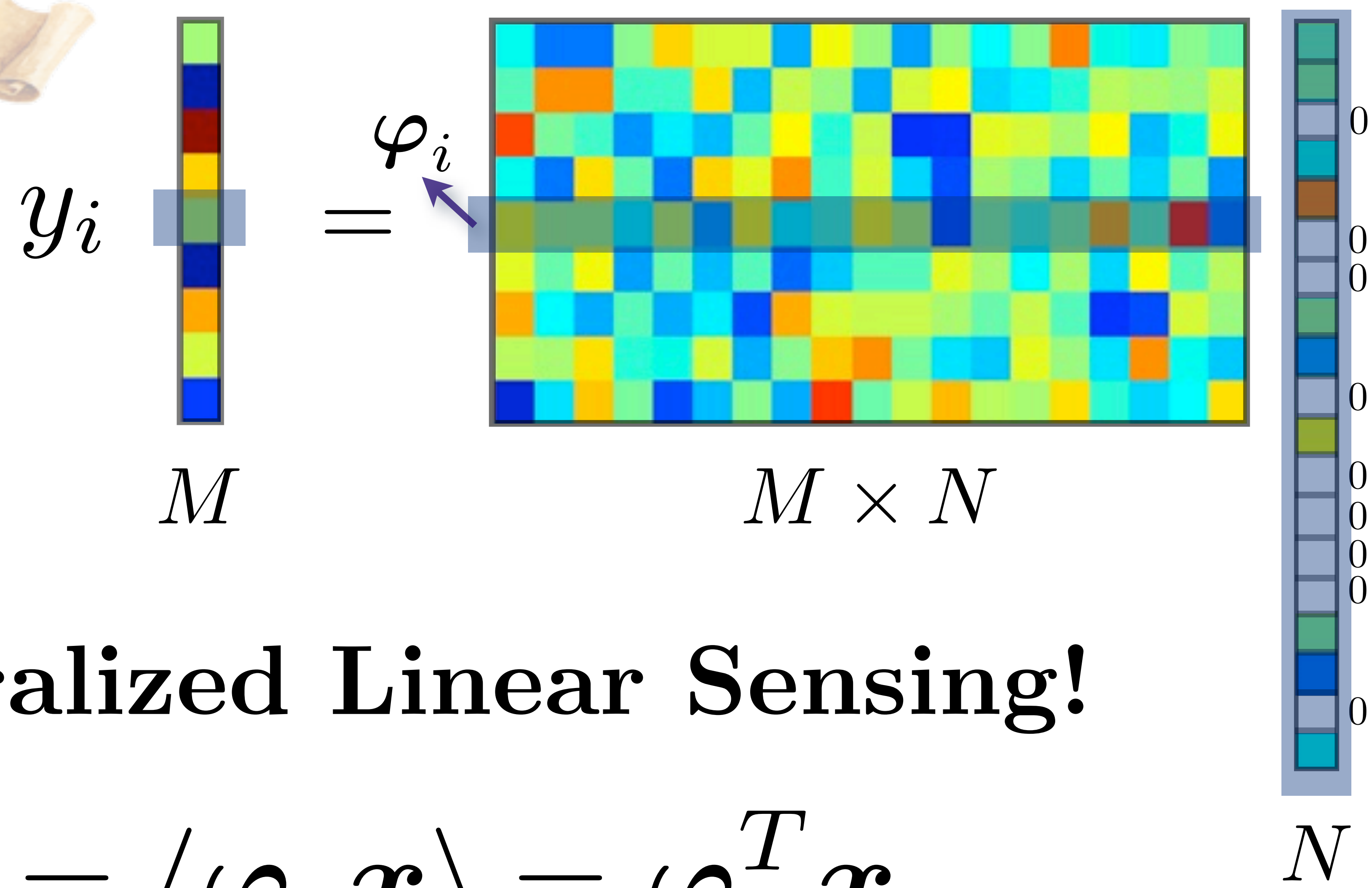
Generalized Linear Sensing!

Compressed Sensing

Then ...

M questions — Sensing method — Signal

$$\boldsymbol{y} \qquad \boldsymbol{\Phi} \qquad \boldsymbol{x}$$



Sparsity Prior ($\Psi = \mathrm{Id}$)

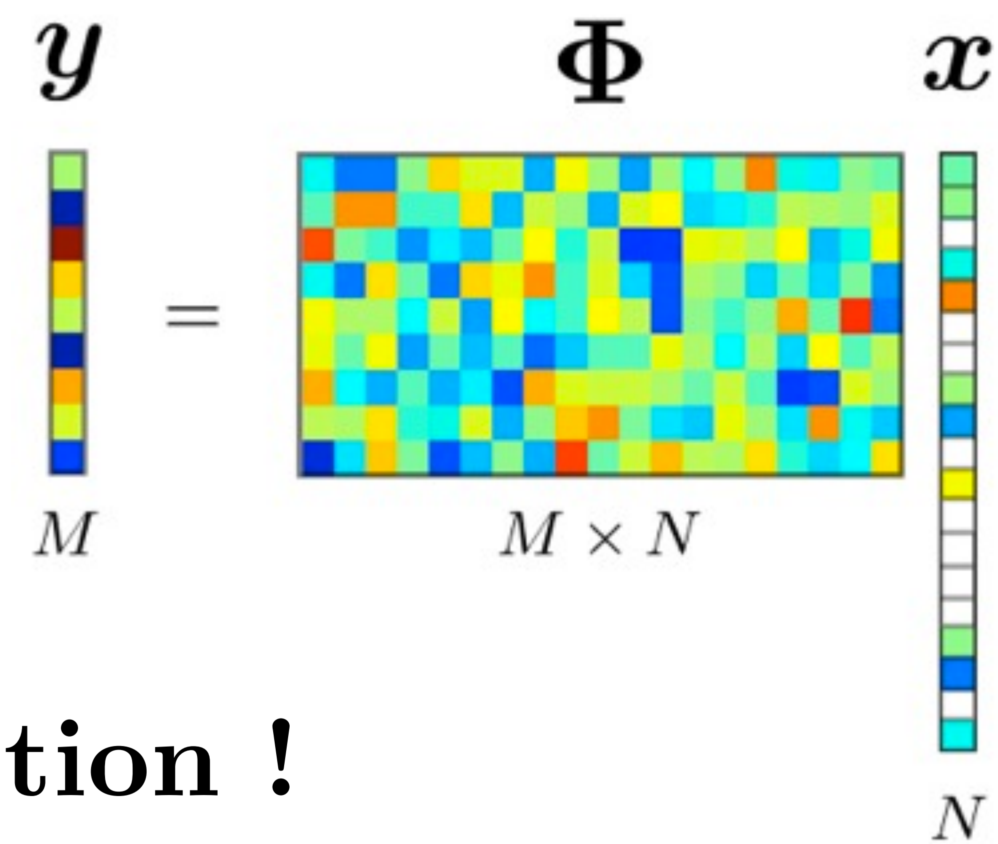
A signal in this discrete world

Generalized Linear Sensing!

$$y_i = \langle \boldsymbol{\varphi}, \boldsymbol{x} \rangle = \boldsymbol{\varphi}^T \boldsymbol{x}$$

$$1 \leq i \leq M$$

Compressed Sensing



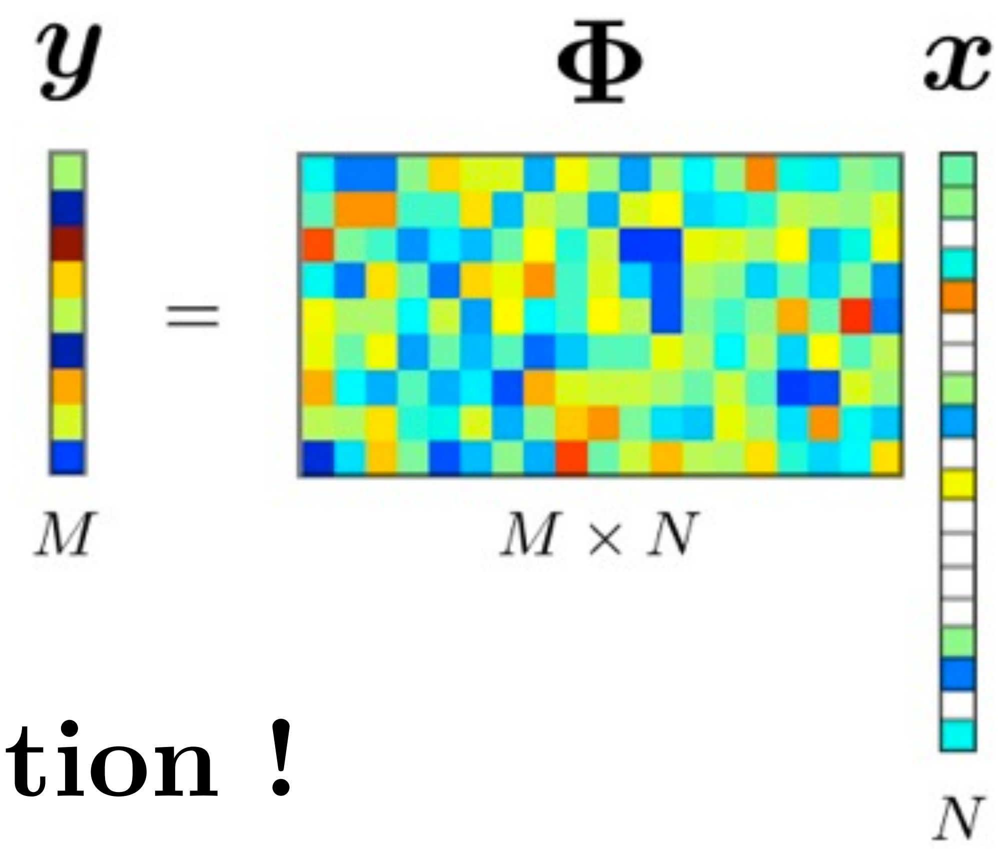
$+$ Non-linear reconstruction !

If $\boldsymbol{x}$ is K-sparse and if $\boldsymbol{\Phi}$ well "conditioned"
then:

$$\boldsymbol{x}^* = \underset{\boldsymbol{u} \in \mathbb{R}^N}{\arg\min} \ \|\boldsymbol{u}\|_0 \text{ s.t. } \boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{u}$$

$\|\boldsymbol{u}\|_0 = \#\{j : u_j \neq 0\}$

Compressed Sensing



$+$ Non-linear reconstruction !

If $\boldsymbol{x}$ is K-sparse and if $\boldsymbol{\Phi}$ well "conditioned" then:

(relax.)

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{u}\in\mathbb{R}^N} \|\boldsymbol{u}\|_{\cancel{0}\;\mathbf{1}} \text{ s.t. } \boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{u}$$

$\|\boldsymbol{u}\|_1 = \sum_j |u_j|$

(Basis Pursuit) [Chen, Donoho, Saunders, 1998]

Compressed Sensing

Simplifying assumption

$\exists\ \delta \in (0,1)$ Restricted Isometry Property

$$\sqrt{1-\delta}\,\|\boldsymbol{v}\|_2 \ \leqslant\ \|\boldsymbol{\Phi v}\|_2 \ \leqslant\ \sqrt{1+\delta}\,\|\boldsymbol{v}\|_2$$

for all $2K$ sparse signals $\boldsymbol{v}$.

any subset of $2K$ columns
is an *isometry*

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then:

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if $\delta < \sqrt{2}-1$ [Candes 08]

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<u>Examples:</u>

+ Gaussian
+ Bernoulli
+ Random Fourier
+ ....

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Examples:

+ Gaussian
+ Bernoulli
+ Random Fourier
+ ....

$$M = O(K \log N/K) \ll N$$

$$\boldsymbol{\Phi} \in \mathbb{R}^{M\times N},\quad \Phi_{ij} \sim_{\text{iid}} \mathcal{N}(0,1)$$

If $\boldsymbol{x}$ is K-sparse and if $\boldsymbol{\Phi}$ well "conditioned"

then:

(relax.)

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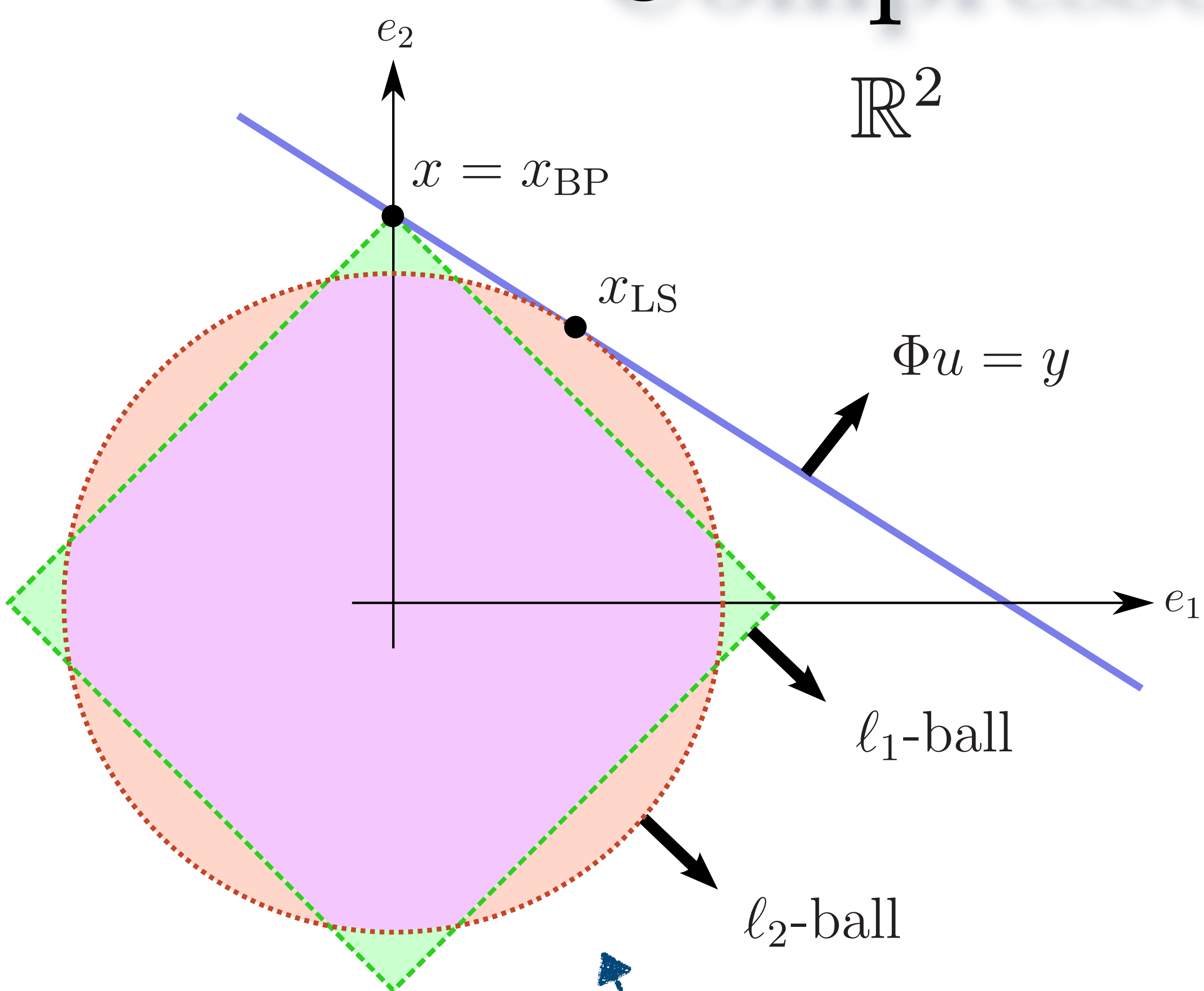

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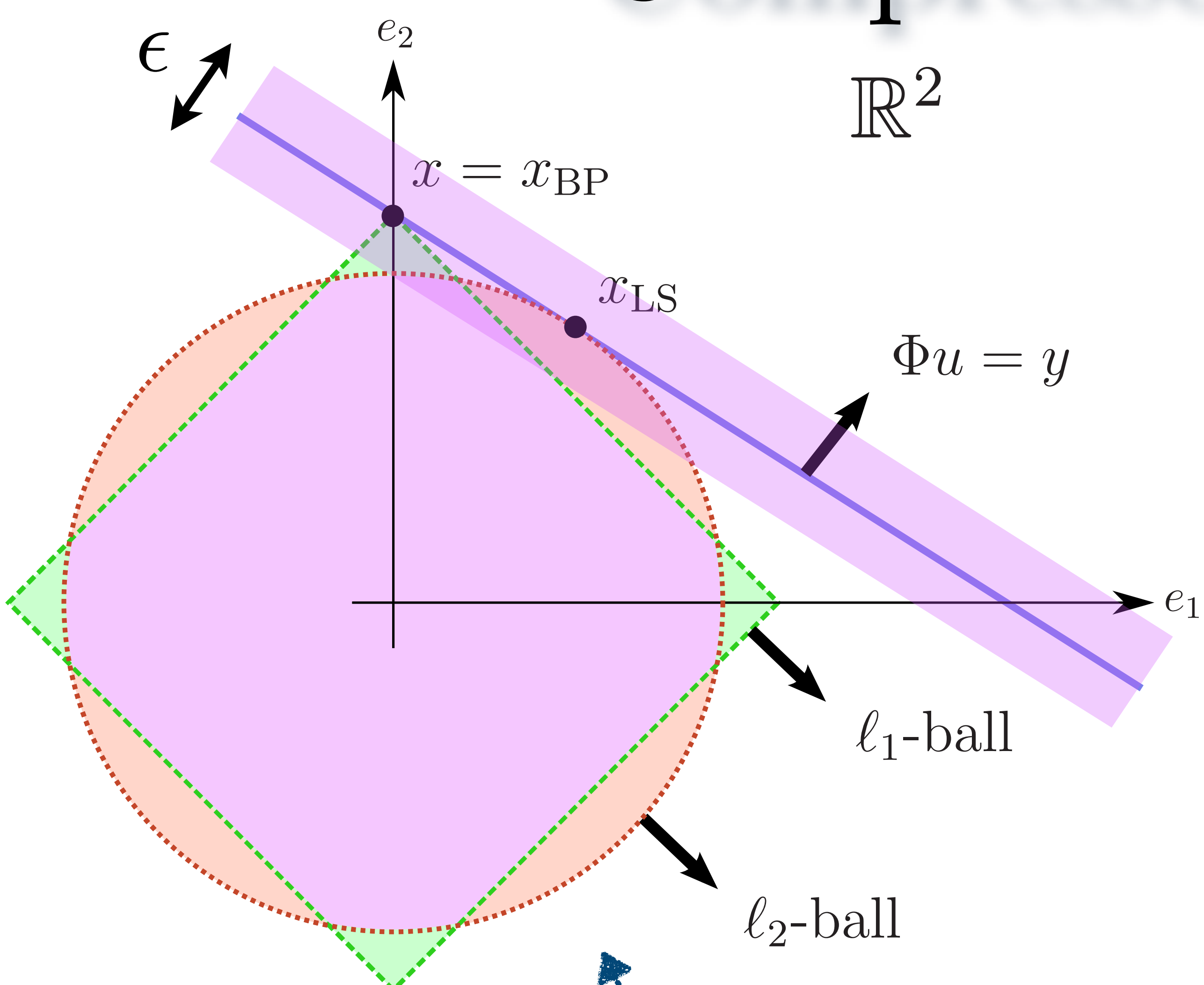
<u>Solvers:</u>

Linear Programming,
Interior Point Method,
Proximal Methods,
... **Tons** of toolboxes ...

Compressed Sensing



If $\boldsymbol{x}$ is K-sparse and if $\boldsymbol{\Phi}$ well "conditioned"

then: $\|\boldsymbol{y} - \boldsymbol{\Phi u}\| \leqslant \epsilon$

(relax.)

$$\boldsymbol{x}^* = \underset{\boldsymbol{u} \in \mathbb{R}^N}{\arg\min} \ \|\boldsymbol{u}\|_{\cancel{2}\,1} \ \text{s.t.} \ \boldsymbol{y} = \boldsymbol{\Phi u}$$

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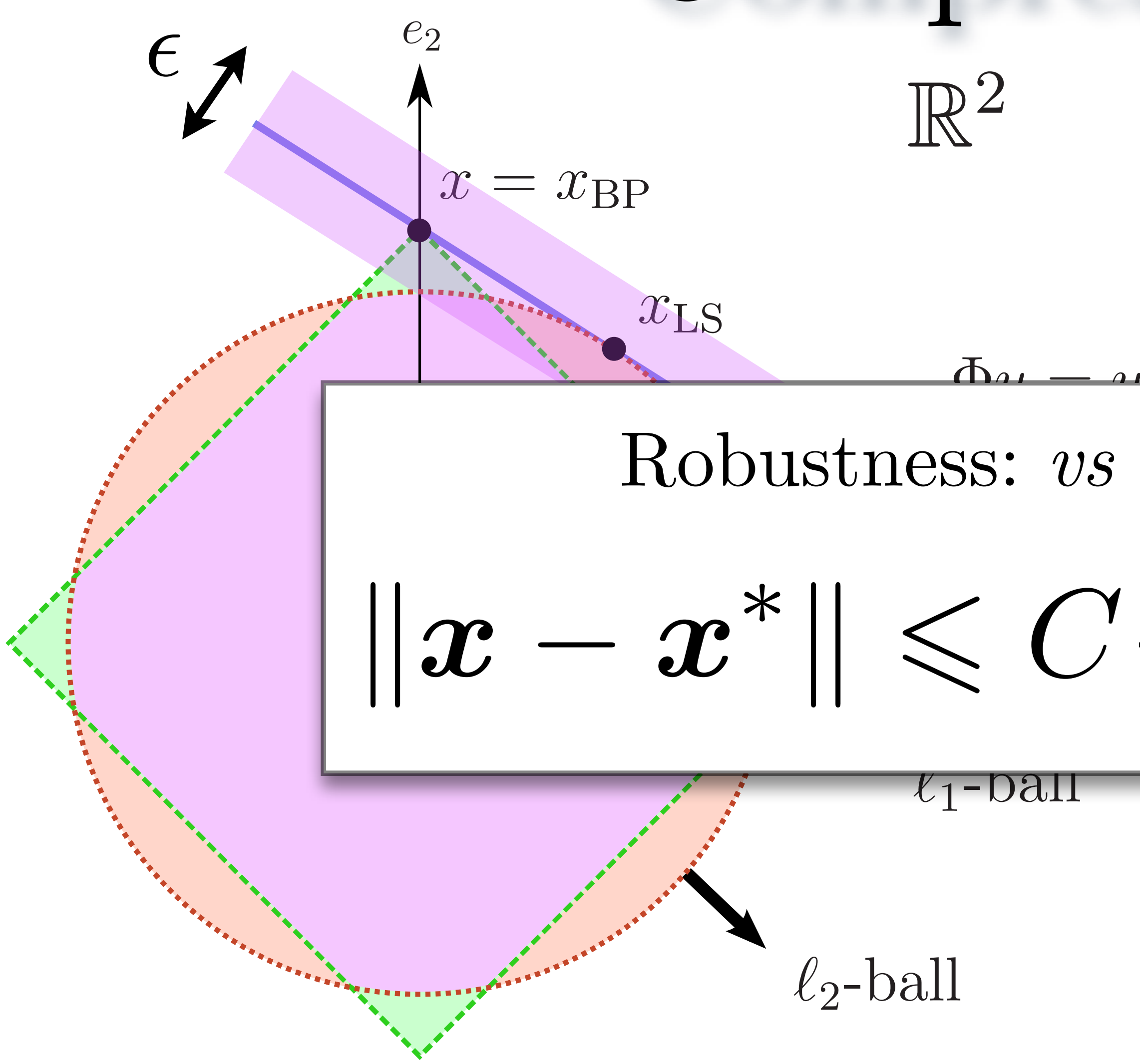

Robustness: *vs* sparse deviation + noise.

$$\|\boldsymbol{x} - \boldsymbol{x}^*\| \leqslant C\,\frac{1}{\sqrt{K}}\,\|\boldsymbol{x} - \boldsymbol{x}_K\|_1 + D\epsilon$$

If $\boldsymbol{x}$ is K-sparse and if $\boldsymbol{\Phi}$ well "conditioned" then:

$$\|\boldsymbol{y} - \boldsymbol{\Phi u}\| \leqslant \epsilon$$

(relax.)

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Part 1

When quantization meets compressed sensing

Outline:

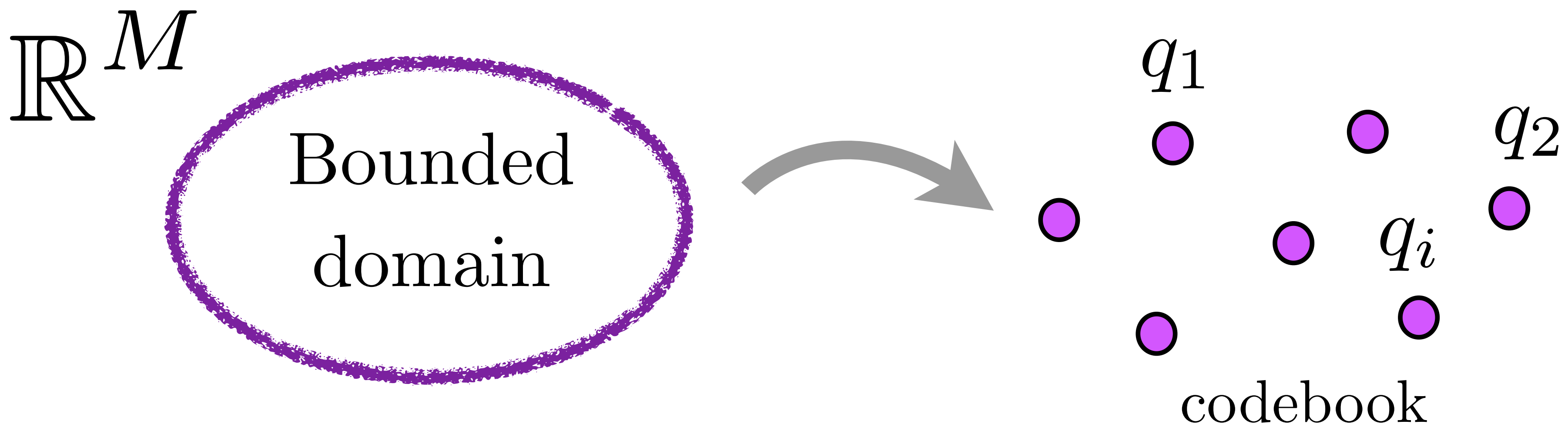
1. Context
2. Former QCS methods and performance limits
3. Consistent Reconstructions
4. Sigma-Delta quantization in CS
5. To saturate or not? And how much?

1. Context

What is quantization?

- <u>Generality</u>:

  Intuitively: "*Quantization maps a bounded continuous domain to a set of finite elements (or codebook)*"



$$\mathcal{Q}[x] \in \{q_1, q_2, \cdots\}$$

- Oldest example: rounding off $\lfloor x \rfloor, \lceil x \rceil, \ldots \quad \mathbb{R} \to \mathbb{Z}$

Example 1: scalar quantization

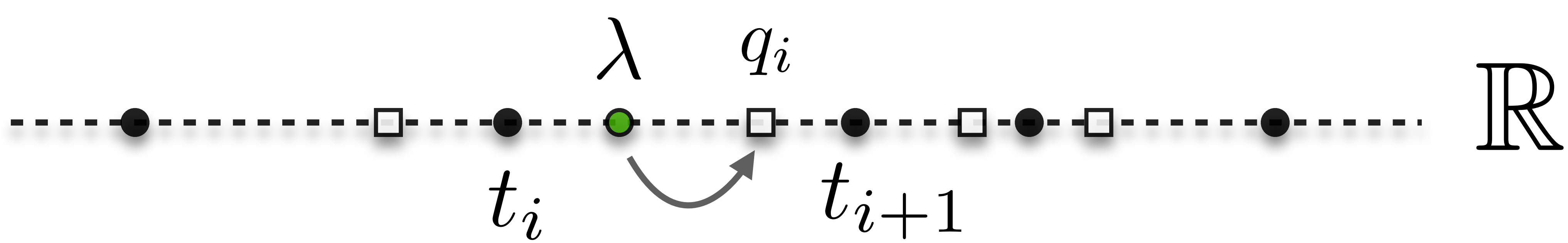
- In $\mathbb{R}^M$, on each component of M- dimensional vectors:

$$\Omega = \{q_i \in \mathbb{R} : 1 \leqslant i \leqslant 2^B\}, \qquad \text{(levels)} \quad \square$$

$$\mathcal{T} = \{t_i \in \overline{\mathbb{R}} : 1 \leqslant i \leqslant 2^B + 1, t_i \leqslant t_{i+1}\} \quad \text{(thresholds)} \quad \bullet$$

$$\forall \lambda \in \mathbb{R}, \qquad \mathcal{Q}[\lambda] = q_i \;\Leftrightarrow\; \lambda \in \mathcal{R}_i \triangleq [t_i, t_{i+1}), \quad \text{1-D quantization cell}$$

$$\forall u \in \mathbb{R}^M, \quad (\mathcal{Q}[u])_j = \mathcal{Q}[u_j]$$



<u>other names:</u>

Pulse Code Modulation - PCM
Memoryless Scalar Quantization - MSQ

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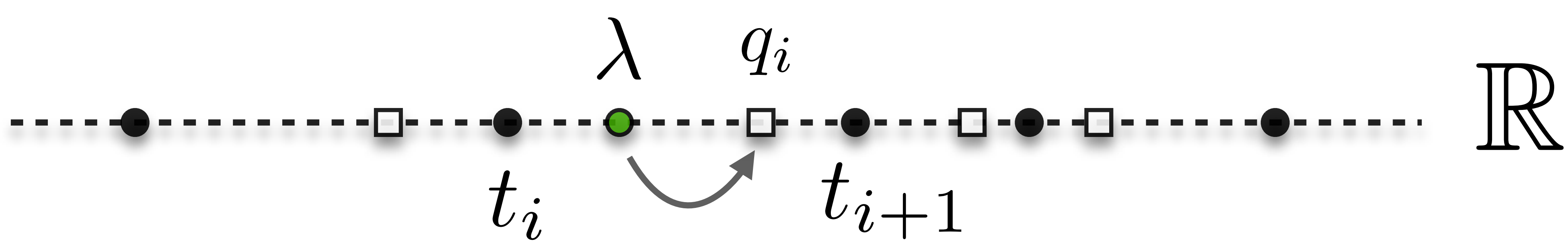
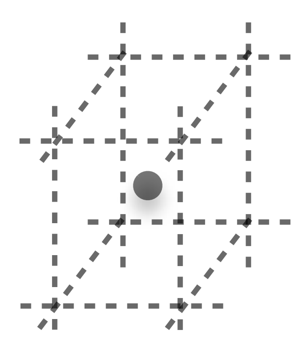


- Globally:

$$\mathcal{Q}[\boldsymbol{z}] = \boldsymbol{q} \in \Omega^M \;\Leftrightarrow\; \boldsymbol{z} \in$$



M - D quantization cell

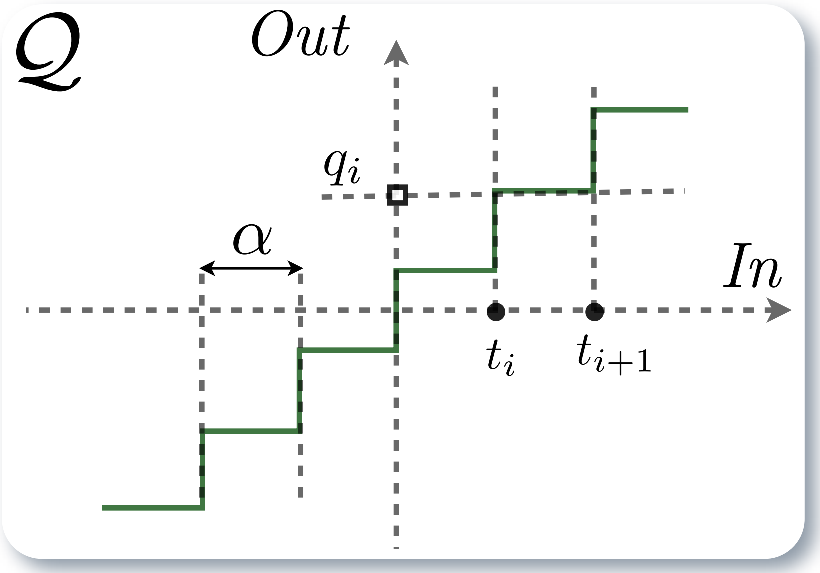
$$\mathcal{R}_{i_1} \times \mathcal{R}_{i_2} \times \cdots \times \mathcal{R}_{i_M}$$

$$:= \mathcal{Q}^{-1}[\boldsymbol{q}]$$

Example 1: scalar quantization

- Regular uniform

$$q_k = (k + 1/2)\alpha$$
$$t_k = k\alpha$$



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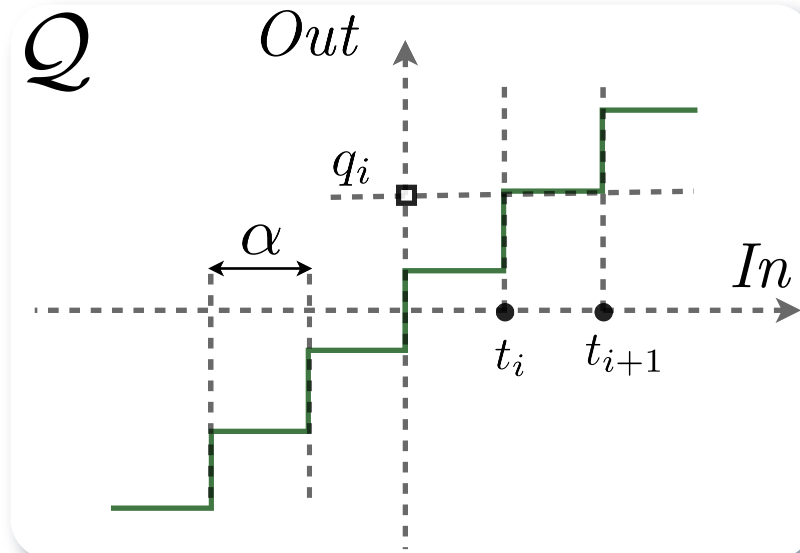
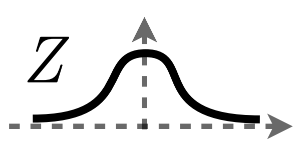


$$q_k = (k + 1/2)\alpha$$
$$t_k = k\alpha$$

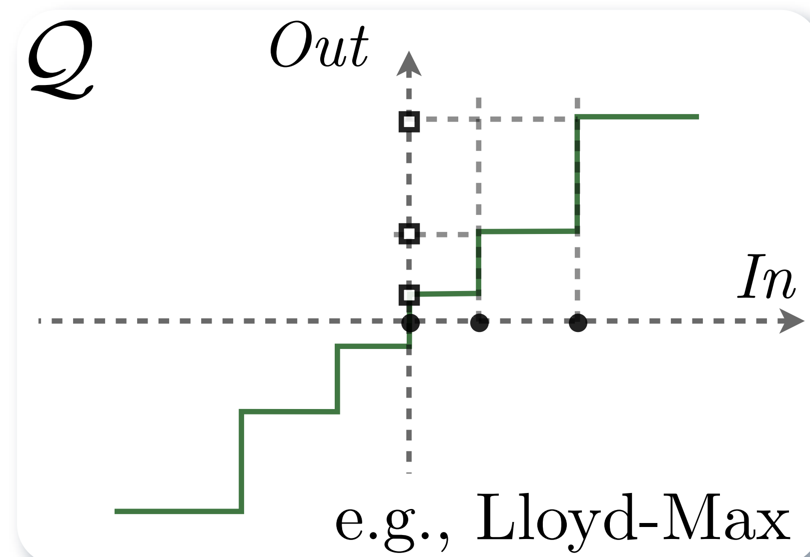
- Regular non-uniform

Ω and $\mathcal{T}$ optimized

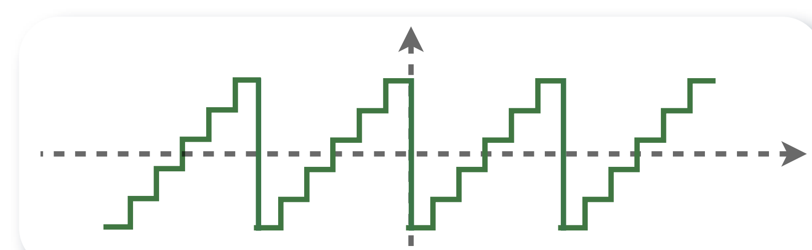
e.g., wrt an input distribution Z
find minimum distortion, *i.e.*,



$$\underset{\mathcal{T},\Omega}{\text{argmin}}\ \mathbb{E}_Z\|Z - \mathcal{Q}[Z]\|^2$$



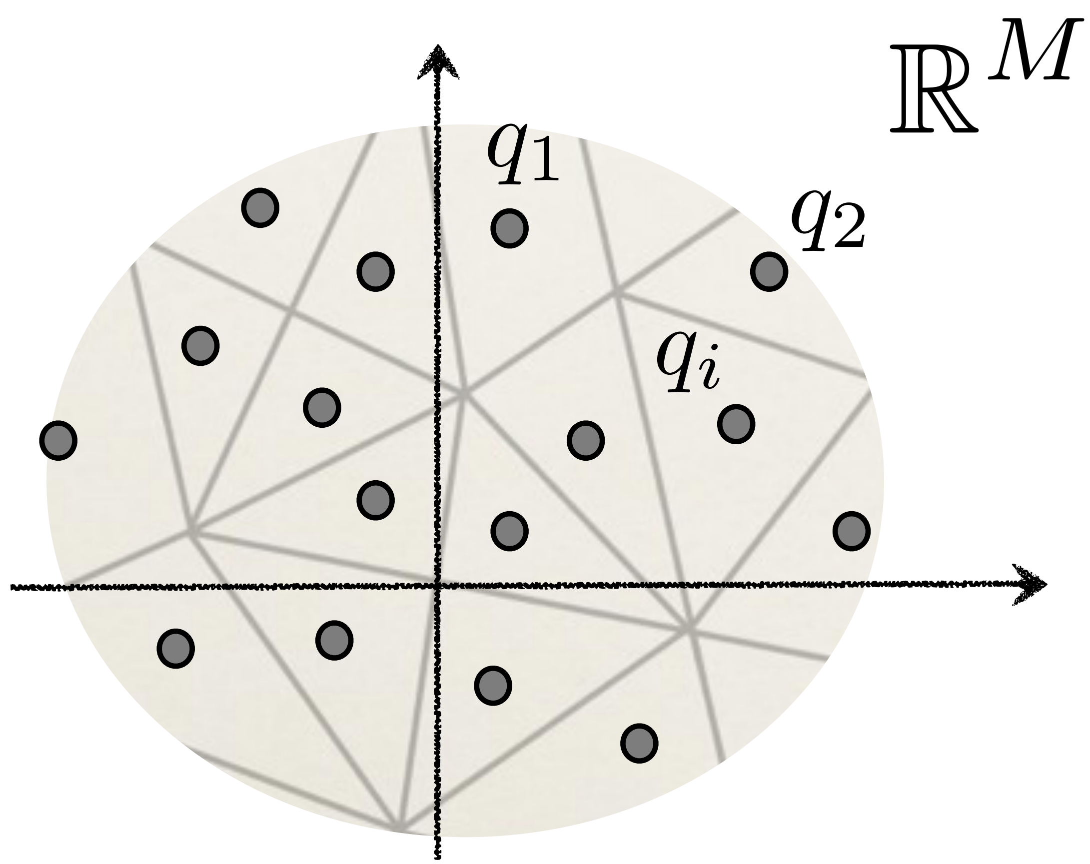
- $\exists$ Non-regular (P. Boufounos)



Example 2: vector quantization

(**caveat**: not really covered in this tutorial, ... except $\Sigma\Delta$, see later)

$$\text{Quantization} = \text{codebook } \boldsymbol{\Omega} + \text{quantization cells } \boldsymbol{\mathcal{R}} = \{\mathcal{R}_i \subset \mathbb{R}^M\}$$

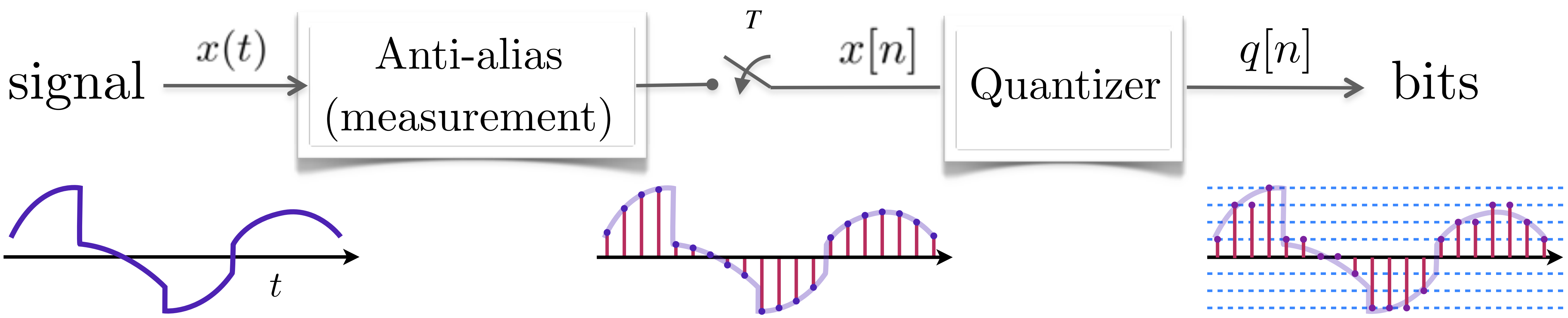


(non-separable quantization)

$$\text{e.g., } \underset{\boldsymbol{\Omega},\boldsymbol{\mathcal{R}}}{\operatorname{argmin}} \ \mathbb{E}_Z \|\boldsymbol{Z} - \mathcal{Q}[\boldsymbol{Z}]\|^2$$

Classical Sampling and Quantization

For acquisition:



Classical Sampling and Quantization

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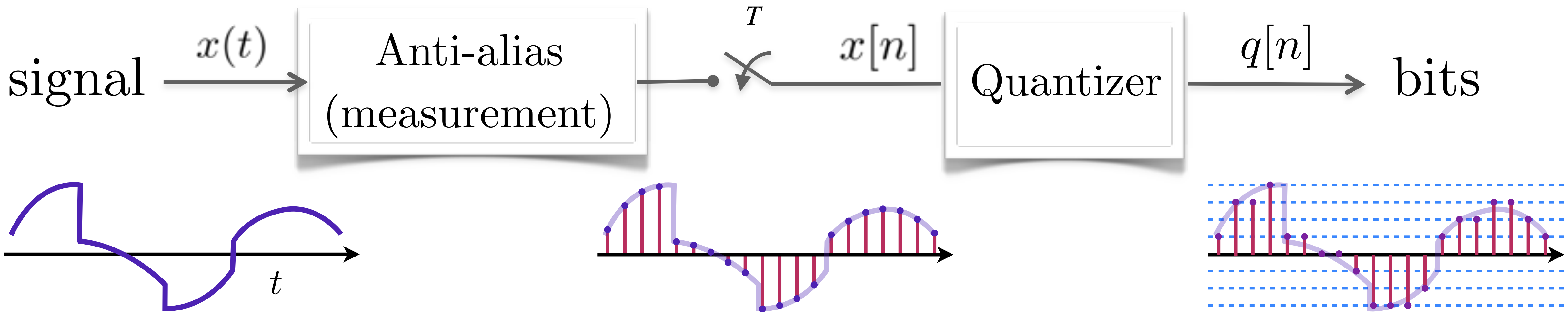
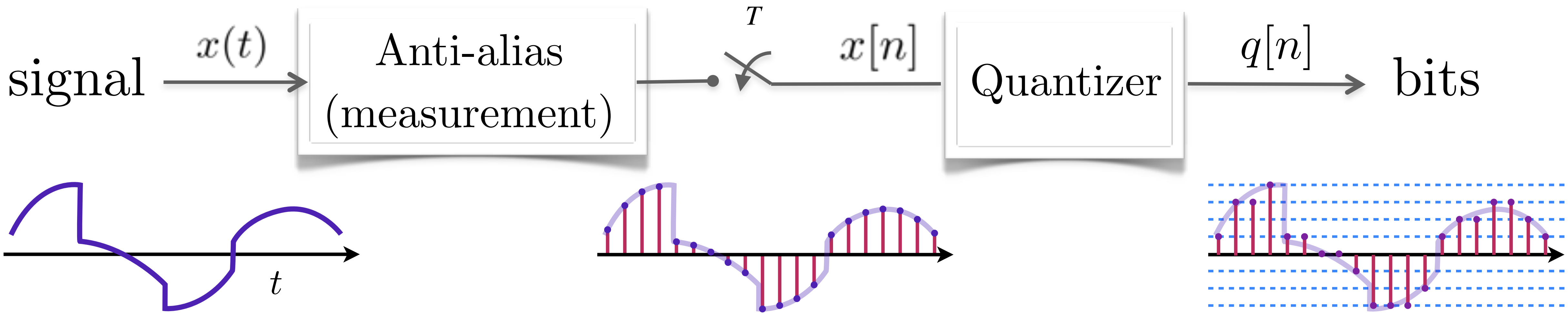


For reconstruction:

Classical Sampling and Quantization

For acquisition:



For reconstruction:

Sampling: discretization in time ⇨ Lossless at the Nyquist rate

Quantization: discretization in amplitude ⇨ Always lossy

Need both for digital data acquisition

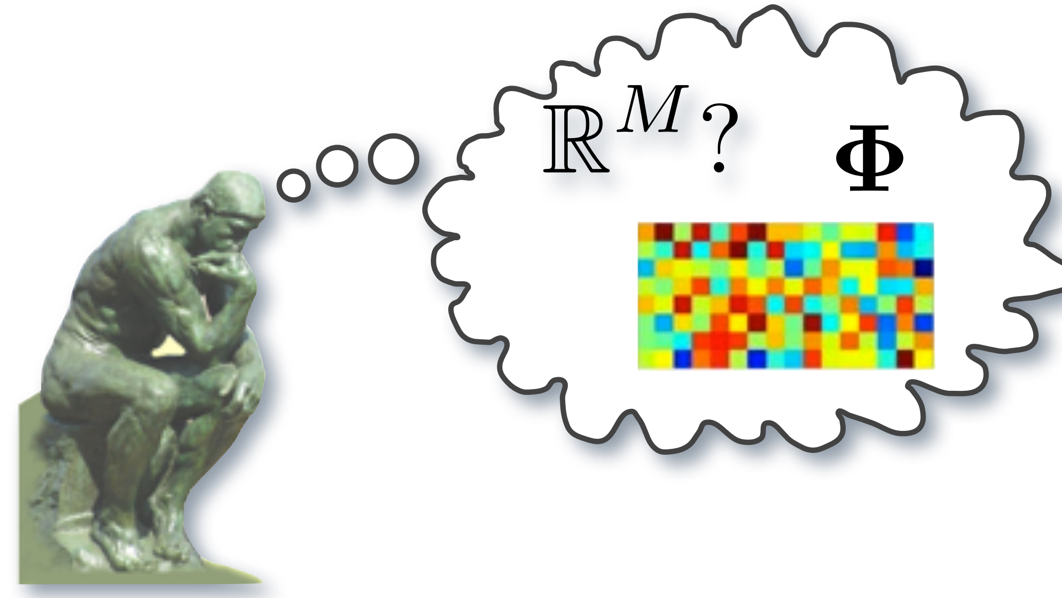
Compressive Sampling and Quantization

Compressed sensing theory says:

"Linearly sample a signal

at a rate function of

its intrinsic dimensionality"

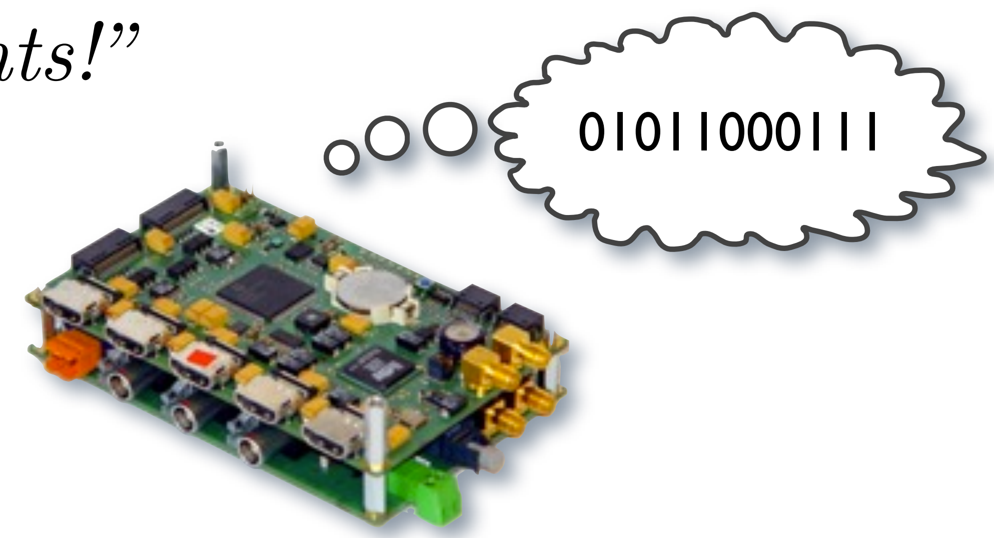


Information theory and sensor designer say:

"Okay, but I need to

quantize/digitize my measurements!"

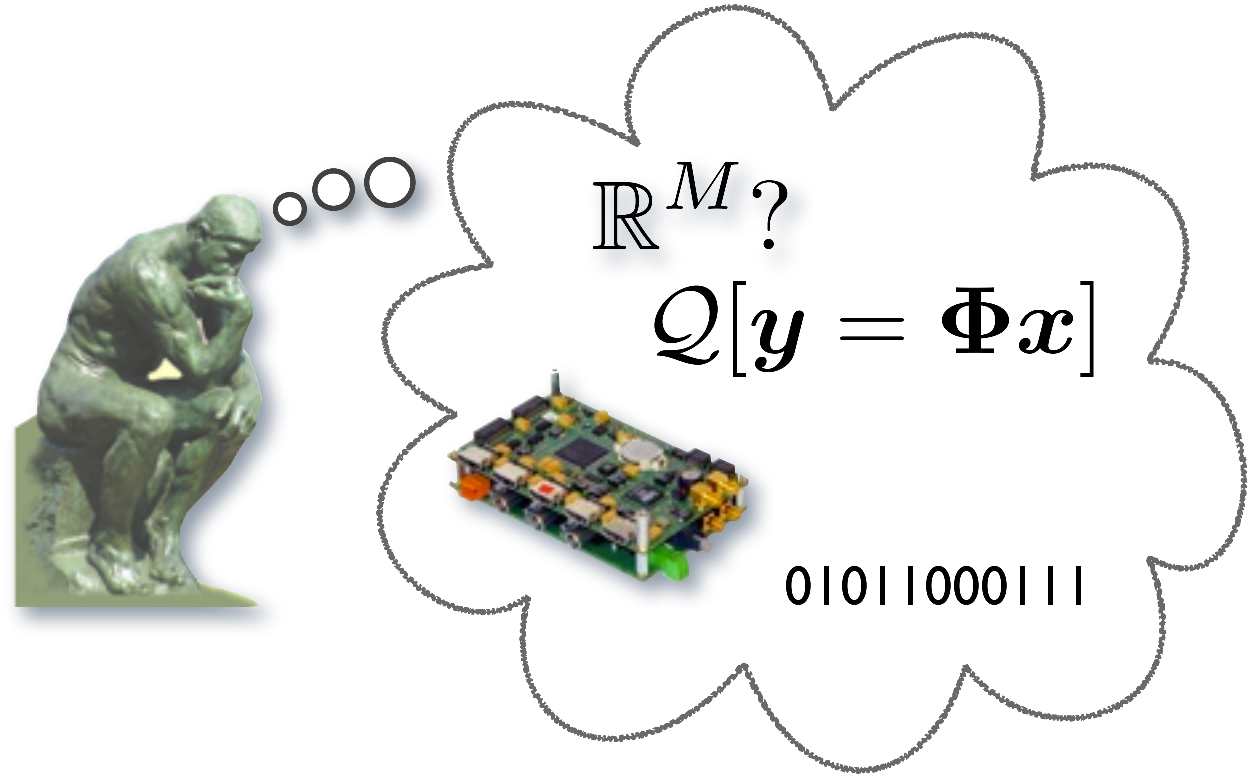
(e.g., in ADC)



The Quantized CS Problem (QCS)

Natural questions:

- How to integrate quantization in CS?

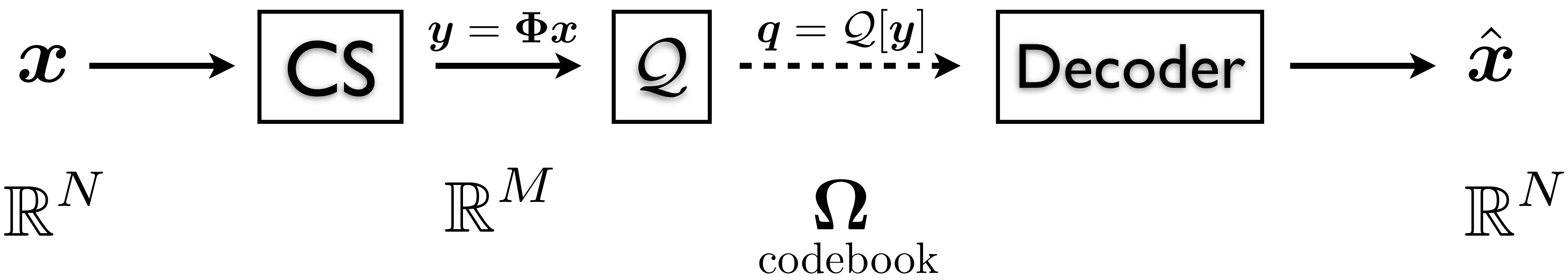


- What do we loose?
- Are they some theoretical limitations?
(related to information theory? geometry?)
- How to minimize quantization effects in the reconstruction?

QCS: a system view

With no additional noise:

e.g., basis pursuit,
greedy methods, ...



scalar or vector
quantization

QCS: a system view

With no additional noise:

e.g., basis pursuit, greedy methods, ...

$$\boldsymbol{x} \longrightarrow \boxed{\text{CS}} \xrightarrow{\boldsymbol{y}\,=\,\boldsymbol{\Phi x}} \boxed{\mathcal{Q}} \xrightarrow{\boldsymbol{q}\,=\,\mathcal{Q}[\boldsymbol{y}]} \boxed{\text{Decoder}} \longrightarrow \hat{\boldsymbol{x}}$$

$\mathbb{R}^N$ $\mathbb{R}^M$ $\boldsymbol{\Omega}$ codebook $\mathbb{R}^N$

scalar or vector quantization

Finite codebook $\Rightarrow \hat{\boldsymbol{x}} \neq \boldsymbol{x}$

(i.e., impossibility to encode continuous domain in a finite number of elements)

QCS: a system view

With no additional noise:

e.g., basis pursuit,
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$\mathbb{R}^N$ $\quad$ $\mathbb{R}^M$ $\quad$ $\boldsymbol{\Omega}$ codebook $\quad$ $\mathbb{R}^N$

Finite codebook $\Rightarrow \hat{\boldsymbol{x}} \neq \boldsymbol{x}$

Objective: Minimize $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|$

given a certain number of:

bits, measurements, or bits/meas.

How?

Where to act?

Change CS, Q or decoder?

Some of them? all?

2. Former QCS methods and performance limits

Scalar quantization in CS

Turning measurements into bits $\rightarrow$ scalar quantization

$$q_i = \mathcal{Q}[(\boldsymbol{\Phi}\boldsymbol{x})_i] = \mathcal{Q}[\langle \boldsymbol{\phi}_i, \boldsymbol{x}\rangle] \ \in \Omega \subset \mathbb{R}$$

$$\boldsymbol{q} \ = \ \mathcal{Q}\big[\boldsymbol{\Phi}\boldsymbol{x}\big] \ \in \boldsymbol{\Omega} = \Omega^M,$$

<u>Important points:</u>

- Definition of $\boldsymbol{\Phi}$ independent of M (e.g., $\Phi_{ij} \sim_{\rm iid} \mathcal{N}(0,1)$)
  $\rightarrow$ preserves measurement dynamic!
- B bits per measurement
- Total bit budget: $R = BM$
- No further encoding (e.g., entropic)

Former solution (Candès, Tao, ...)

- Quantization is like a noise

$$\boxed{\boldsymbol{q} \;=\; \mathcal{Q}\big[\boldsymbol{\Phi x}\big] = \boldsymbol{\Phi x} + \boldsymbol{n}}$$

quantization distortion

Former solution (Candès, Tao, ...)

- Quantization is like a noise

$$\boxed{\boldsymbol{q} \;=\; \mathcal{Q}\big[\boldsymbol{\Phi x}\big] = \boldsymbol{\Phi x} + \boldsymbol{n}}$$

and CS is robust (e.g., with *basis pursuit denoise*)

$$\hat{\boldsymbol{x}} = \operatorname*{argmin}_{\boldsymbol{u}\in\mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{\Phi u} - \boldsymbol{q}\| \leqslant \epsilon \quad \text{(BPDN)}$$

$\ell_2 - \ell_1$ instance optimality:

If $\|\boldsymbol{n}\| \leqslant \epsilon$ and $\frac{1}{\sqrt{M}}\boldsymbol{\Phi}$ is RIP$(\delta, 2K)$ with $\delta \leqslant \sqrt{2} - 1$, then

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \;\leqslant\; C\frac{\epsilon}{\sqrt{M}} + D\, e_0(K),$$

for some $C, D > 0$ and $e_0(K) = \|\boldsymbol{x} - \boldsymbol{x}_K\|_1/\sqrt{K}$.

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How to find it?

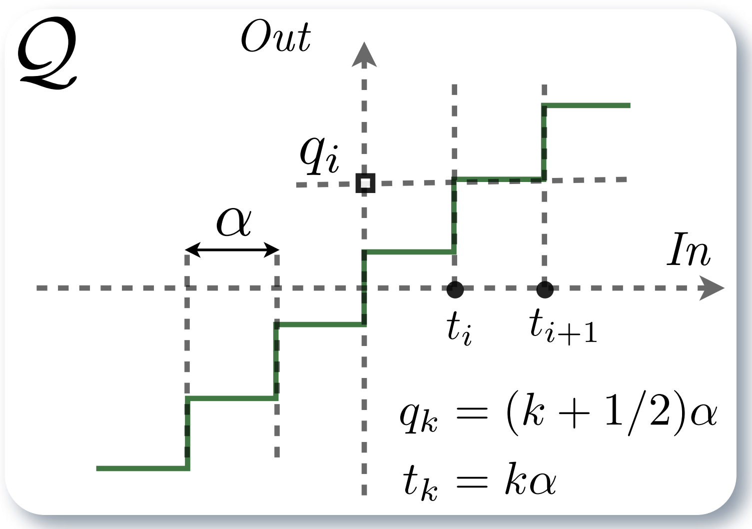
$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \ \leqslant \ C\frac{\epsilon}{\sqrt{M}} + D\, e_0(K),$$

for some $C, D > 0$ and $e_0(K) = \|\boldsymbol{x} - \boldsymbol{x}_K\|_1/\sqrt{K}$.

Former solution (Candès, Tao, ...)

1. For uniform quantization, by construction:

ϵ?



$$\begin{aligned} n_i &= \mathcal{Q}[(\boldsymbol{\Phi x})_i] - (\boldsymbol{\Phi x})_i \\ &\in q_{k_i} - \mathcal{R}_{k_i} = [-\alpha/2, \alpha/2] \\ &\Rightarrow \|\boldsymbol{n}\|_\infty \leq \alpha/2 \end{aligned}$$

$$\Rightarrow \|\boldsymbol{n}\|^2 \leqslant M\|\boldsymbol{n}\|_\infty^2 \leqslant M\alpha^2/4$$

and plug this upper bound in BPDN

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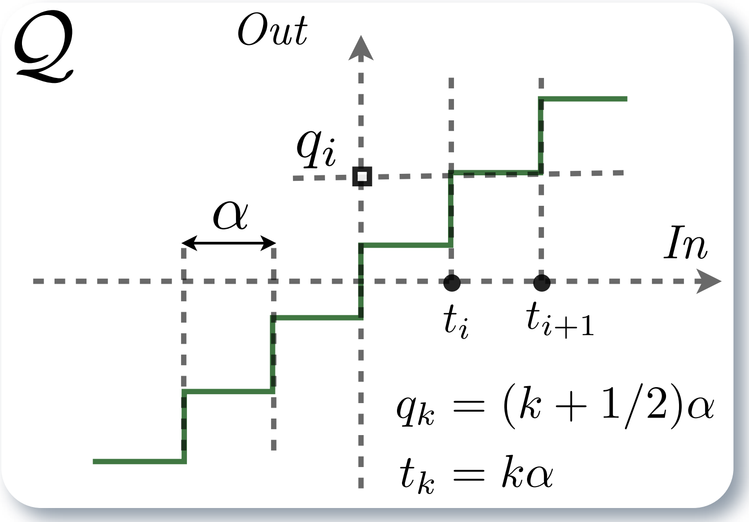
ϵ?



$$\begin{aligned} n_i &= \mathcal{Q}[(\boldsymbol{\Phi x})_i] - (\boldsymbol{\Phi x})_i \\ &\in q_{k_i} - \mathcal{R}_{k_i} = [-\alpha/2, \alpha/2] \\ &\Rightarrow \|\boldsymbol{n}\|_\infty \leq \alpha/2 \end{aligned}$$

$$\Rightarrow \|\boldsymbol{n}\|^2 \leqslant M\|\boldsymbol{n}\|_\infty^2 \leqslant M\alpha^2/4$$

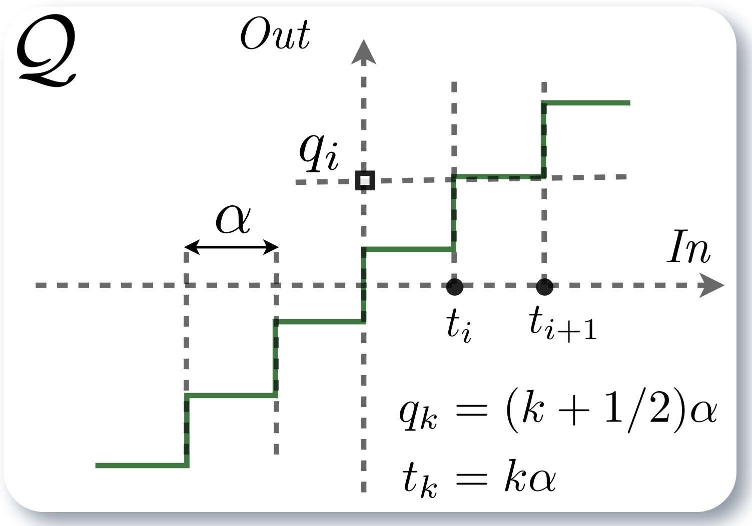
and plug this upper bound in BPDN

can be improved!

Former solution (Candès, Tao, ...)

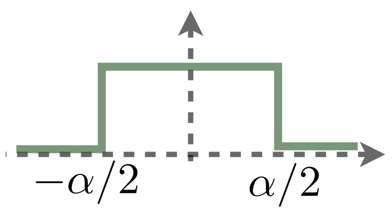
2. For uniform quantization, uniform model!

ϵ?



$$n_i = \mathcal{Q}[(\mathbf{\Phi x})_i] - (\mathbf{\Phi x})_i$$
$$\in q_{k_i} - \mathcal{R}_{k_i} = [-\alpha/2, \alpha/2]$$
$$\sim_{\rm iid} \text{Uniform}([-\alpha/2, \alpha/2])$$

(HRA - high resolution assumption)



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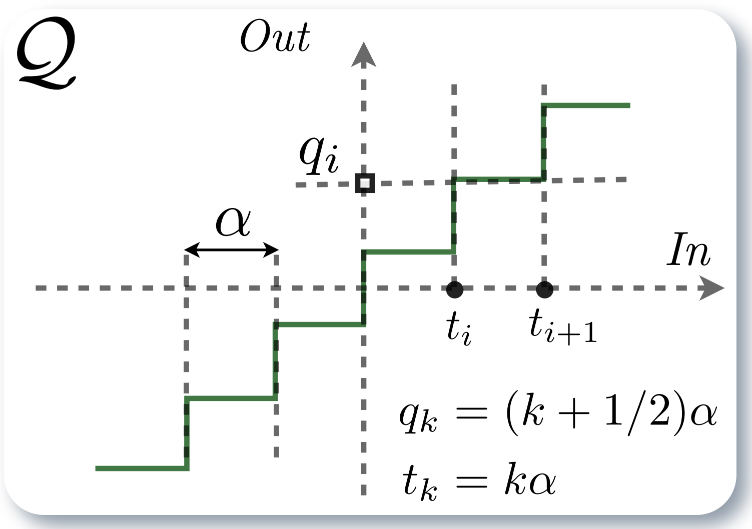

$$
\begin{aligned}
n_i &= \mathcal{Q}[(\boldsymbol{\Phi x})_i] - (\boldsymbol{\Phi x})_i \\
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\end{aligned}
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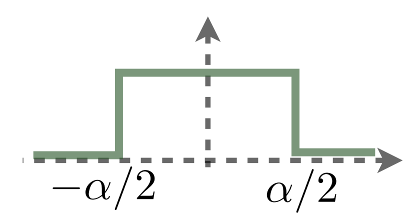

$$\Rightarrow \mathbb{E}|n_i|^2 = \alpha^2/12$$

$$\Rightarrow \|\boldsymbol{n}\|^2 \leqslant \mathbb{E}\|\boldsymbol{n}\|^2 + \kappa\sqrt{\text{Var}\|\boldsymbol{n}\|^2}$$ (Chernoff-Hoeffding, bounded RVs)

$$\boxed{\leqslant M\frac{\alpha^2}{12} + \kappa\sqrt{M}\frac{\alpha^2}{6\sqrt{5}} = \epsilon_2^2 \simeq M\frac{\alpha^2}{12}}$$

with $\Pr > 1 - e^{-2\kappa^2}$

and plug this upper bound in BPDN

Former solution (Candès, Tao, ...)

- Therefore, from BPDN $\ell_2 - \ell_1$ instance optimality:

$$\boxed{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \lesssim C\,\alpha + D\,e_0(K),}$$

for $C,D > 0$

(for BPDN with ϵ_2, under prev. cond.)

Former solution (Candès, Tao, ...)

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(for BPDN with ϵ_2, under prev. cond.)

- Assuming :
  - bounded dynamics: $\|\boldsymbol{\Phi x}\|_\infty = \max_j |(\boldsymbol{\Phi x})_i| \leqslant \rho$ (e.g., by discarding saturation) (see later)
  - B bits per measurements $\Rightarrow \alpha \simeq \frac{2\rho}{2^B}$

$$\boxed{\Rightarrow \text{BPDN RMSE} \lesssim C'\,2^{-B} + D\,e_0(K)}$$

for $C'{,}D > 0$

as soon as RIP holds: $M = O(K \log N/K)$

- Equivalently: BPDN RMSE $\simeq O(2^{-R/M}) + e_0(K)$

  for a rate $R = BM$ bits (total "bid budget" for all meas.)

RMSE Lower bound?

- Let a fixed K-sparse $\boldsymbol{x} \in \mathbb{R}^N$

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- Let a fixed K-sparse $\boldsymbol{x} \in \mathbb{R}^N$
- Oracle: you know $T = \operatorname{supp} \boldsymbol{x}$

RMSE Lower bound?

- Let a fixed K-sparse $\boldsymbol{x} \in \mathbb{R}^N$
- Oracle: you know $T = \operatorname{supp} \boldsymbol{x}$
- Noisy measurements (random noise):

  Given $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$ with $\Phi_{ij} \sim_{\text{iid}} N(0,1)$

  $\boldsymbol{y} = \boldsymbol{\Phi}_T \boldsymbol{x} + \boldsymbol{n}$, with $\mathbb{E}\, \boldsymbol{n}\boldsymbol{n}^T = \sigma^2 \mathbf{Id}_{M \times M}$

- Assume: $\frac{1}{\sqrt{M}} \boldsymbol{\Phi}$ is $\text{RIP}(K, \delta_K)$ and $\text{RIP}(1, \delta_1)$
- Compute LS solution: $\hat{\boldsymbol{x}}_T = \boldsymbol{\Phi}_T^{\dagger} \boldsymbol{y} = (\boldsymbol{\Phi}_T^* \boldsymbol{\Phi}_T)^{-1} \boldsymbol{\Phi}_T^* \boldsymbol{y}$ pseudo-inverse

  $\hat{\boldsymbol{x}}_{T^c} = 0$

- Then: $\boxed{\text{MSE} = \mathbb{E}_{\boldsymbol{n}} \|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2 \geqslant r^{-1}\, \sigma^2 \left(\frac{1-\delta_1}{1+\delta_K}\right)}$

  (as for BPDN) & $\text{MSE} \leqslant \frac{1}{1-\delta_K} \sigma^2$ from [Needell, Tropp, 08]

  for oversampling factor $r = M/K$

- for QCS: $\boxed{\Rightarrow \text{RMSE} = \Omega(r^{\cancel{-1/2}} 2^{-B})}$ & $\text{RMSE} = O(2^{-B})$

3. Consistent Reconstructions

Consistent reconstructions in CS?

- Problem in previous case: if $\hat{\boldsymbol{x}}$ solution of BPDN,

  - no Quantization Consistency (QC): $\mathcal{Q}[\boldsymbol{\Phi}\hat{\boldsymbol{x}}] \neq Q[\boldsymbol{\Phi}\boldsymbol{x}]$

$$\underbrace{\|\boldsymbol{\Phi}\hat{\boldsymbol{x}} - \mathcal{Q}[\boldsymbol{\Phi}\boldsymbol{x}]\| \leqslant \epsilon_2}_{\text{(from BPDN constraint)}} \quad \nRightarrow \mathcal{Q}[\boldsymbol{\Phi}\hat{\boldsymbol{x}}] = Q[\boldsymbol{\Phi}\boldsymbol{x}]$$

$\Longrightarrow$ sensing information is fully not exploited!

  - ℓ_2 constraint $\approx$ Gaussian distribution (MAP - cond. log. lik.)

But why looking for consistency?

First,

Proposition (Goyal, Vetterli, Thao, 98) *If T is known (with $|T| = K$), the best decoder* $\mathrm{Dec}()$ *provides a* $\hat{\boldsymbol{x}} = \mathrm{Dec}(\boldsymbol{y}, \boldsymbol{\Phi})$ *such that:*

$$\mathrm{RMSE} = (\mathbb{E}\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2)^{1/2} \gtrsim r^{-1}\alpha,$$

where $\mathbb{E}$ *is wrt a probability measure on* $\boldsymbol{x}_T$ *in a bounded set* $\mathcal{S} \subset \mathbb{R}^K$.

This bound is achieved, at least, for $\boldsymbol{\Phi}_T = \mathrm{DFT} \in \mathbb{R}^{M \times K}$, *when* $\mathrm{Dec}()$ *is* **consistent.**

V. K Goyal, M. Vetterli, N. T. Thao, "Quantized Overcomplete Expansions in R^N: Analysis, Synthesis, and Algorithms" , IEEE Tran. IT, 44(1), 1998

But why looking for consistency?

Second,

If $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$ is a (random) frame in $\mathbb{R}^N$ $(M \geqslant N)$,

Then, for $\mathcal{Q}(\boldsymbol{y}) = \boldsymbol{y} + \boldsymbol{n}$ with $n_i \sim \mathcal{U}([-\frac{1}{2}\alpha, \frac{1}{2}\alpha])$, and $\hat{\boldsymbol{x}}$ consistent,

$$(\mathbb{E}_{\boldsymbol{\Phi},\boldsymbol{n}} \|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2)^{1/2} \ \lesssim\ r^{-1}\alpha,$$

[Powell, Whitehouse, 2013]

(unit norm frame)

and

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \ \lesssim\ r^{-1}\alpha \cdot\ O(\log M, \log N, \log \eta),$$

[LJ 2014]

(Gaussian frame)

with $\Pr \geqslant 1 - \eta$.

or $\frac{K}{M}\alpha \cdot\ O(\log K, \log M, \log N, \log \eta)$ in K sparse case

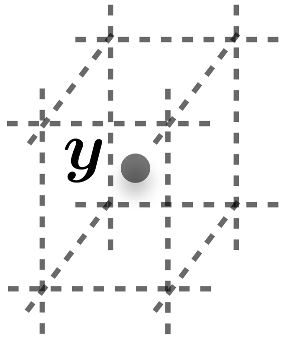
In quest of consistency... $\ell_2 \to \ell_\infty$

- Modify BPDN [W. Dai, O. Milenkovic, 09]

$$\hat{\boldsymbol{x}} = \operatorname*{argmin}_{\boldsymbol{u}\in\mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \mathcal{Q}[\boldsymbol{\Phi u}] = \boldsymbol{q}$$

$+$ modified greedy algo:
"*subspace pursuit*"

$$\Leftrightarrow \boldsymbol{\Phi u} \in \underbrace{\mathcal{Q}^{-1}[\boldsymbol{q}]}_{\text{convex set in } \mathbb{R}^M}$$



$$\Leftrightarrow \|\boldsymbol{\Phi u} - \boldsymbol{q}\|_\infty \leq \alpha/2$$

(if uniform quant.)

$\exists$ numerical methods

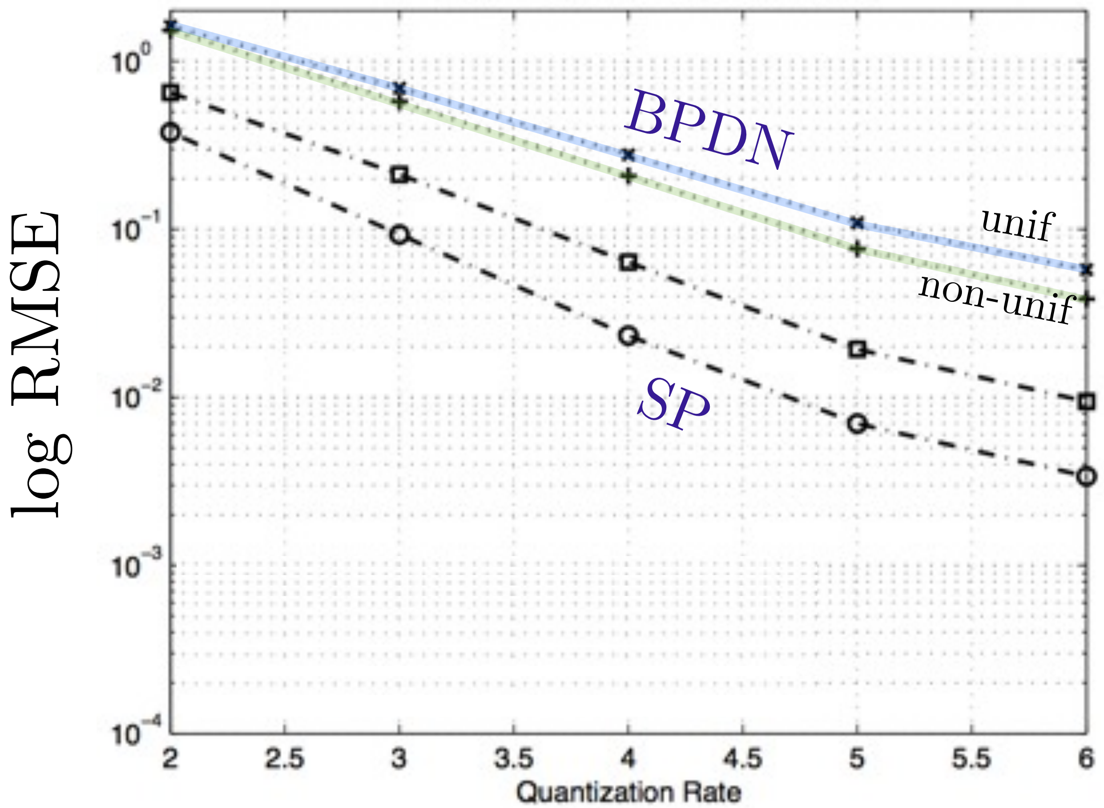
In quest of consistency...

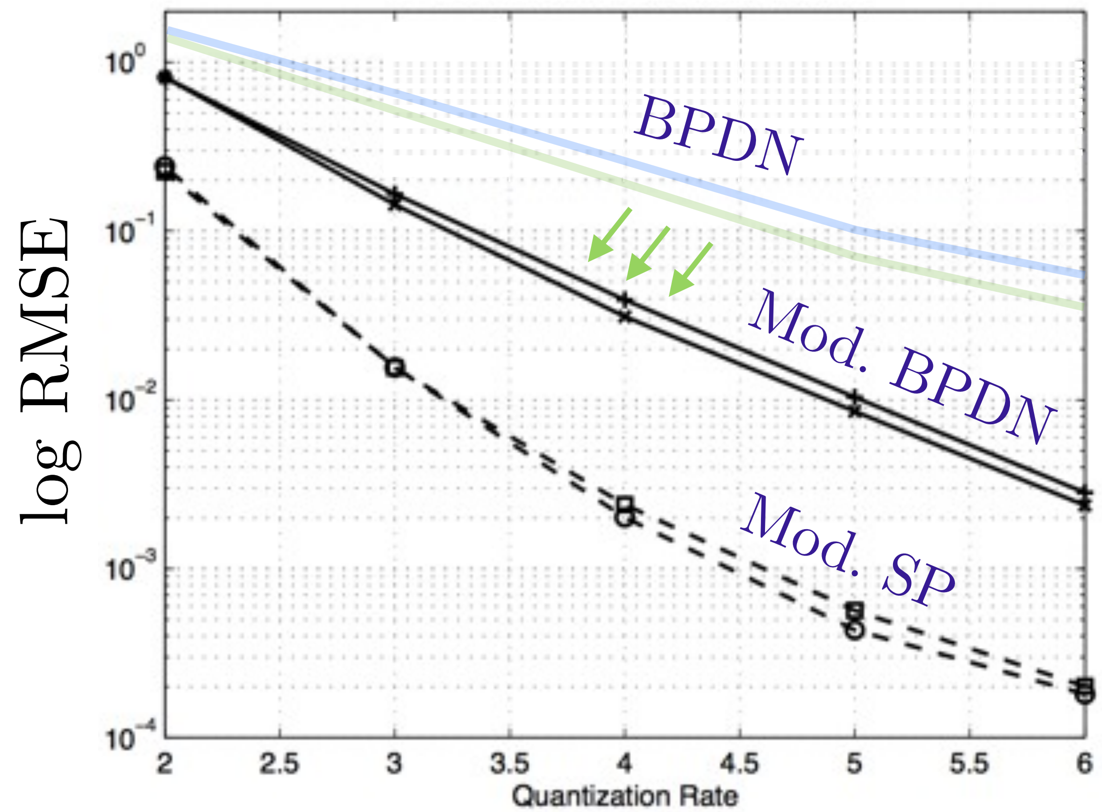
$\ell_2 \to \ell_\infty$

- Modify BPDN [W. Dai, O. Milenkovic, 09]

$$\hat{\boldsymbol{x}} = \operatorname*{argmin}_{\boldsymbol{u}\in\mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \mathcal{Q}[\boldsymbol{\Phi u}] = \boldsymbol{q}$$

Simulations: $M = 128, N = 256, K = 6, 1000$ trials $\Rightarrow \lambda \simeq 20$





W. Dai, H. V. Pham, and O. Milenkovic, "Quantized Compressive Sensing", preprint, 2009

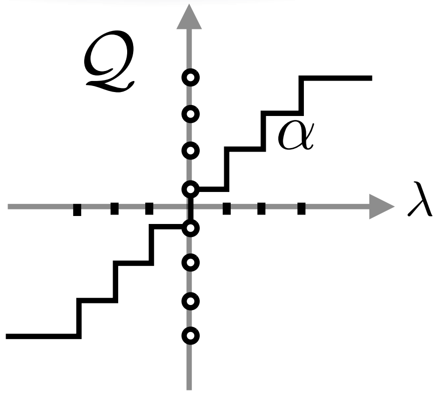
Dequantizing CS?

$\ell_2 \to \ell_p \ (p \geq 2)$

[LJ, Hammond, Fadili, 2009, 2011]

- Distortion model:

$$\boldsymbol{q} = \mathcal{Q}[\boldsymbol{\Phi x}] = \boldsymbol{\Phi x} + \boldsymbol{n}, \quad n_i \sim U(-\tfrac{\alpha}{2}, \tfrac{\alpha}{2})$$



- Observation: $\|\boldsymbol{\Phi x} - \boldsymbol{q}\|_\infty \leq \alpha/2$
- Reconstruction: Generalizing BPDN with BPDQ

$$\boxed{\hat{\boldsymbol{x}} \ = \ \arg\min_{\boldsymbol{u} \in \mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{q} - \boldsymbol{\Phi u}\|_p \ \leq \epsilon_p}$$

Towards $p = \infty$

Related to GGD MAP

Dequantizing CS?

[LJ, Hammond, Fadili, 2009, 2011]

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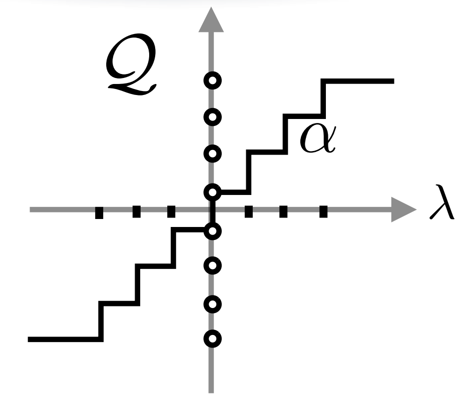

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Related to GGD MAP

How to find it? again, uniform model:

Dequantizing CS?

[LJ, Hammond, Fadili, 2009, 2011]

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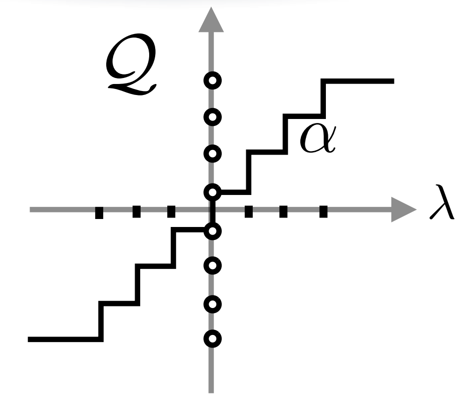

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Towards $p = \infty$

Related to GGD MAP

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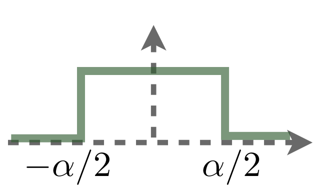


Estimating p^{th} moment:

$$\epsilon_p(\alpha) = \frac{\alpha}{2(p+1)^{1/p}} \left(M + \kappa(p+1)\sqrt{M}\right)^{1/p}$$

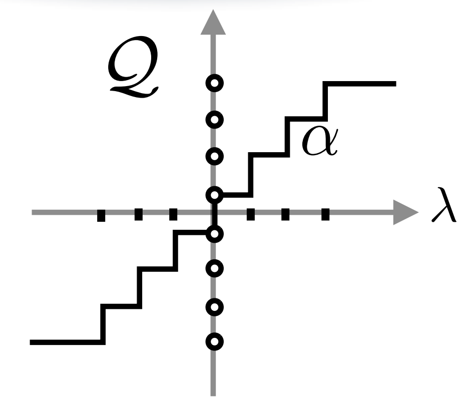
works with $\Pr \geq 1 - e^{-2\kappa^2}$

Note: $\epsilon_p(\alpha) \xrightarrow[p\to\infty]{} \frac{\alpha}{2} = \text{QC}\,!$

Dequantizing CS?

[LJ, Hammond, Fadili, 2009, 2011]

$\ell_2 \to \ell_p \; (p \geq 2)$



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Towards $p = \infty$

Related to GGD MAP

BPDQ Stability ?

Dequantizing CS?

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[LJ, Hammond, Fadili, 2009, 2011]

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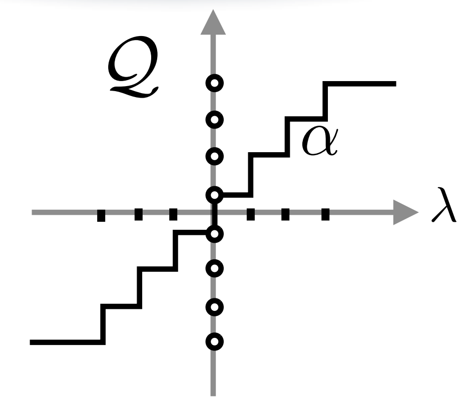

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Towards $p = \infty$

Related to GGD MAP

BPDQ Stability ?

Ok, if $\boldsymbol{\Phi}$ is RIP_p of order K, *i.e.*,

$\exists\ \mu_p > 0,\ \delta \in (0,1)$,

$$\sqrt{1-\delta}\,\|\boldsymbol{v}\|_2 \leqslant \frac{1}{\mu_p}\|\boldsymbol{\Phi v}\|_p \leqslant \sqrt{1+\delta}\,\|\boldsymbol{v}\|_2,$$

for all K sparse signals $\boldsymbol{v}$.

Dequantizing CS?

$\ell_2 \to \ell_p \; (p \geq 2)$

[LJ, Hammond, Fadili, 2009, 2011]

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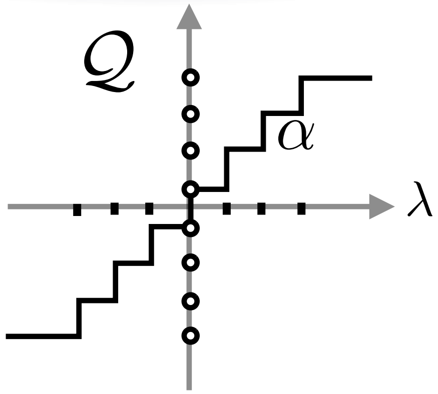

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Towards $p = \infty$

Related to GGD MAP

Gain over BPDN (for tight $\epsilon_p(\alpha, M)$)

$$\Rightarrow \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(\epsilon_p/\mu_p)$$

$$\boxed{\Rightarrow \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(\alpha/\sqrt{p+1})}$$

But no free lunch: for $\boldsymbol{\Phi}$ Gaussian

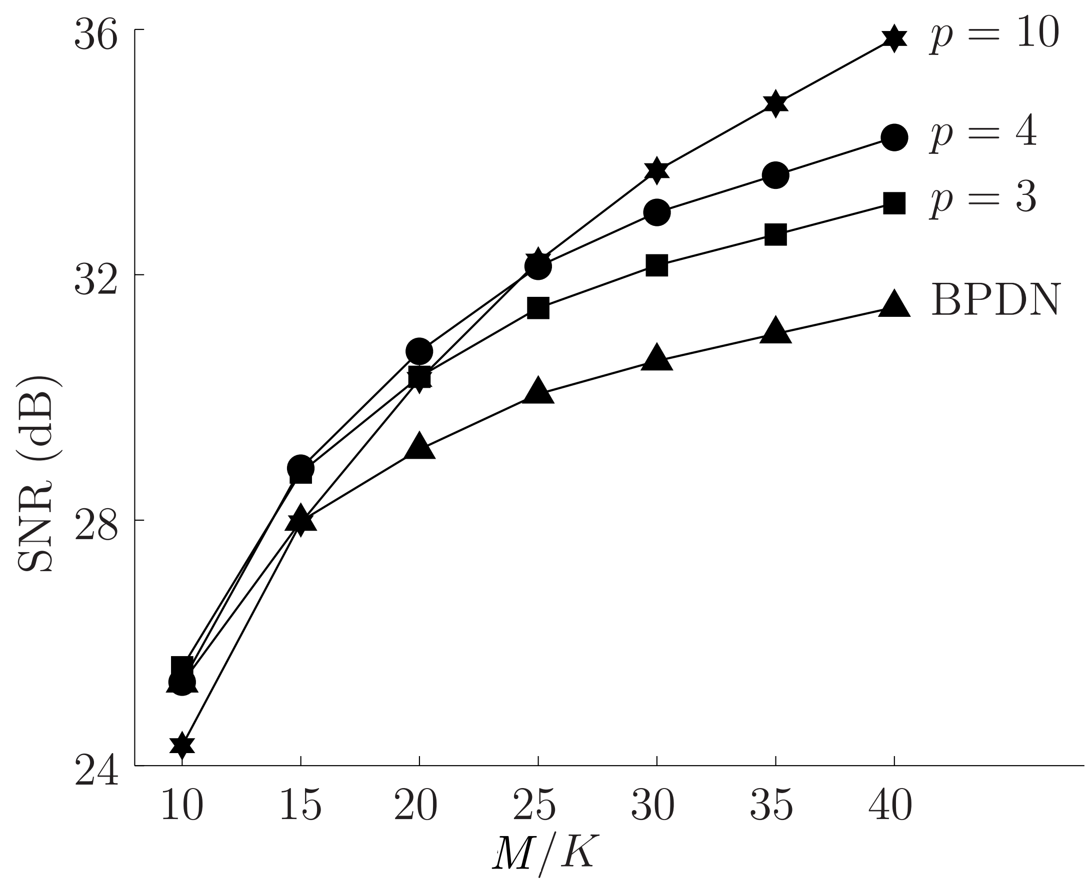
$$\boxed{M = O\big((K \log N/K)^{p/2} \big)}$$

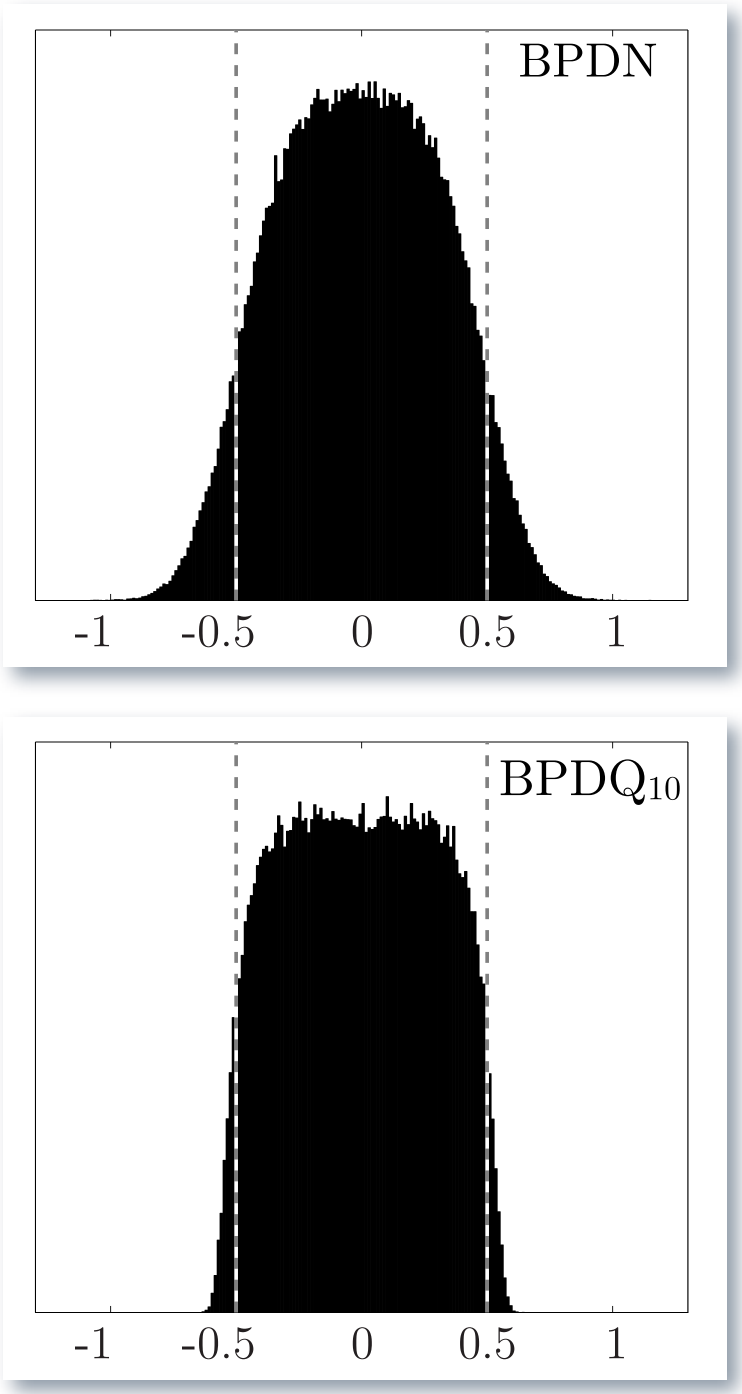
$\Longrightarrow$ Another reading: limited range of valid p for a given M (and K)!

Dequantizing CS?

[LJ, Hammond, Fadili, 2009, 2011]

Histograms of
$$\alpha^{-1}(\boldsymbol{q} - \boldsymbol{\Phi}\hat{\boldsymbol{x}})_i$$



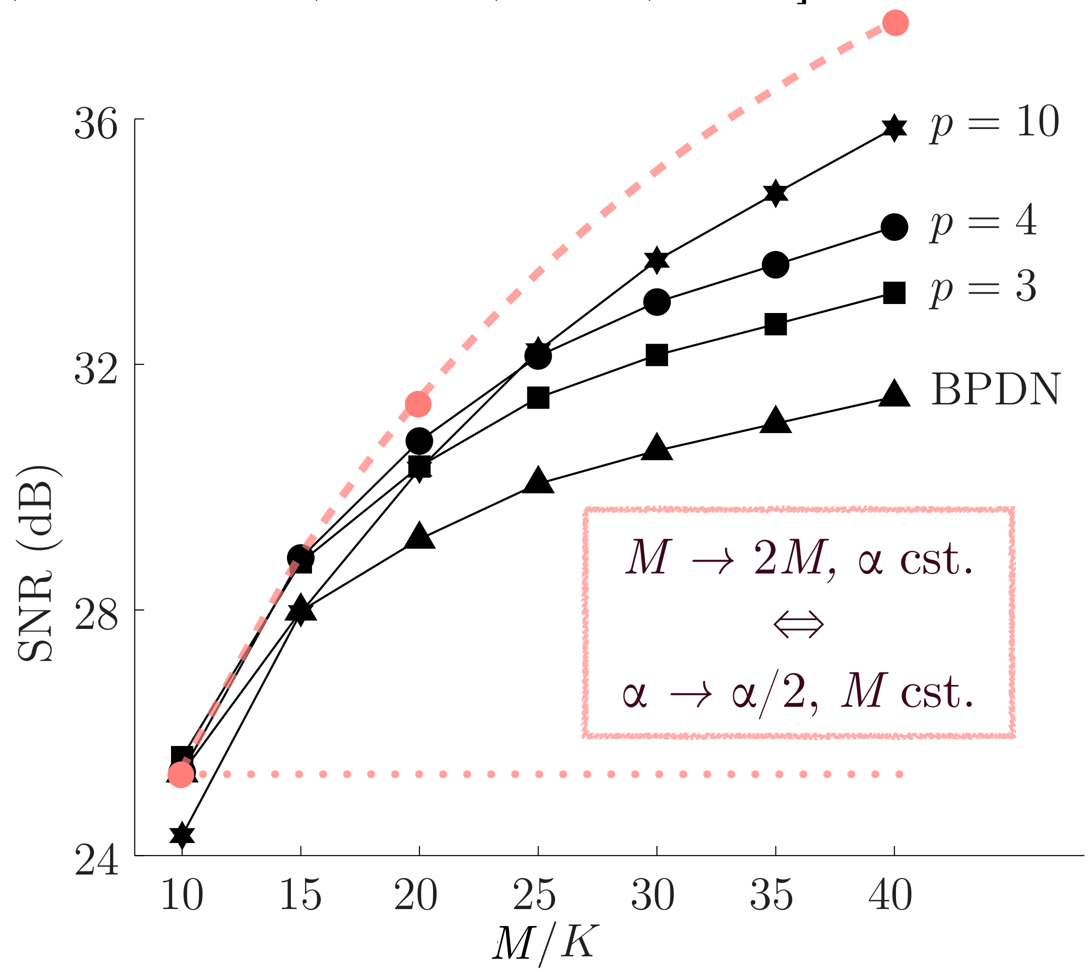


- N=1024, K=16, Gaussian $\boldsymbol{\Phi}$
- 500 K-sparse (canonical basis)
- Non-zero components follow $\mathcal{N}(0,1)$
- Quantiz. bin width $\alpha = \|\boldsymbol{\Phi x}\|_\infty/40$

LJ, D. Hammond, J. Fadili "Dequantizing compressed sensing: When oversampling and non-gaussian constraints combine." Information Theory, IEEE Transactions on, 57(1), 559-571.

Dequantizing CS?

[LJ, Hammond, Fadili, 2009, 2011]



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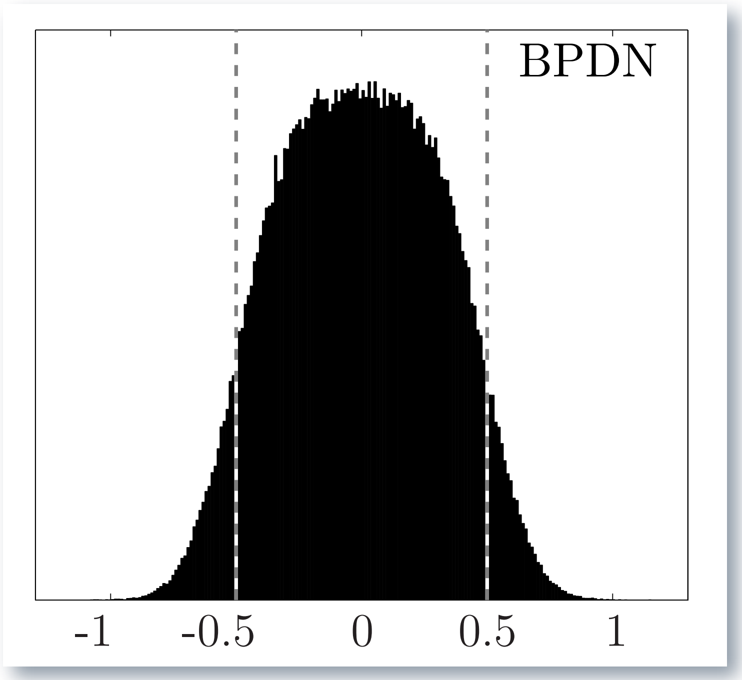
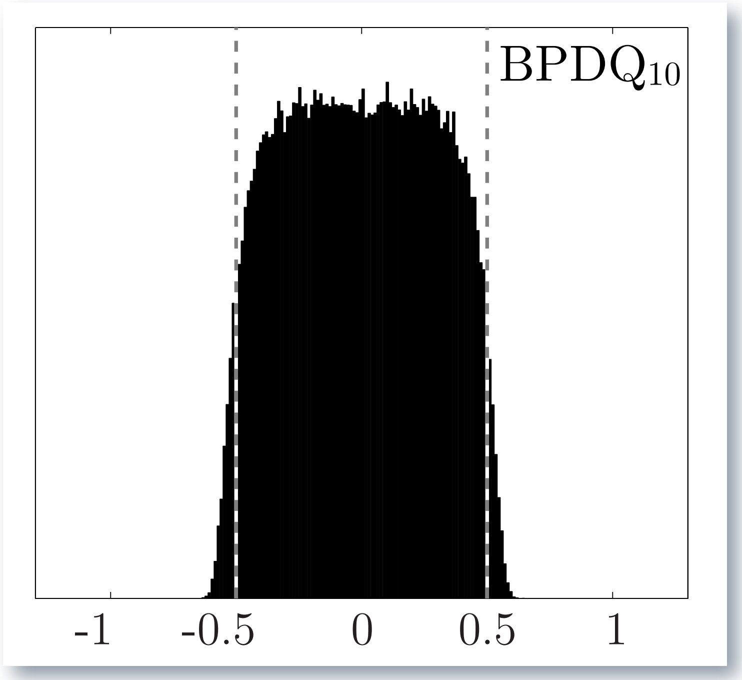
$$\alpha^{-1}(\boldsymbol{q} - \boldsymbol{\Phi}\hat{\boldsymbol{x}})_i$$





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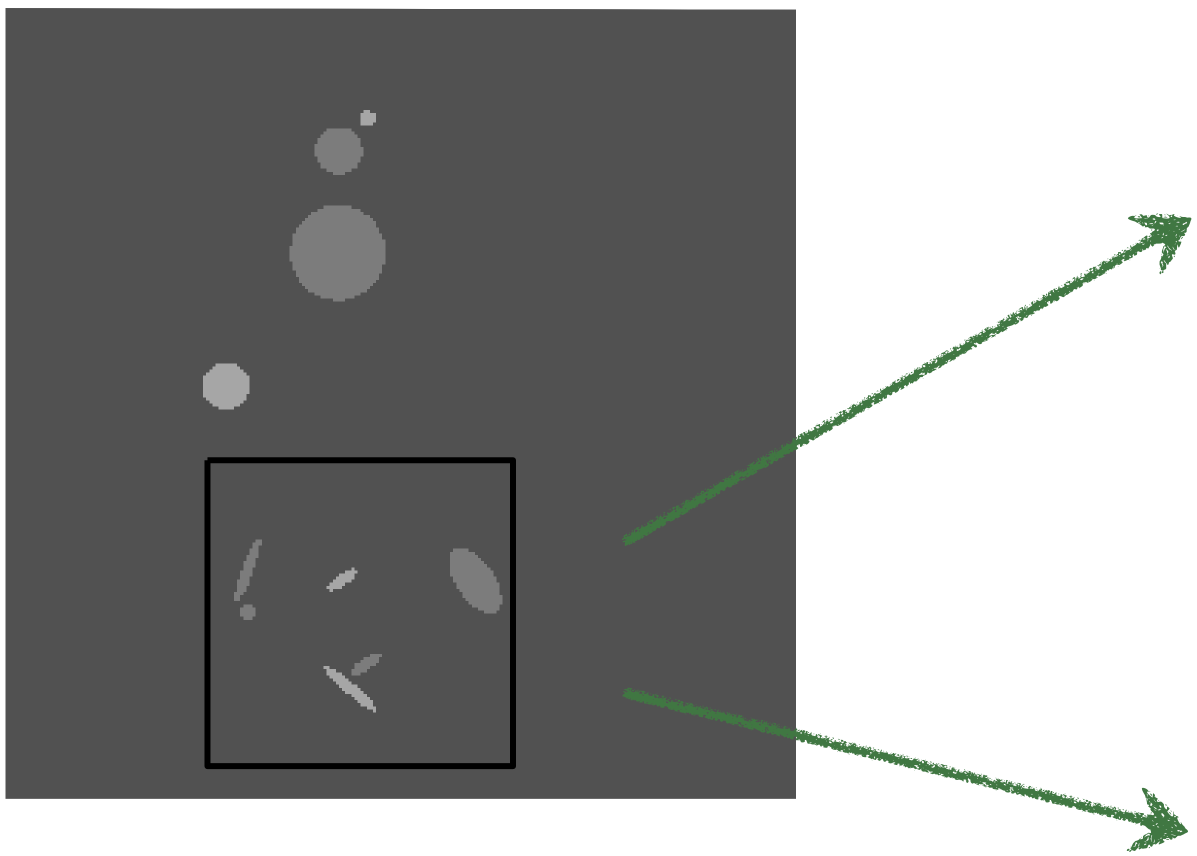
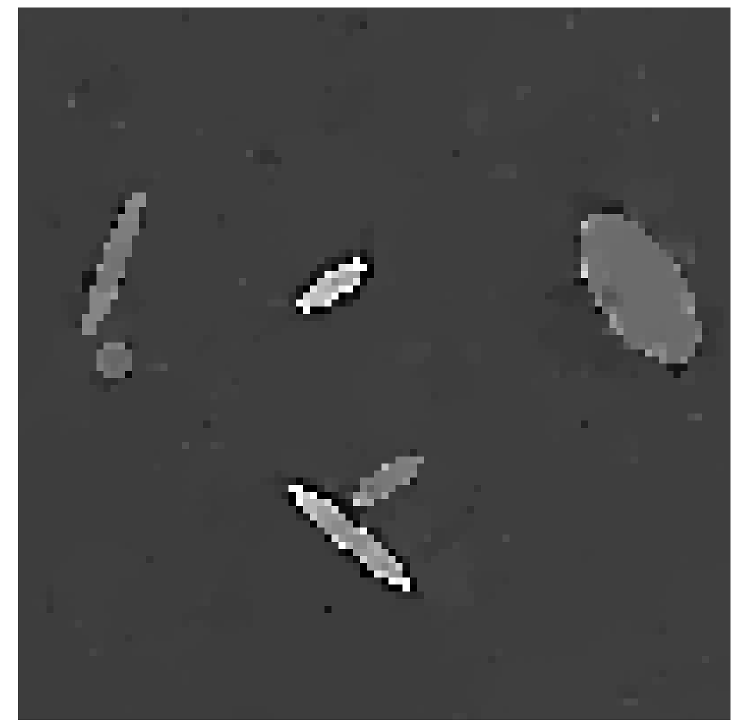
Dequantizing CS?

[LJ, Hammond, Fadili, 2009, 2011]

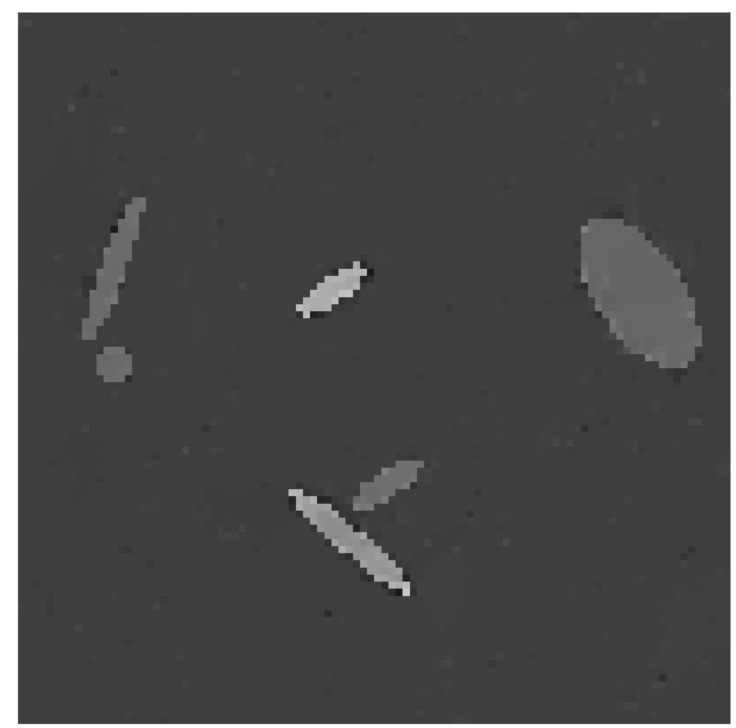
A bit outside the theory...





BPDN-TV

SNR: 8.96 dB



BPDQ_{10} -TV

SNR: 12.03 dB

* Synthetic Angiogram [Michael Lustig 07, SPARCO],
* $\mathbf{\Phi}$: **Random Fourier Ensemble**
* $N/M = 8$
* Decoder: $\Delta_{TV,p}(y, \epsilon_p)$
* Quantiz. bin width $= 50$ (i.e. 12 bins)

LJ, D. Hammond, J. Fadili "Dequantizing compressed sensing: When oversampling and non-gaussian constraints combine." Information Theory, IEEE Transactions on, 57(1), 559-571.

4. Sigma-Delta quantization in CS

Context:

- **Former attempts:** (see prev. slides)

  CS + uniform scalar quantization (or pulse code modulation - PCM)

For K-sparse signals: $\|\mathcal{Q}_\alpha[\boldsymbol{\Phi x}] - \boldsymbol{\Phi x}\|_2 \leqslant c\sqrt{M}\alpha \;\Rightarrow\; \|\boldsymbol{x}^* - \boldsymbol{x}\| \leqslant C\alpha$ (with RIP)

and for high λ, $\|\mathcal{Q}_\alpha[\boldsymbol{\Phi x}] - \boldsymbol{\Phi x}\|_p \leqslant cM^{1/p}\alpha \;\Rightarrow\; \|\boldsymbol{x}^* - \boldsymbol{x}\| \leqslant C\alpha/\sqrt{p+1}$ (with RIP$_p$)

- **No (real) improvement if M increases!**
- **Can we do better?**

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- **No (real) improvement if M increases!**
- **Can we do better?**

  Can we have $\|\boldsymbol{x}^* - \boldsymbol{x}\| \leqslant O(r^{-s}\alpha)$ for some $s > 0$?

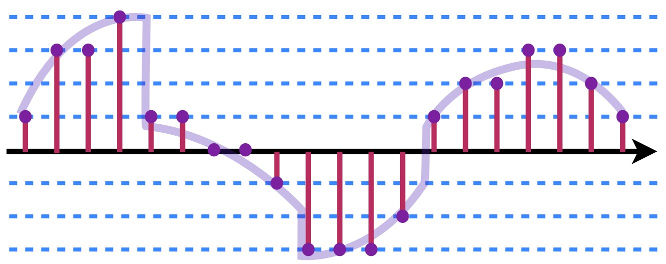
- **Staying with PCM,** $s \leqslant 1$ (Goyal-Vetterli-Thao lower bound)
- **Solution: replacing PCM by $\Sigma\Delta$ quantization!**

  [S. Güntürk, A. Powell, R. Saab, Ö. Yılmaz]

$\Sigma\Delta$ quantization (reminder)

- PCM: Signal sensing + unif. quantization (step α)

$$\boldsymbol{x} \in \mathbb{R}^K \quad \longrightarrow \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \in \mathbb{R}^M$$



$\boldsymbol{q} = \mathcal{Q}_{\rm PCM}[\boldsymbol{y}]$ with

$$q_k = \mathcal{Q}_{\rm PCM}[y_k] := \operatorname*{argmin}_{u \in \alpha\mathbb{Z}} |y_k - u|, \quad 1 \leqslant k \leqslant M$$

Let $\boldsymbol{A}^\#$, a left inverse of $\boldsymbol{A}$, *i.e.*, $\boldsymbol{A}^\#\boldsymbol{A} = \mathbf{Id}$.

Then, $\hat{\boldsymbol{x}} := \boldsymbol{A}^\#\boldsymbol{q} \Rightarrow \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = \|\boldsymbol{A}^\#\underbrace{(\boldsymbol{y} - \boldsymbol{q})}_{\text{quant. noise}}\|$

$\rightarrow$ Goal: $\displaystyle\operatorname*{minimize}_{\boldsymbol{A}^\#\boldsymbol{A} = \mathbf{Id}.} \|\boldsymbol{A}^\#(\boldsymbol{y} - \boldsymbol{q})\|$!

$\rightarrow$ Taking (Moore-Penrose) pseudo-inverse: $\boldsymbol{A}^\# = \boldsymbol{A}^\dagger = (\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*$
(or canonical dual of the frame $\boldsymbol{A}$)

- In CS, this could be used if signal support was known (see before)

$\Sigma\Delta$ quantization (reminder)

- $\Sigma\Delta \equiv$ noise shaping! Enjoy of:
  - freedom to pick $\boldsymbol{q} \in \alpha\mathbb{Z}^M$
  - freedom to take another left inverse $\boldsymbol{A}^{\#}$

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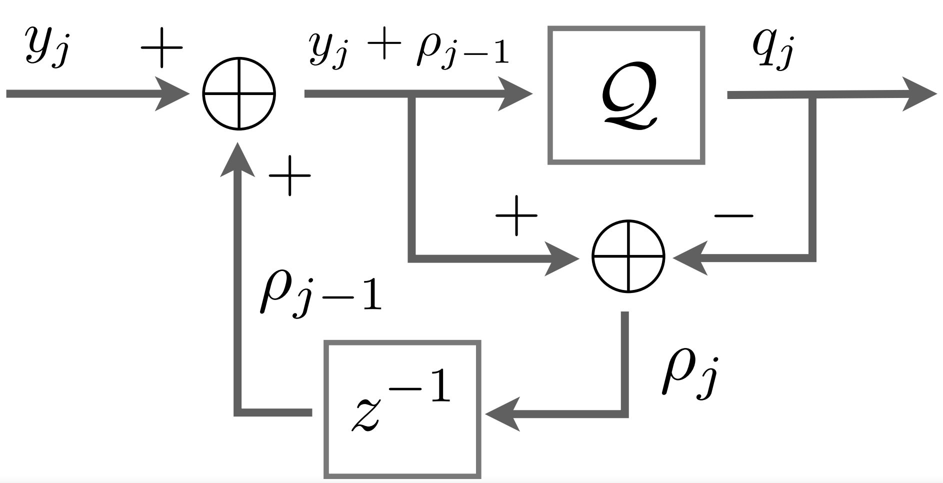
  - freedom to take another left inverse $\boldsymbol{A}^{\#}$
- 1st order $\Sigma\Delta$: (in 1-D) Quantizing the sequence $\{y_j : j \geqslant 0\}$

Use of state variables $\{\rho_j\}$ (1-step memory):

find q_j: $\quad q_j = \mathcal{Q}^{(1)}_{\Sigma\Delta}[y_j] := \operatorname{argmin}_{u \in \alpha\mathbb{Z}} |\rho_{j-1} + y_j - u|$ $\quad = \mathcal{Q}_{\mathrm{PCM}}[\rho_{j-1} + y_j]$

find ρ_j: $\quad (\Delta\rho)_j = \rho_j - \rho_{j-1} = y_j - q_j \quad$ (difference eq.)



with: $\quad |\rho_j| \leqslant \alpha$

$|y_j - q_j| \leqslant 2\alpha$

bigger than α but still $O(\alpha)$

$\Sigma\Delta$ quantization (reminder)

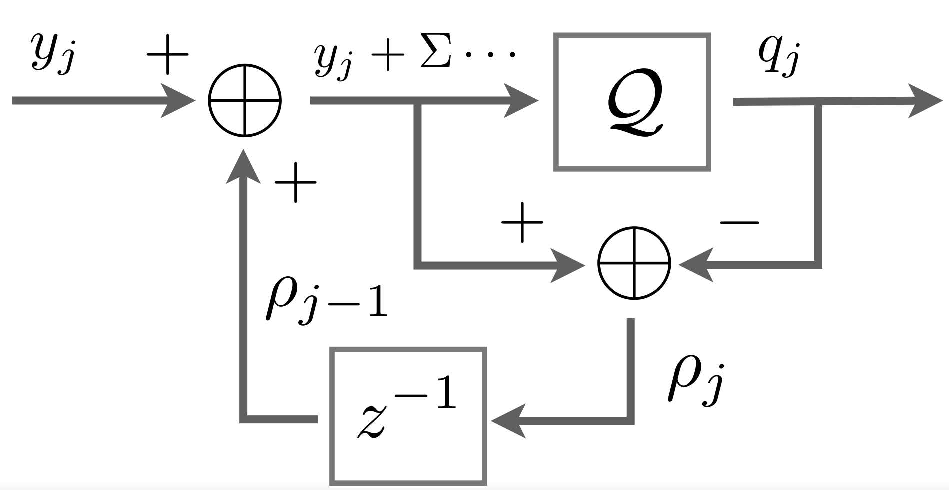
- $\Sigma\Delta \equiv$ noise shaping! Enjoy of:
  - freedom to pick $\boldsymbol{q} \in \alpha\mathbb{Z}^M$
  - freedom to take another left inverse $\boldsymbol{A}^{\#}$
- s^{th} order $\Sigma\Delta$: (in 1-D) Quantizing the sequence $\{y_j : j \geqslant 0\}$

  Use of state variables $\{\rho_j\}$ (s-step memory):

Remark:
PCM is 0^{th} order $\Sigma\Delta$

find q_j: $\quad q_j = \mathcal{Q}^{(s)}_{\Sigma\Delta}[y_j] := \operatorname{argmin}_{u\,\in\,\alpha\mathbb{Z}} \big|\sum_{i=1}^{s}(-1)^{i-1}\binom{s}{i}\rho_{j-n} + y_j - u\big|$

find ρ_j: $\quad (\Delta^s\rho)_j = y_j - q_j \quad$ (s^{th} order difference eq.)



with: $|\rho_j| \leqslant \alpha$

$|y_j - q_j| \leqslant 2^{s-1}\alpha$

bigger than α but still $O(\alpha)$

$\Sigma\Delta$ quantization (reminder)

- $\Sigma\Delta \equiv$ noise shaping! Enjoy of:
  - freedom to pick $\boldsymbol{q} \in \alpha\mathbb{Z}^M$
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- s^{th} order $\Sigma\Delta$:

Most important fact: $(\Delta^s \rho)_j = y_j - q_j \;\Leftrightarrow\; \boldsymbol{D}^s \boldsymbol{\rho} = \boldsymbol{y} - \boldsymbol{q}$

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$$\hat{\boldsymbol{x}} := \boldsymbol{A}^\# \boldsymbol{q} \Rightarrow \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = \|\boldsymbol{A}^\# \boldsymbol{D}^s (\boldsymbol{y} - \boldsymbol{q})\|$$

$\Sigma\Delta$ quantization (reminder)

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$$\hat{\boldsymbol{x}} := \boldsymbol{A}^{\#}\boldsymbol{q} \Rightarrow \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = \|\boldsymbol{A}^{\#}\boldsymbol{D}^s(\boldsymbol{y} - \boldsymbol{q})\|$$

minimize $\|\boldsymbol{A}^{\#}\boldsymbol{D}^s(\boldsymbol{y} - \boldsymbol{q})\|$!

~~Pseudo-inverse~~
$\cancel{\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*}$

Sobolev duals
$\boldsymbol{A}_{\mathrm{sob},s} = (\boldsymbol{D}^{-s}\boldsymbol{A})^{\dagger}\boldsymbol{D}^{-s}$

$\Sigma\Delta$ quantization (reminder)

- $\Sigma\Delta$ ≡ noise shaping! Enjoy of:
  - freedom to pick $\boldsymbol{q} \in \alpha\mathbb{Z}^M$
  - freedom to take another left inverse $\boldsymbol{A}^{\#}$
- s^{th} order $\Sigma\Delta$:

Most important fact: $(\Delta^s \rho)_j = y_j - q_j \;\Leftrightarrow\; \boldsymbol{D}^s \boldsymbol{\rho} = \boldsymbol{y} - \boldsymbol{q}$

$$\boxed{\hat{\boldsymbol{x}} = \boldsymbol{A}_{\text{sob},s}\boldsymbol{q}} \qquad \boldsymbol{A}_{\text{sob},s} = (\boldsymbol{D}^{-s}\boldsymbol{A})^{\dagger}\boldsymbol{D}^{-s}$$

Proposition *Let* $\boldsymbol{A} \in \mathbb{R}^{M\times K}$ *with* $A_{ij} \sim_{\text{iid}} \mathcal{N}(0,1)$.
For any $\kappa \in (0,1)$, *if* $r := M/K \geqslant c(\log M)^{1/(1-\kappa)}$, *then with* $Pr > 1 - e^{-c'M/r^{\kappa}}$,

$$\boxed{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \;\leqslant\; C_s\, r^{-\kappa(s-\frac{1}{2})}\alpha,}$$

for some $c, c', C_s > 0$.

proof: show that
$\sigma_{\min}(\boldsymbol{D}^{-s}\boldsymbol{A}) > C'_s\, r^{\kappa(s-\frac{1}{2})}\sqrt{M}$

$\Sigma\Delta$ quantization in CS

$$\boldsymbol{x} \in \Sigma_K \subset \mathbb{R}^N \;\longrightarrow\; \boldsymbol{y} = \boldsymbol{\Phi x} \in \mathbb{R}^M \;\longrightarrow\; \boldsymbol{q} = \mathcal{Q}^{(s)}_{\Sigma\Delta}[\boldsymbol{y}]$$

$$\|\boldsymbol{y} - \boldsymbol{q}\| \leqslant 2^{s-1}\,\alpha\,\sqrt{M}$$

Two-steps procedure:

remark: Recent dev. don't require these!

1. find the support T of $\boldsymbol{x}$: coarse approx. with BPDN
2. compute $\hat{\boldsymbol{x}} := (\boldsymbol{\Phi}_T)_{\mathrm{sob},s}\boldsymbol{q} = (\boldsymbol{D}^{-s}\boldsymbol{\Phi}_T)^{\dagger}\boldsymbol{D}^{-s}\boldsymbol{q}$

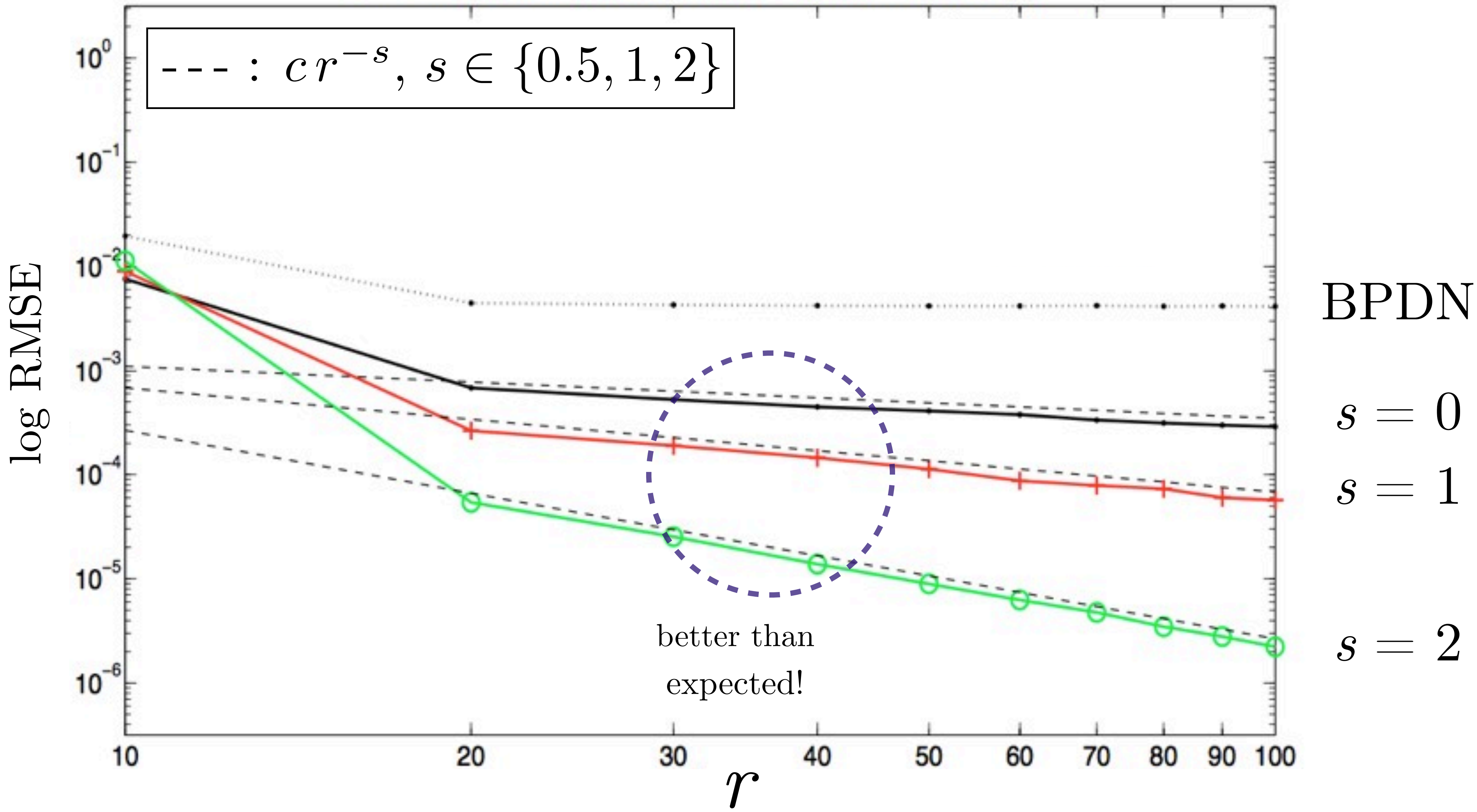
Proposition *Let* $\boldsymbol{\Phi} \in \mathbb{R}^{M\times K}$ *with* $\Phi_{ij} \sim_{\mathrm{iid}} \mathcal{N}(0,1)$. *Suppose* $\kappa \in (0,1)$ *and* $r := M/K \geqslant c(\log M)^{1/(1-\kappa)}$ *for* $c > 0$. *Then,* $\exists\, c', C, C_s > 0$ *such that, with* $Pr > 1 - e^{-c' M/r^{\kappa}}$, *for all* $\boldsymbol{x} \in \Sigma_K$ *s.t.* $\min_{i\in\mathrm{supp}\,\boldsymbol{x}} |x_i| \geqslant C\alpha$,

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \;\leqslant\; C_s\, r^{-\kappa(s-\frac{1}{2})}\alpha.$$

proof: Union bound on any K-column subset of $\boldsymbol{\Phi}$ + proba having good support.

ΣΔ quantization in CS (Simulations)

$M \in \{100, 200, \cdots, 1000\}$, $K = 10$ and 1000 trials ($x_i \in \{0, \pm 1/\sqrt{K}\}$, $\|\boldsymbol{x}\| \simeq 1, \alpha = 10^{-2}$)



Güntürk, C. S., Lammers, M., Powell, A. M., Saab, R., & Yılmaz, Ö. (2013). **Sobolev duals for random frames and ΣΔ quantization of compressed sensing measurements**. Foundations of Computational Mathematics, 13(1), 1-36.

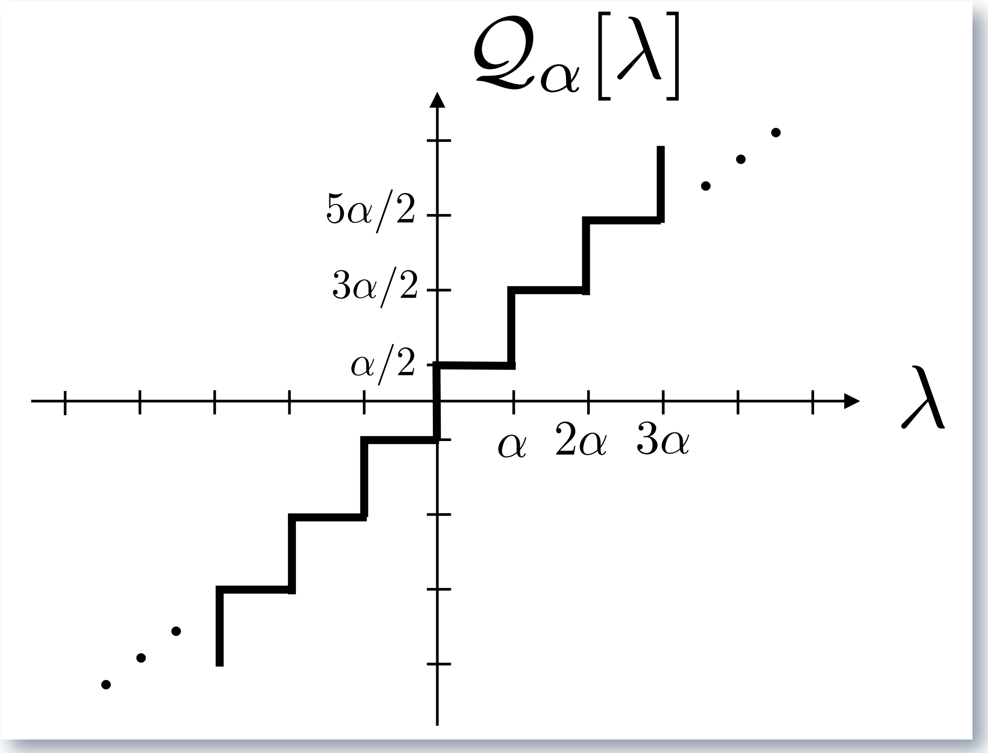
5. To saturate or not? And how much?

Saturation phenomenon:

Uniform quantization:

- α quantization interval
- error per measurement bounded:

$$|\lambda - \mathcal{Q}_\alpha[\lambda]| \leqslant \alpha/2$$



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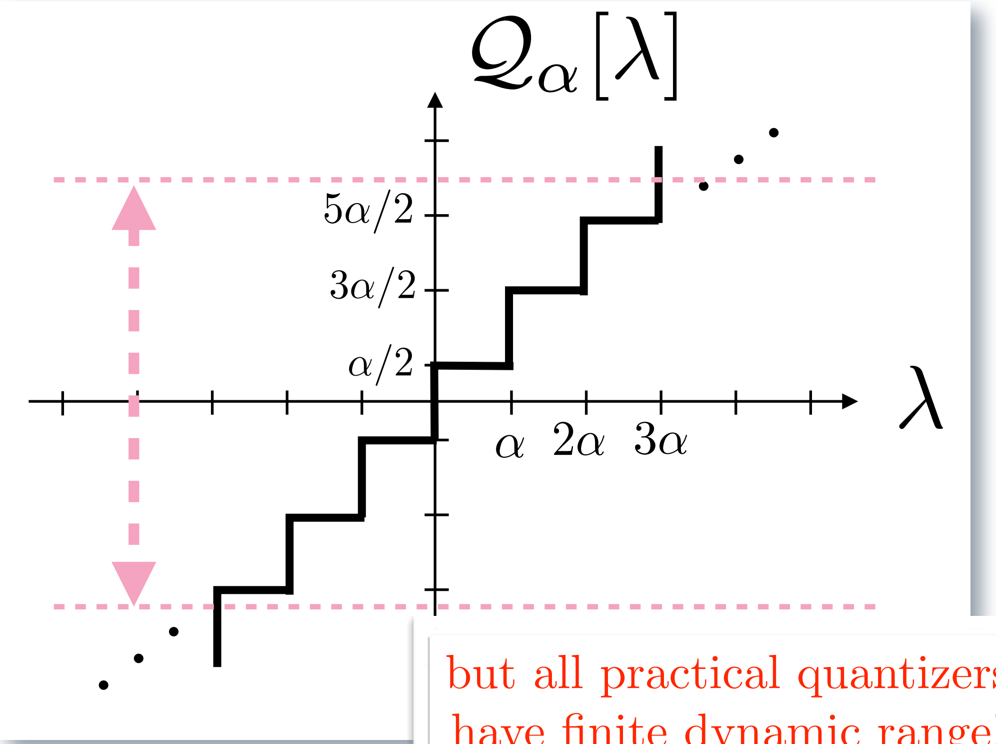

but all practical quantizers have finite dynamic range!

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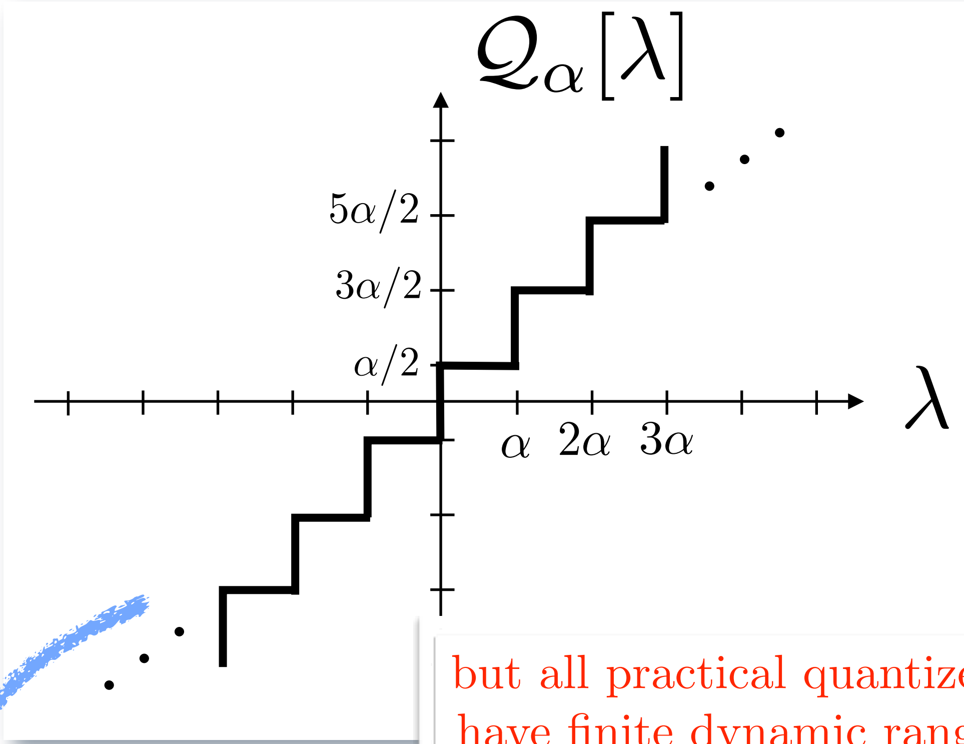
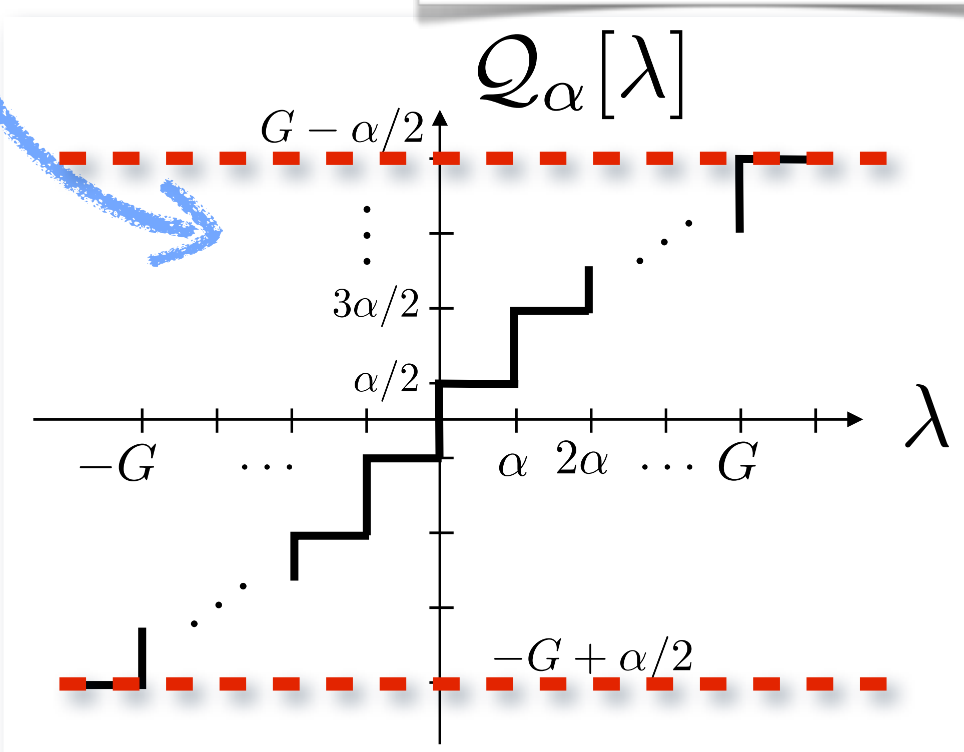


but all practical quantizers have finite dynamic range!

Finite Dynamic Range Quantization:

- G "saturation level"
- B bit rate (bits per measurement)
- quantization interval is $\alpha = 2^{-B+1}G$
- measurements above G saturate
- saturation error is *unbounded*

CS guarantees are for bounded errors only!

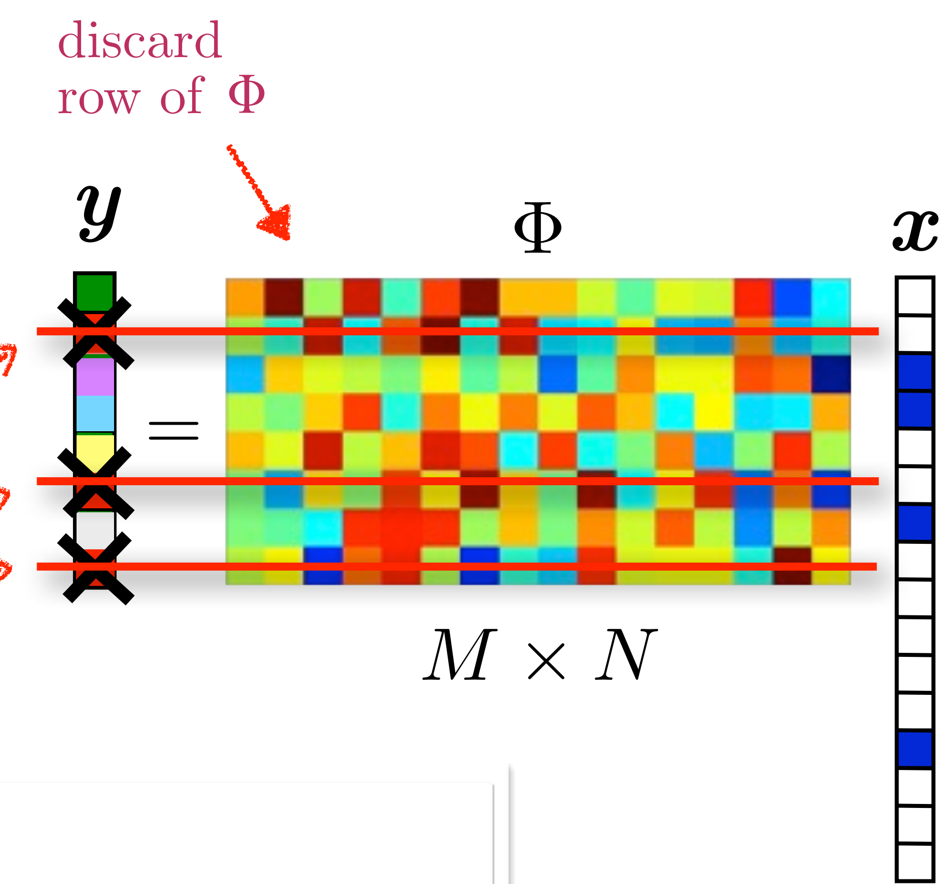


Democracy in Action

[Laska, Boufounos, Davenport, Baraniuk 12]

(i) Saturation Rejection:

Simply discard saturated measurements and corresponding rows of Φ



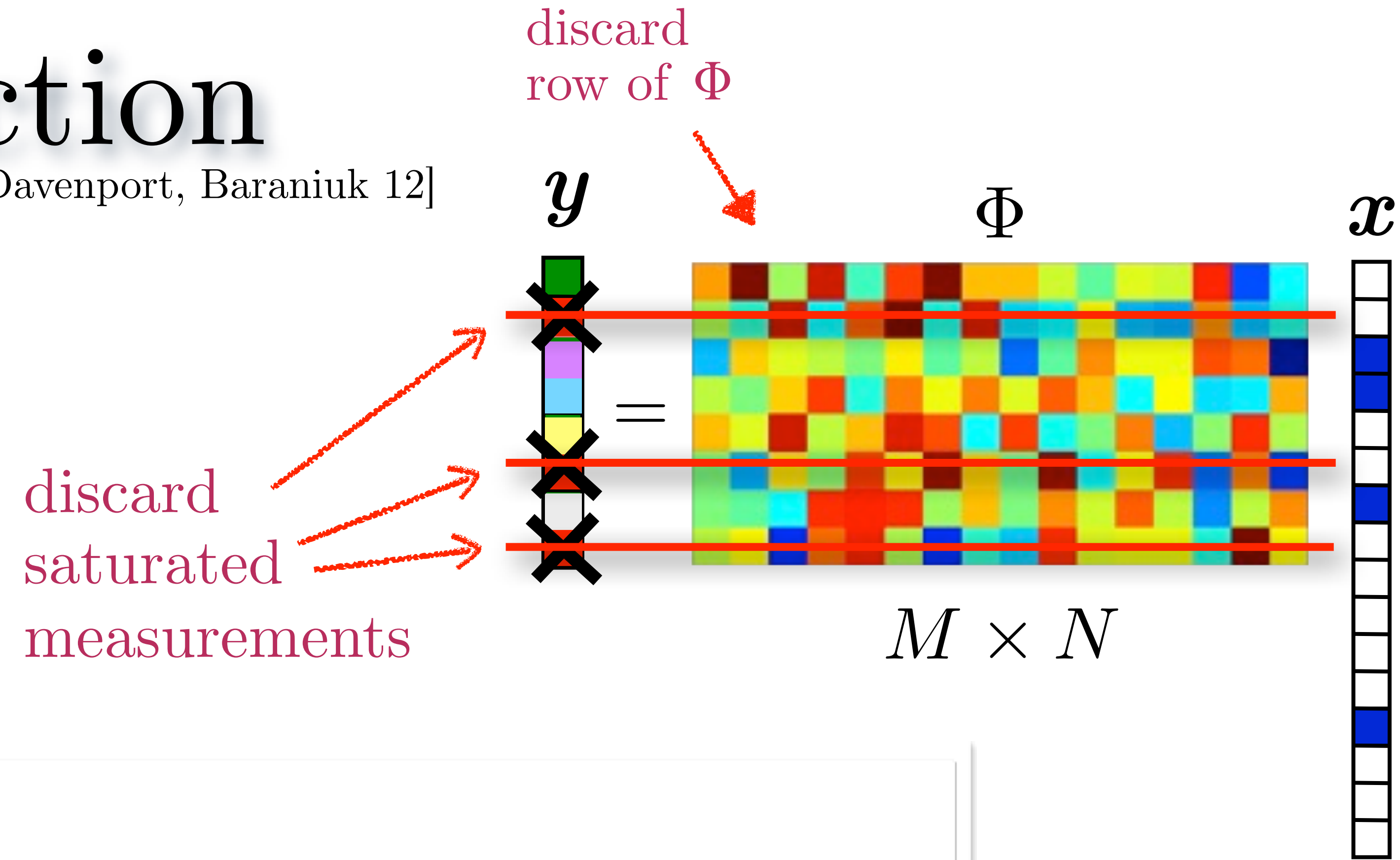
"democratic measurements"

each measurement has roughly same amount of information

RIP holds on row subsets of Φ

Democracy in Action

[Laska, Boufounos, Davenport, Baraniuk 12]



(i) Saturation Rejection:

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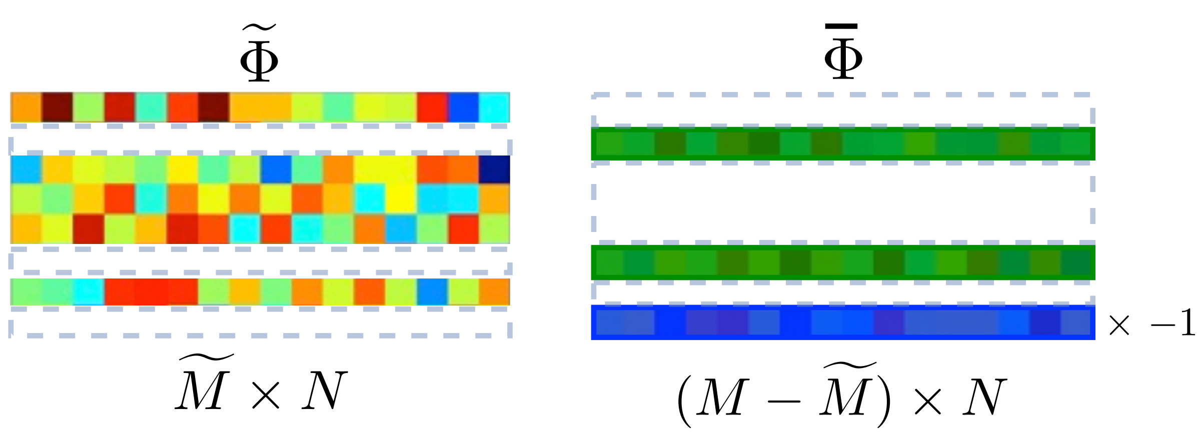
"democratic measurements"

each measurement has roughly same amount of information

RIP holds on row subsets of Φ

(ii) Saturation Consistency:

Include saturated measurements as inequality constraint



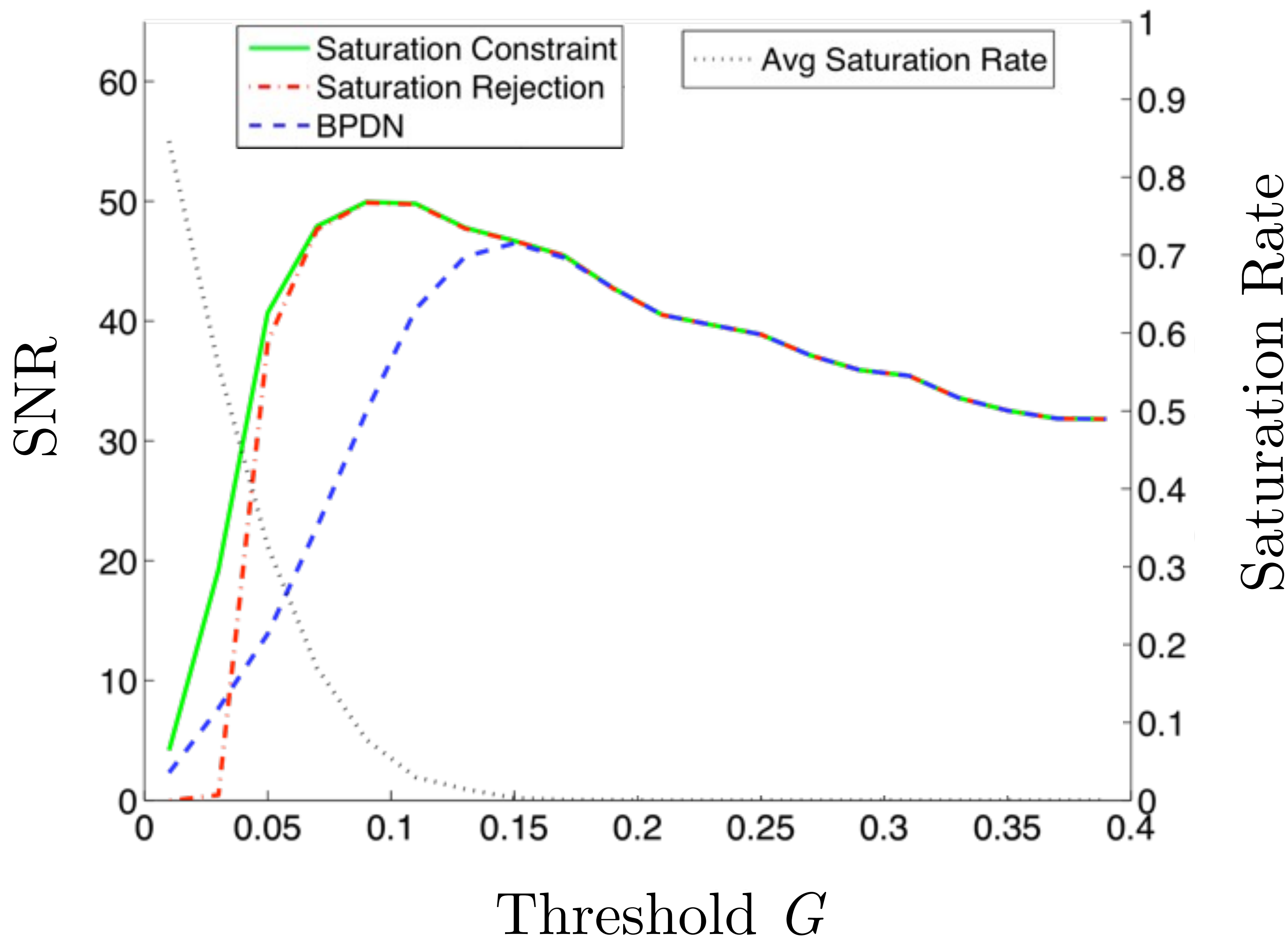
$$\hat{\mathbf{x}} = \mathrm{argmin}_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \|\widetilde{\Phi}\mathbf{x} - \widetilde{\mathbf{y}}\|_2 < \epsilon$$

Measurement error term (quantization)

$$\text{and} \quad \bar{\Phi}\mathbf{x} \geq G \cdot \mathbf{1}$$

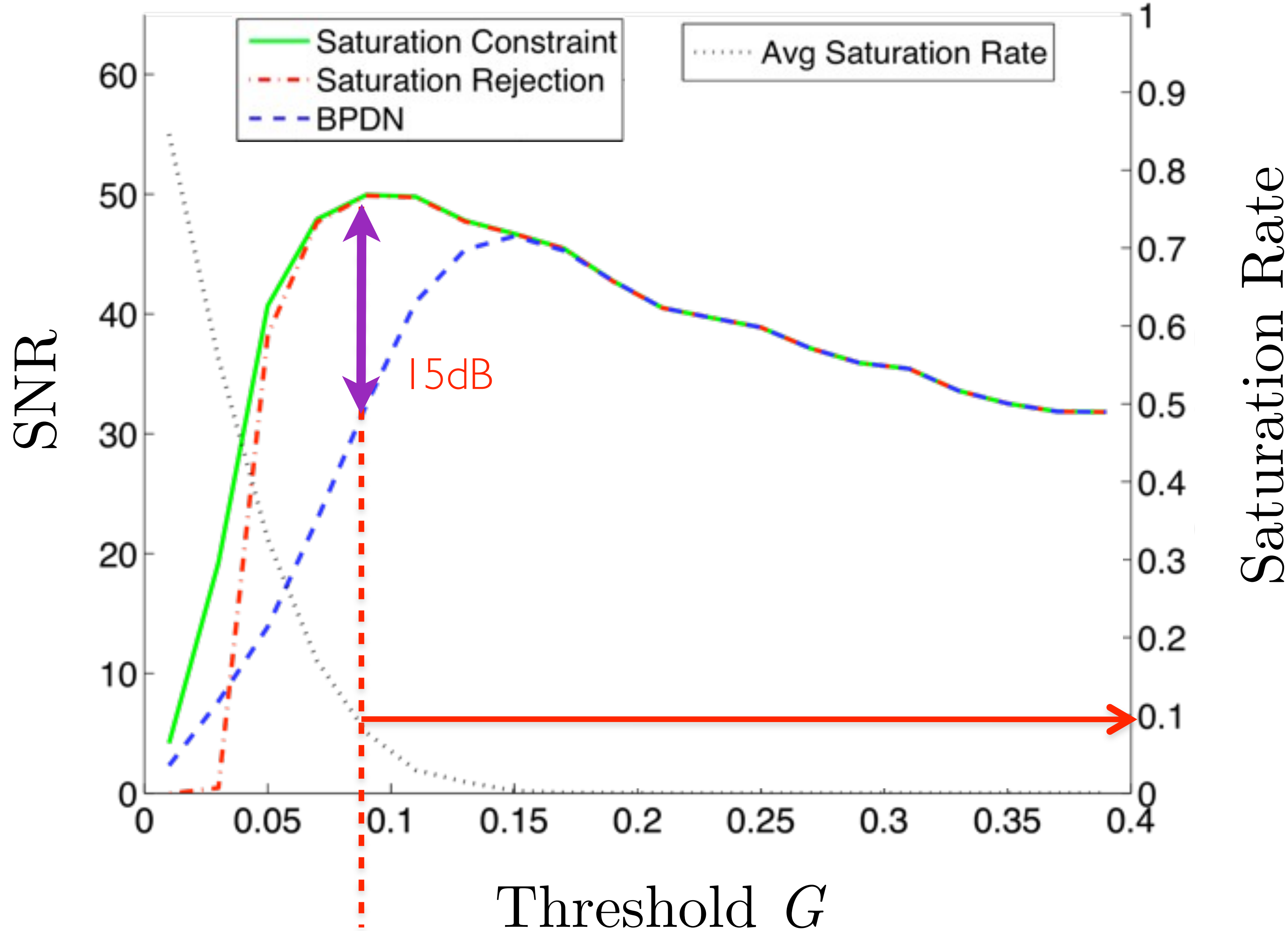
Saturation consistency constraint

Experimental Results



J.N. Laska, P.T. Boufounos, M.A. Davenport, R.G.Baraniuk, "Democracy in action: Quantization, saturation, and compressive sensing". *Applied and Computational Harmonic Analysis*, *31*(3), 429-443. (2011)

Experimental Results

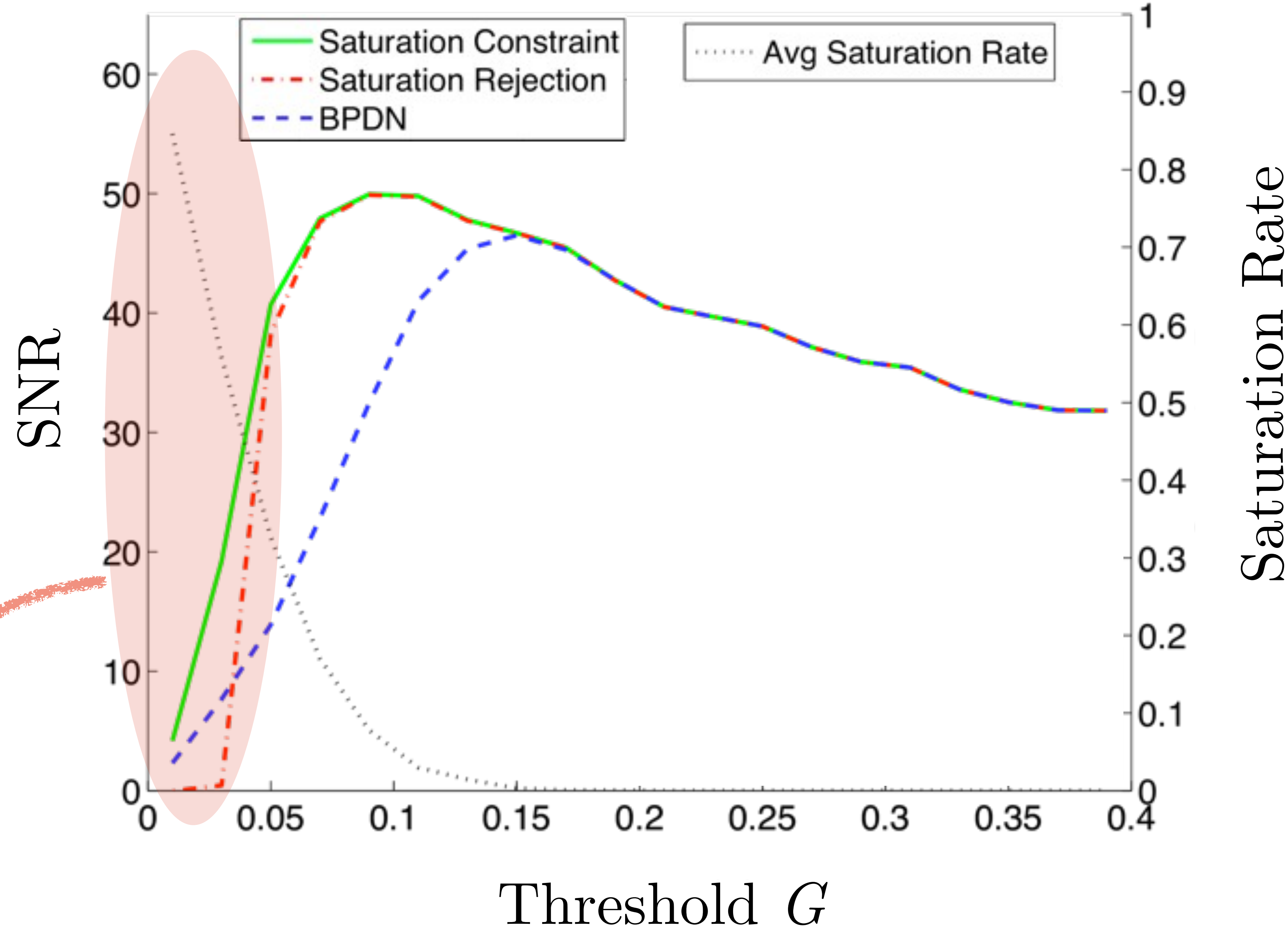


Note: optimal performance requires 10% saturation

J.N. Laska, P.T. Boufounos, M.A. Davenport, R.G.Baraniuk, "Democracy in action: Quantization, saturation, and compressive sensing". *Applied and Computational Harmonic Analysis*, *31*(3), 429-443. (2011)

Experimental Results

The "saturation gap"



- Majority of measurements saturate
- Recovery fails

J.N. Laska, P.T. Boufounos, M.A. Davenport, R.G.Baraniuk, "Democracy in action: Quantization, saturation, and compressive sensing". *Applied and Computational Harmonic Analysis*, *31*(3), 429-443. (2011)

Experimental Results

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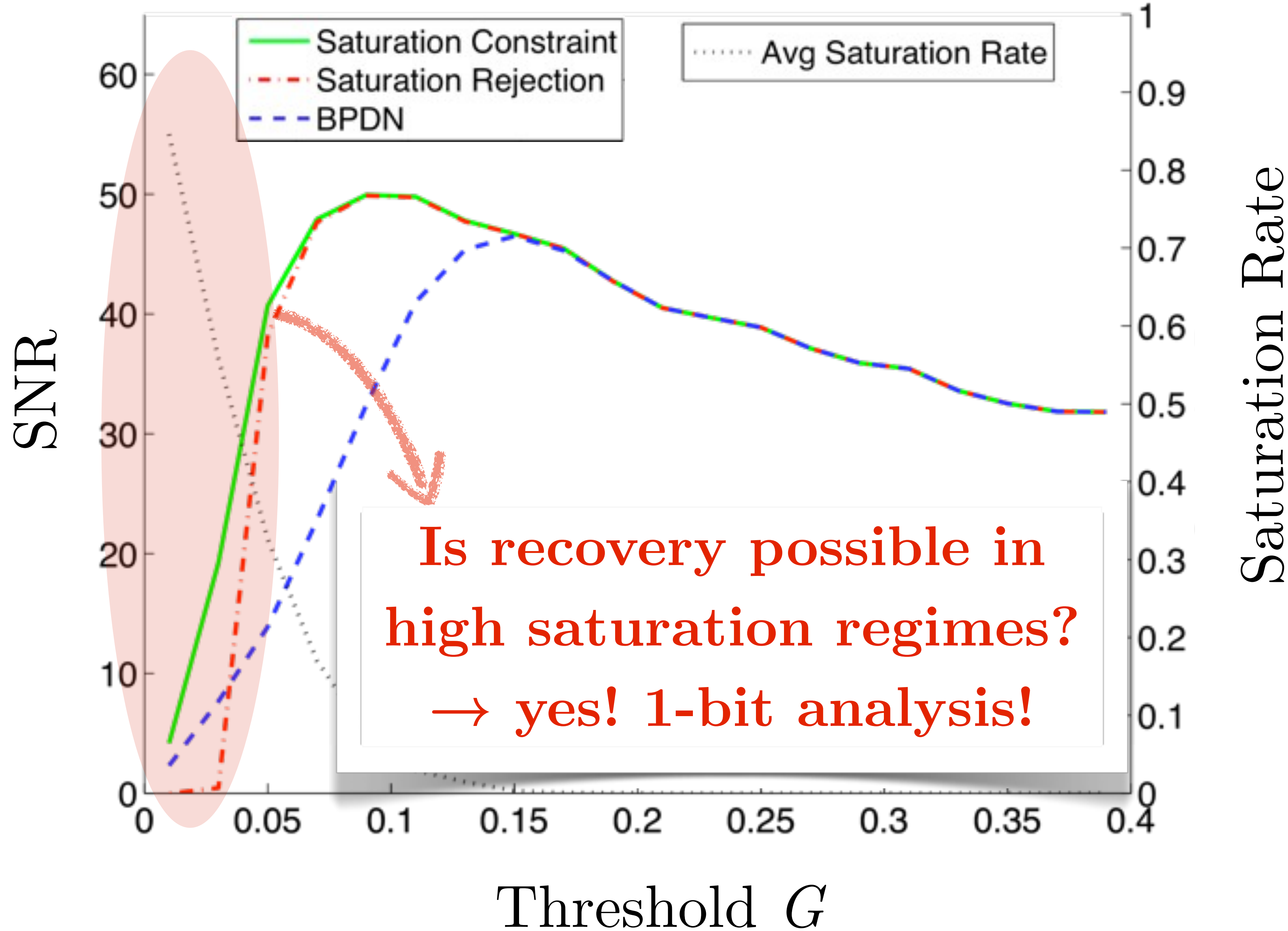

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Further Reading

- V. K Goyal, M. Vetterli, N. T. Thao, "Quantized Overcomplete Expansions in RN: Analysis, Synthesis, and Algorithms", *IEEE Trans. Info. Theory*, 44(1), 1998

- P. T. Boufounos and R. G. Baraniuk, "Quantization of sparse representations," *Rice University ECE Department Technical Report 0701*. Summary appears in *Proc. Data Compression Conference (DCC)*, Snowbird, UT, March 27-29, 2007

- W. Dai, H. V. Pham, and O. Milenkovic, "Quantized Compressive Sensing", preprint, 2009

- L. Jacques, D. Hammond, J. Fadili "Dequantizing compressed sensing: When oversampling and non-gaussian constraints combine." *IEEE Transactions on Information Theory*, 57(1), 559-571, 2011

- J.N. Laska, P.T. Boufounos, M.A. Davenport, R.G.Baraniuk, "Democracy in action: Quantization, saturation, and compressive sensing". *Applied and Computational Harmonic Analysis*, 31(3), 429-443, 2011

- L. Jacques, D. Hammond, J. Fadili, "Stabilizing Nonuniformly Quantized Compressed Sensing with Scalar Companders", arXiv:1206.6003, 2012

- Güntürk, C. S., Lammers, M., Powell, A. M., Saab, R., & Yılmaz, Ö. "Sobolev duals for random frames and $\Sigma\Delta$ quantization of compressed sensing measurements". *Foundations of Computational Mathematics*, 13(1), 1-36, 2013

- A. M. Powell, J.T. Whitehouse, "Error bounds for consistent reconstruction: random polytopes and coverage processes", arXiv:1405.7094, 2013

- L Jacques, "Error Decay of (almost) Consistent Signal Estimations from Quantized Random Gaussian Projections", arXiv:1406.0022, 2014

- P. T. Boufounos, L. Jacques, F. Krahmer, R. Saab, "Quantization and Compressive Sensing", arXiv:1405.1194

Part 2

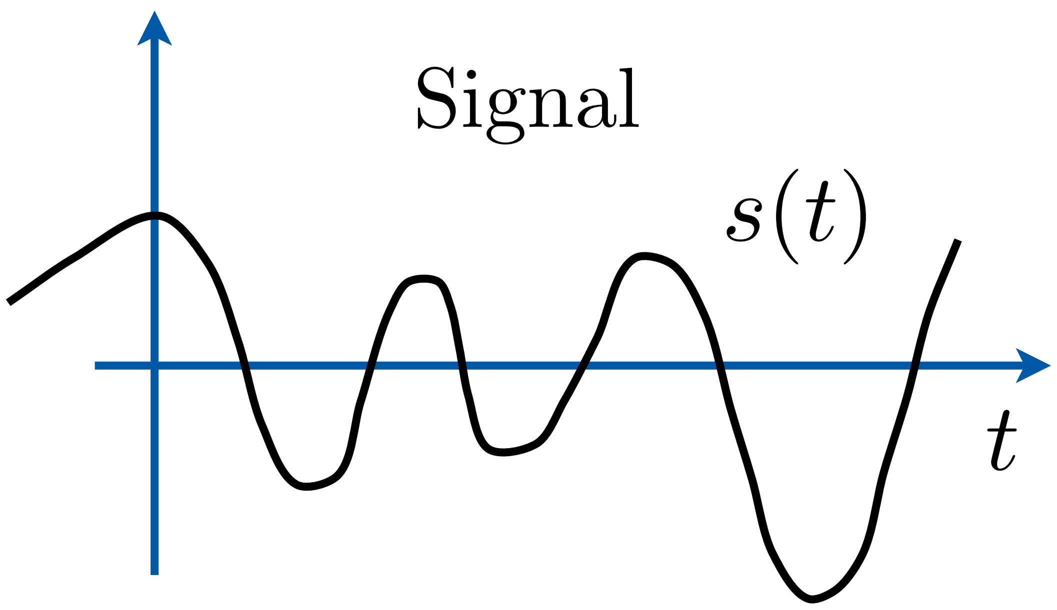
Extreme quantization: 1-bit compressed sensing

Outline:

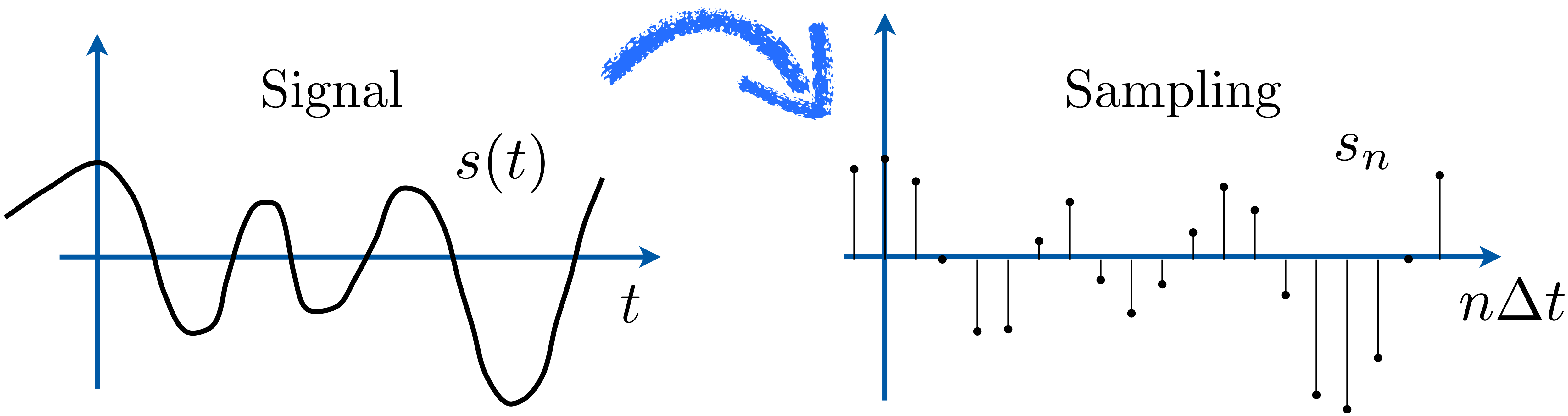
1. Context
2. Theoretical performance limits
3. Stable embeddings: angles are preserved
4. Generalized Embeddings
5. 1-bit CS Reconstructions?
6. Playing with thresholds in 1-bit CS

1. Context

Central question: 1-bit sampling?



Central question: 1-bit sampling?



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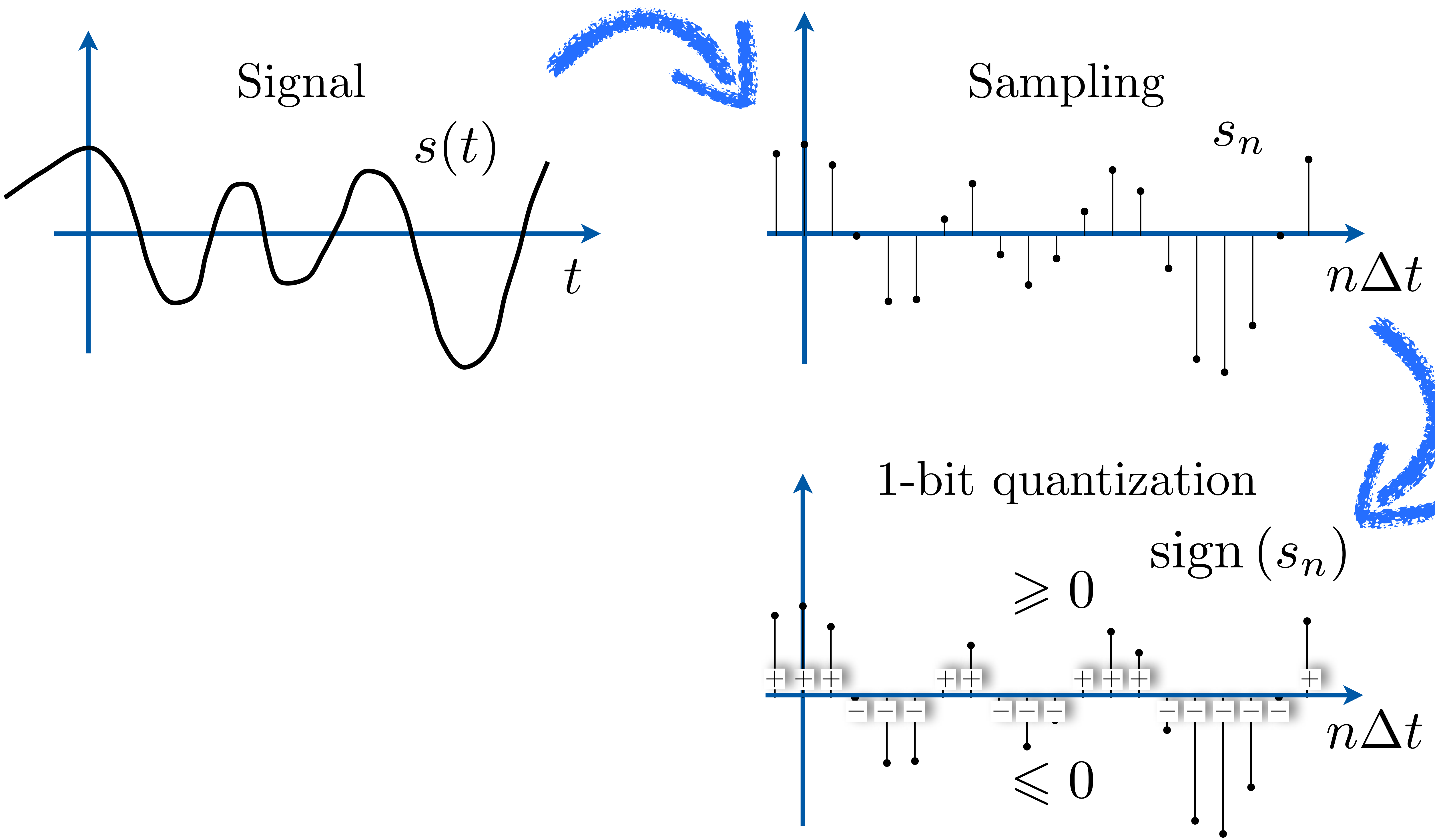
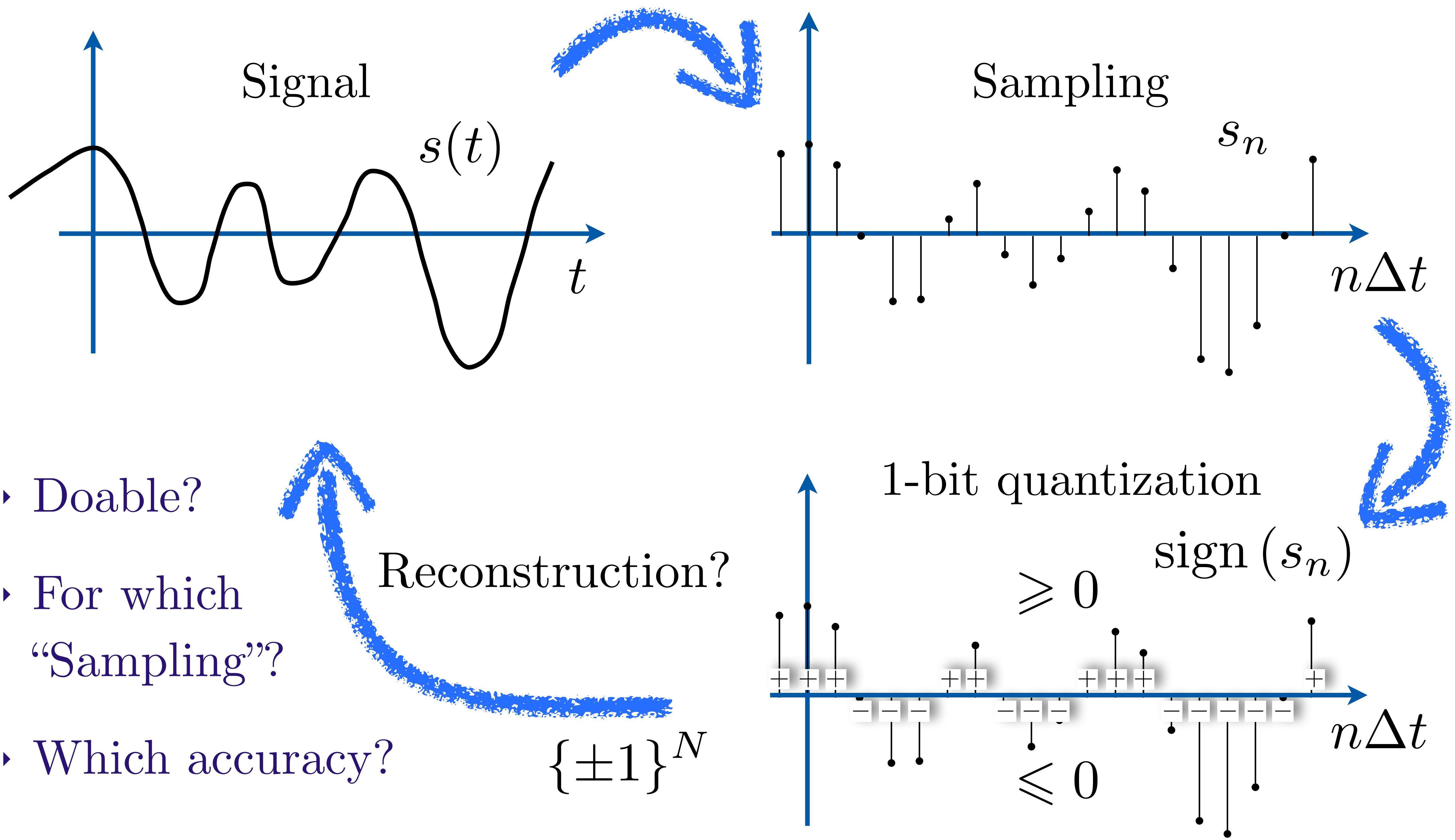
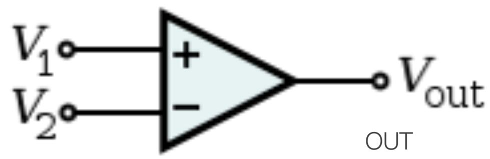


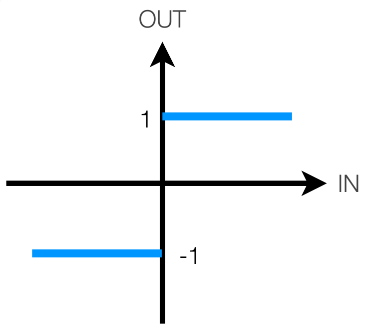
Central question: 1-bit sampling?

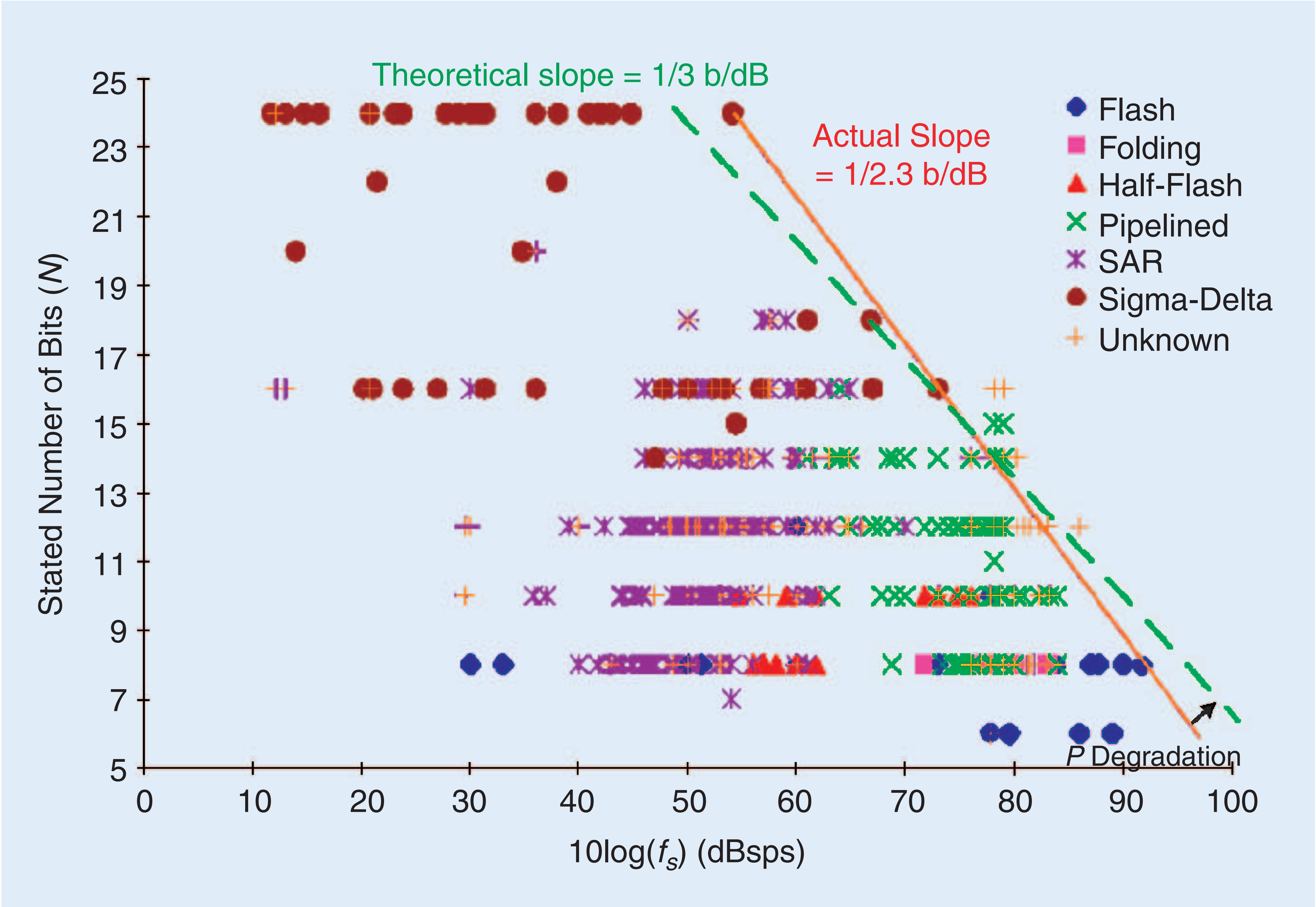


- Doable?
- For which "Sampling"?
- Which accuracy?

Why 1-bit? Very Fast Quantizers!







[FIG1] Stated number of bits versus sampling rate.

[From "Analog-to-digital converters" B. Le, T.W. Rondeau, J.H. Reed, and C.W.Bostian, IEEE Sig. Proc. Magazine, Nov 2005]

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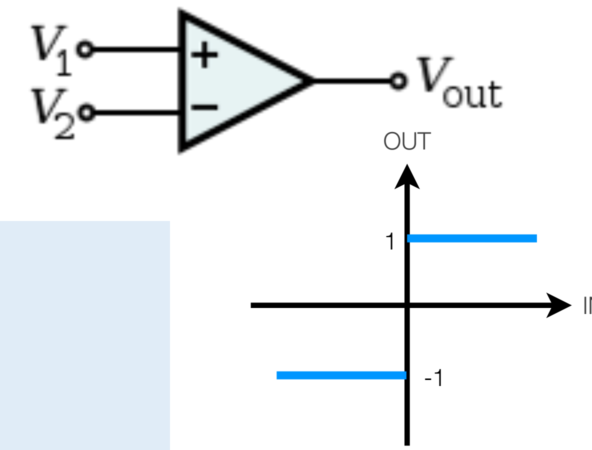
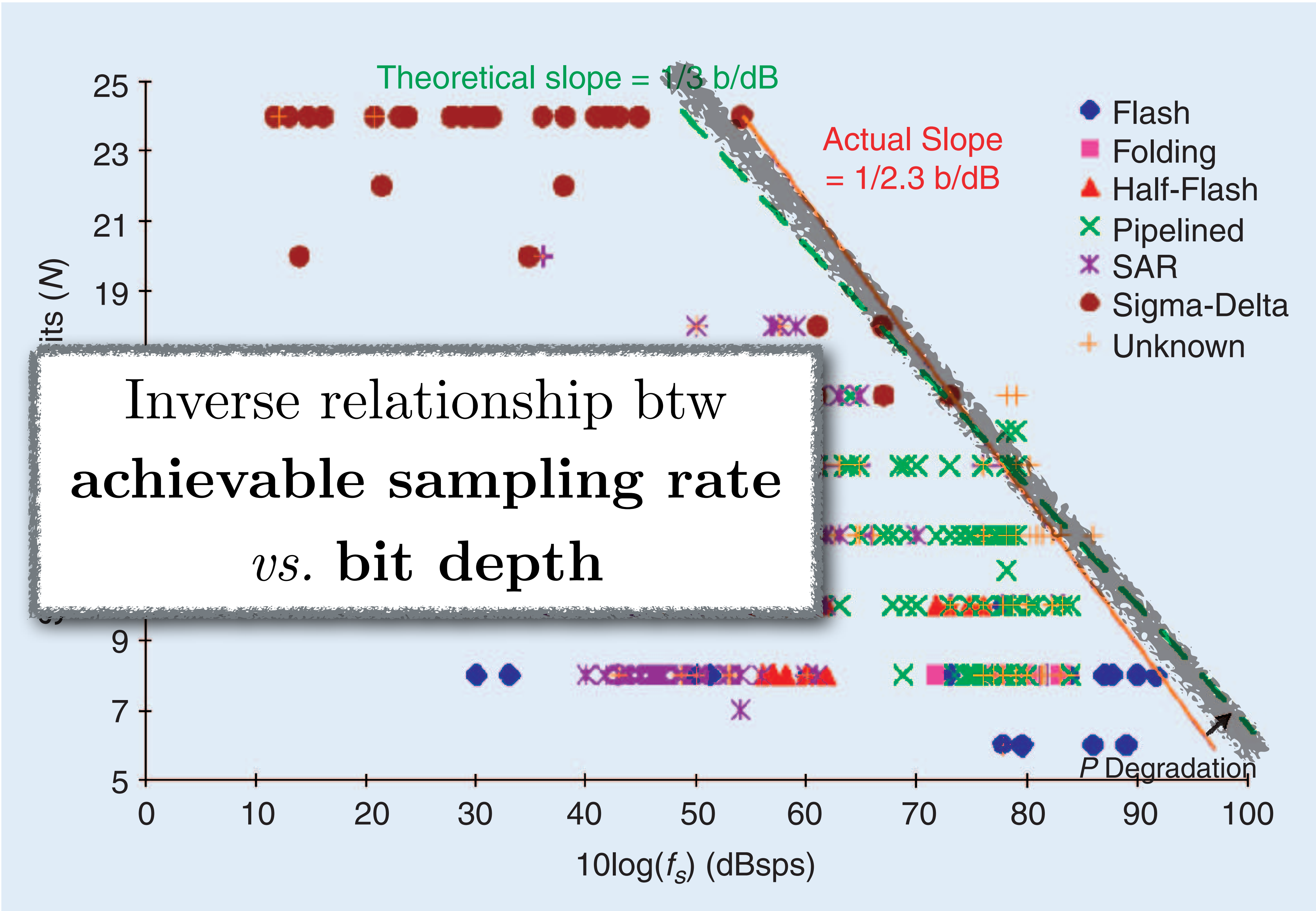


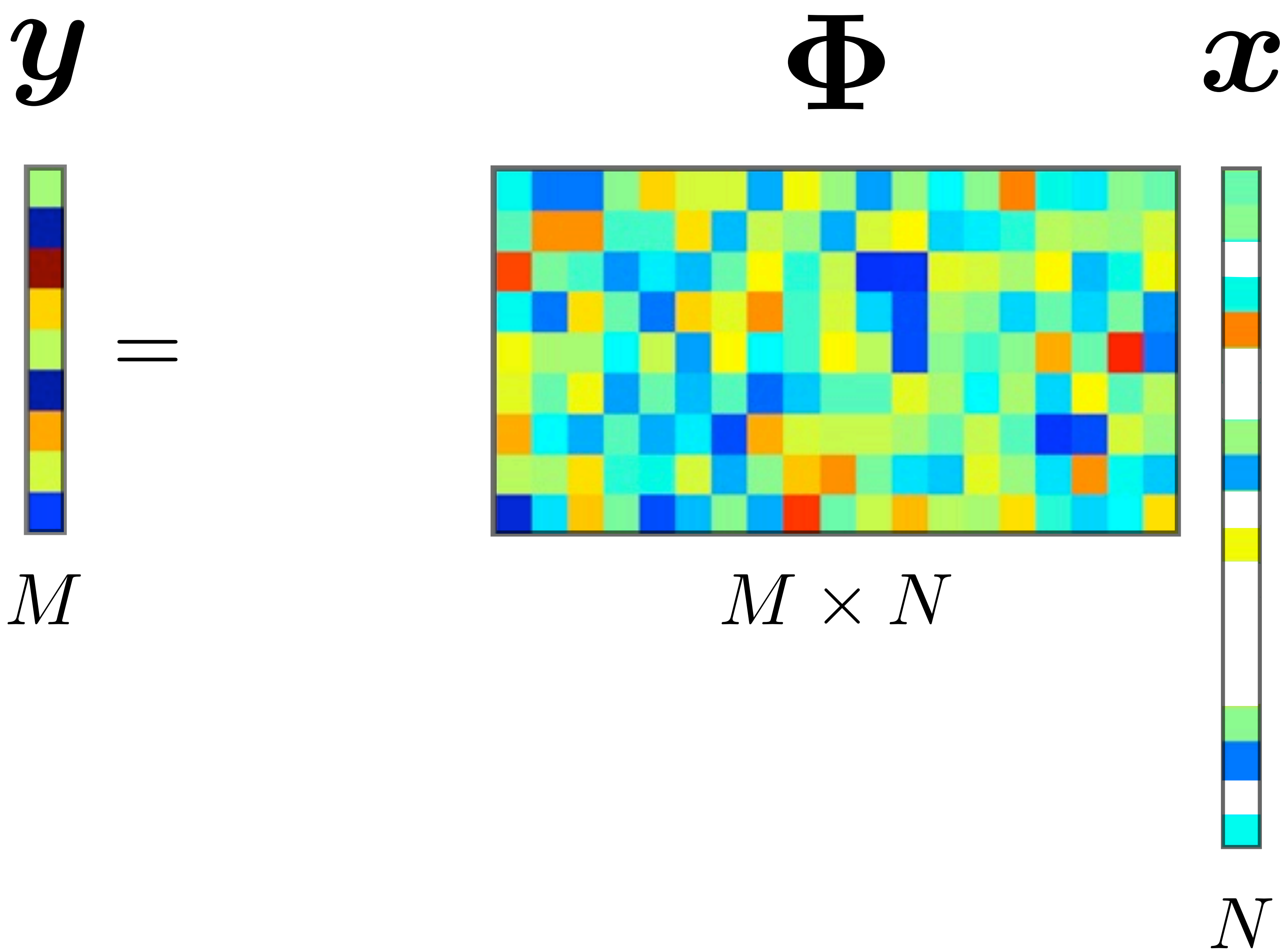


Inverse relationship btw **achievable sampling rate** *vs.* **bit depth**

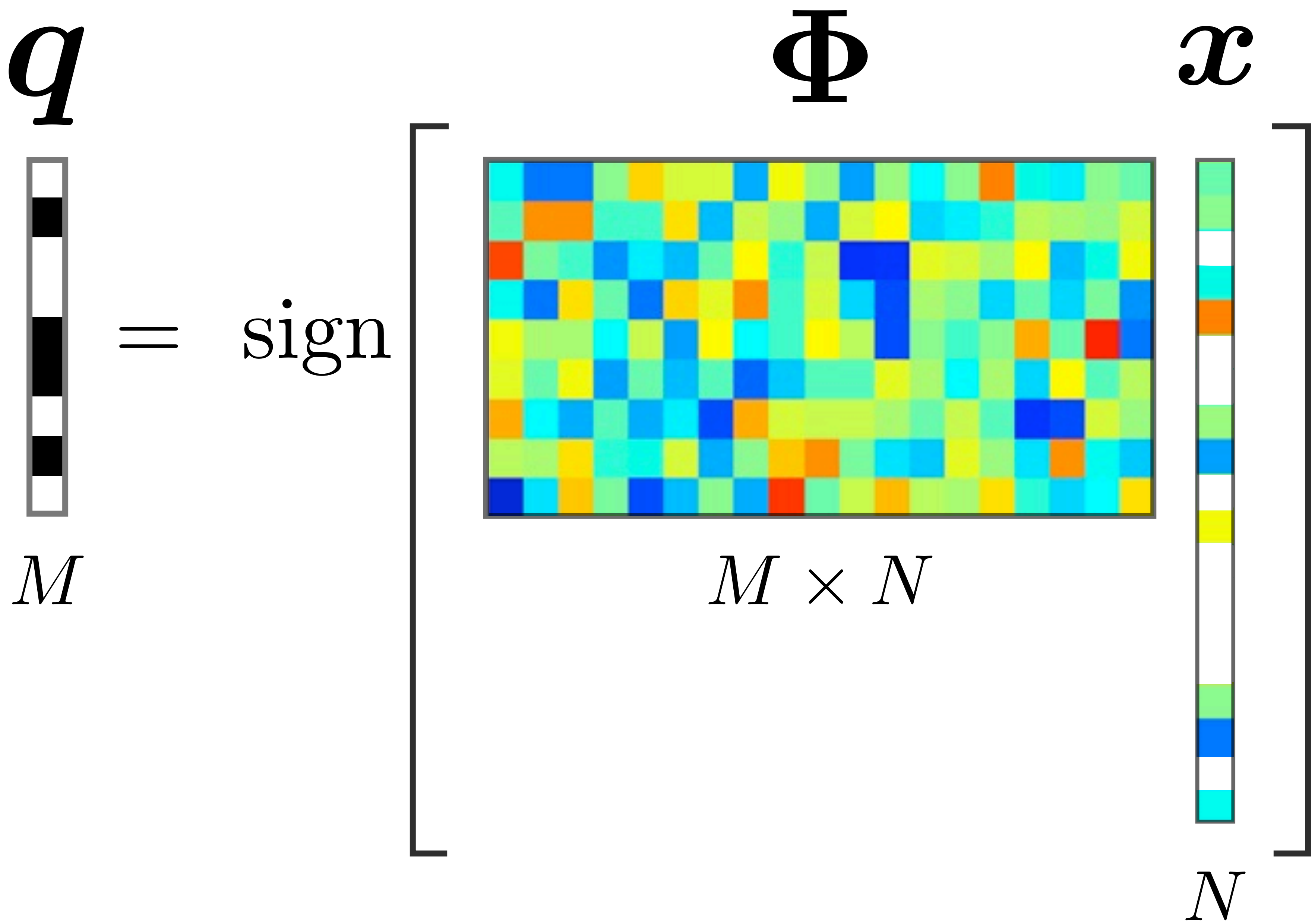
[FIG1] Stated number of bits versus sampling rate.

[From "Analog-to-digital converters" B. Le, T.W. Rondeau, J.H. Reed, and C.W.Bostian, IEEE Sig. Proc. Magazine, Nov 2005]

Compressed Sensing

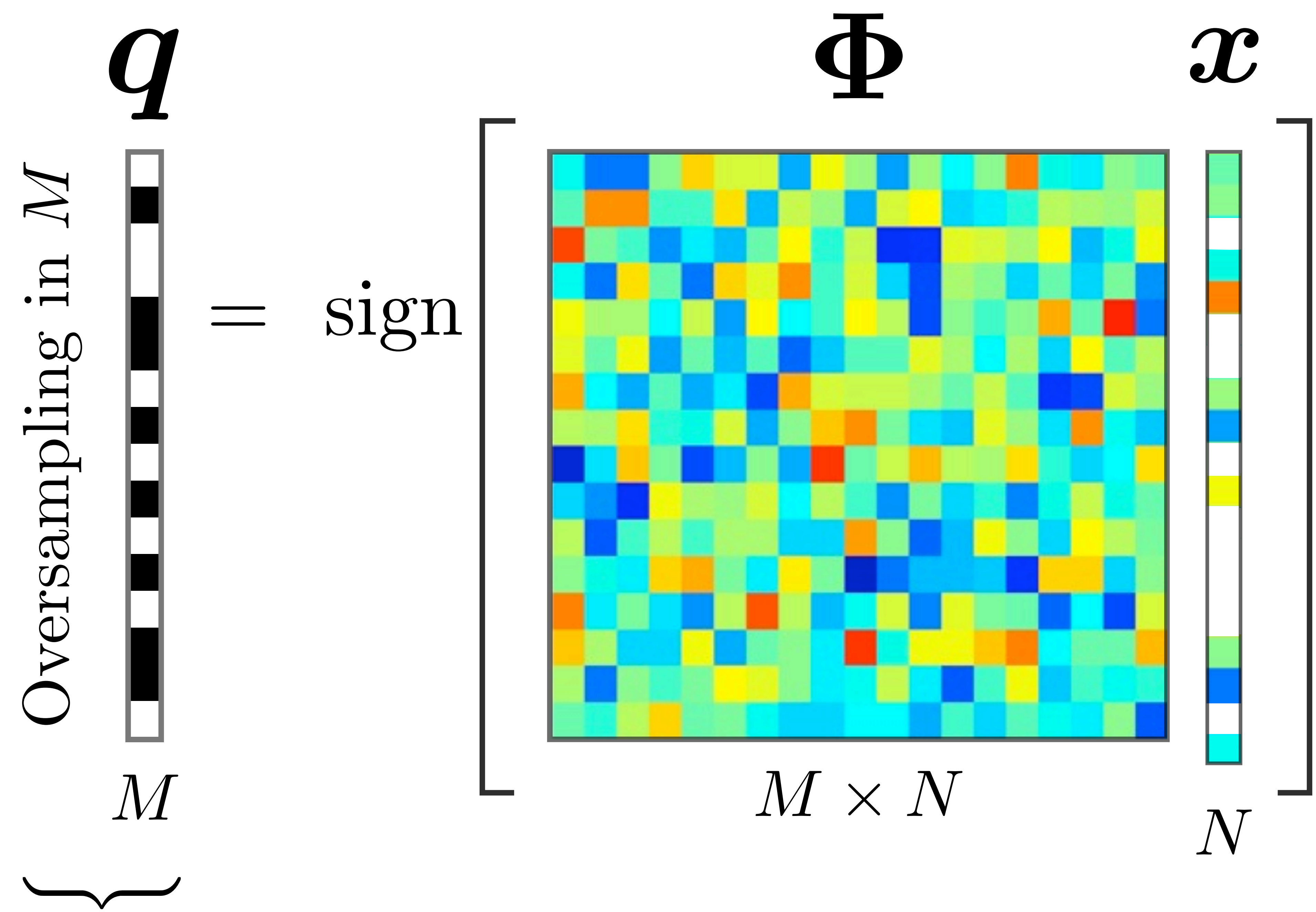


1-bit Compressed Sensing



with: $\mathrm{sign}\, t = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t \leqslant 0 \end{cases}$ component-wise

1-bit Compressed Sensing



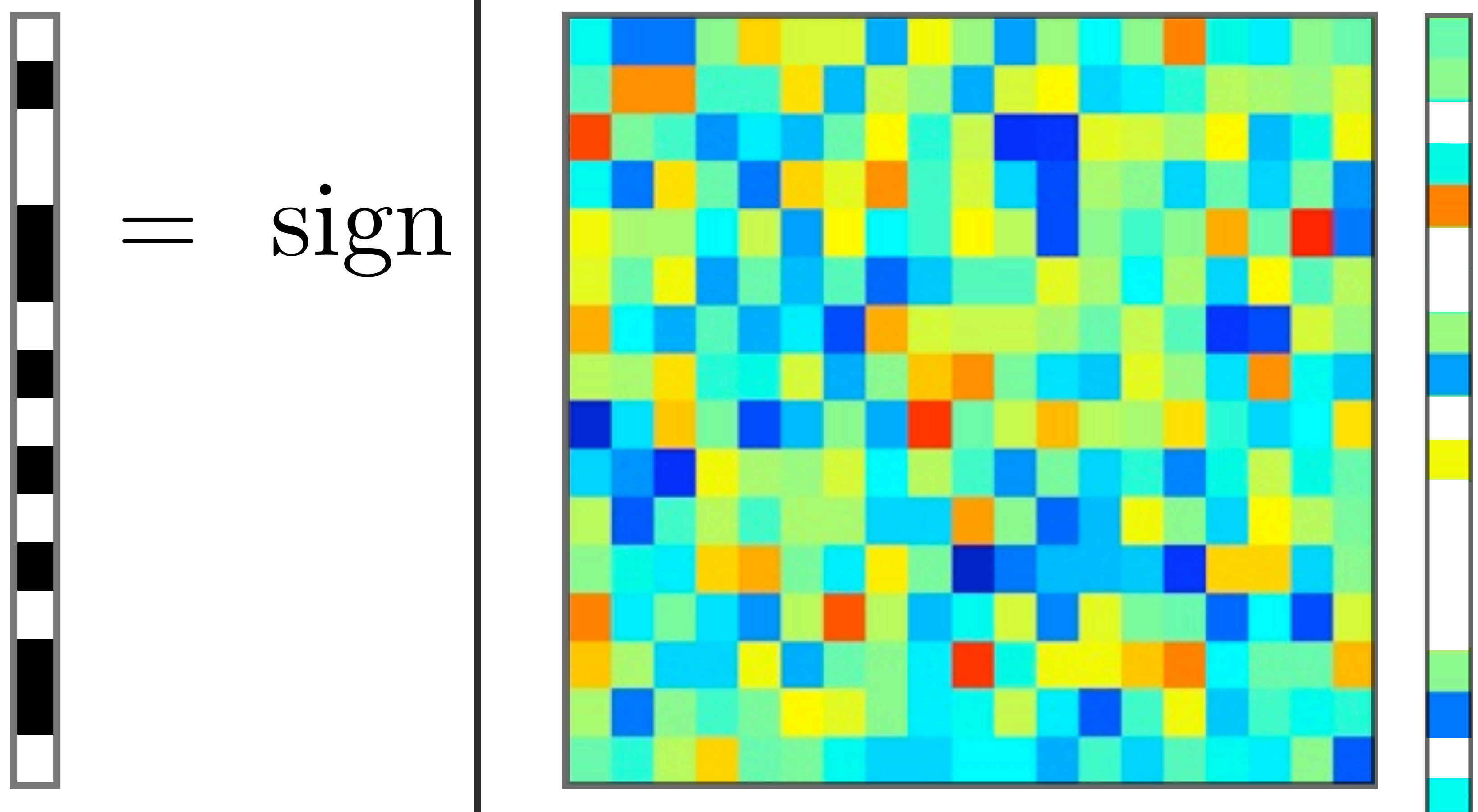
M-bits! But, which information inside $\boldsymbol{q}$?

1-bit Computational ~~Compressed~~ Sensing

bits matter!

$$\boldsymbol{q} = \mathrm{sign}\big[\,\boldsymbol{\Phi}\,\boldsymbol{x}\,\big]$$

Oversampling in M



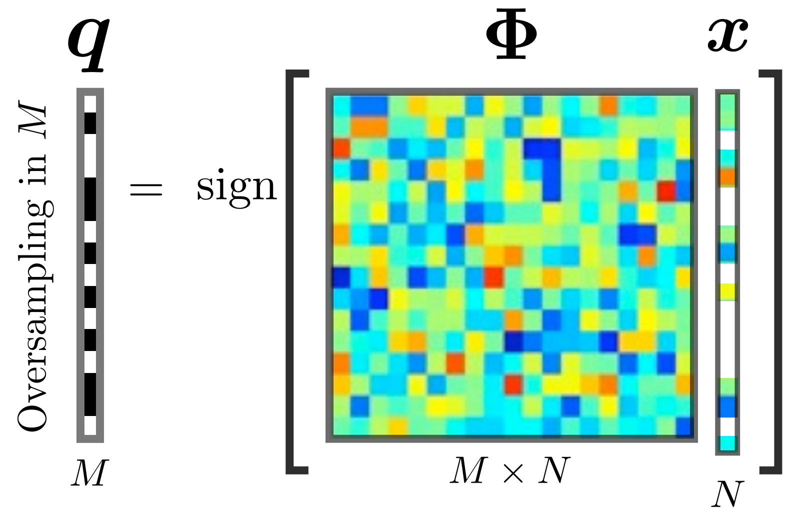
M $\quad M \times N \quad N$

$\underbrace{\qquad}$

M-bits! But, which information inside $\boldsymbol{q}$?

1-bit Computational ~~Compressed~~ Sensing

bits matter!



Warning 1: signal amplitude is lost!

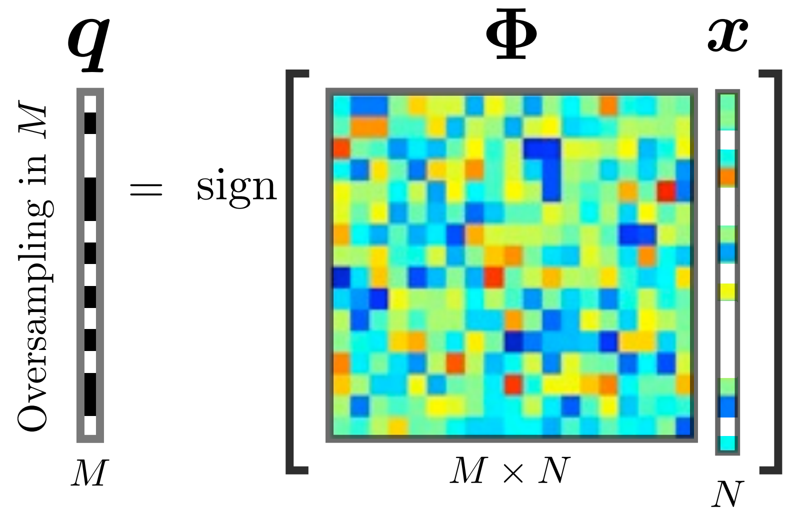
$$\boldsymbol{q} = \operatorname{sign}\left(\boldsymbol{\Phi}(\lambda\boldsymbol{x})\right) = \operatorname{sign}\left(\boldsymbol{\Phi}\boldsymbol{x}\right), \quad \forall\lambda > 0$$

$\Rightarrow$ Amplitude is arbitrarily fixed

Examples : $\|\boldsymbol{x}\| = 1$ or $\|\boldsymbol{\Phi}\boldsymbol{x}\|_1 = 1$

1-bit ~~Compressed~~ Computational Sensing

bits matter!



[Plan, Vershynin, 11]

Warning 2: $\exists$ forbidden sensing!

Let $\boldsymbol{x}_\lambda := (1, \lambda, 0, \cdots, 0)^T \in \mathbb{R}^N$

and $\boldsymbol{\Phi} \in \{\pm 1\}^{M \times N}$ (*e.g.*, Bernoulli).

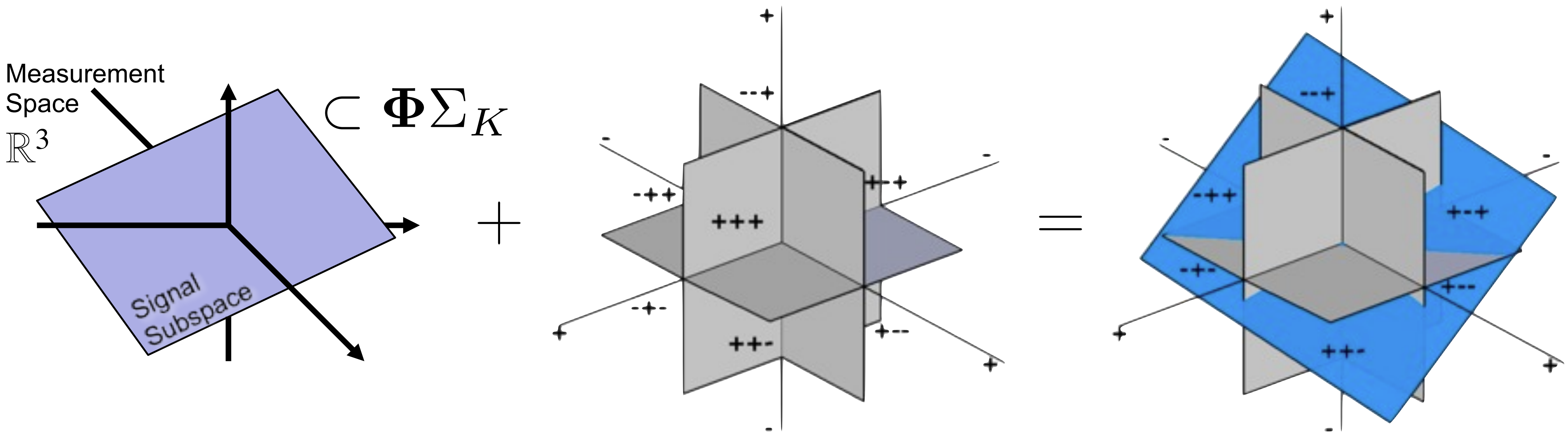
We have $\|\boldsymbol{x}_0 - \boldsymbol{x}_\lambda\| = \lambda$

but $\boldsymbol{q} = \operatorname{sign}(\boldsymbol{\Phi}\boldsymbol{x}_0) = \operatorname{sign}(\boldsymbol{\Phi}\boldsymbol{x}_\lambda), \ \forall |\lambda| < 1$

$\Rightarrow$ No hope to distinguish them by increasing M!

2. Theoretical performance limits

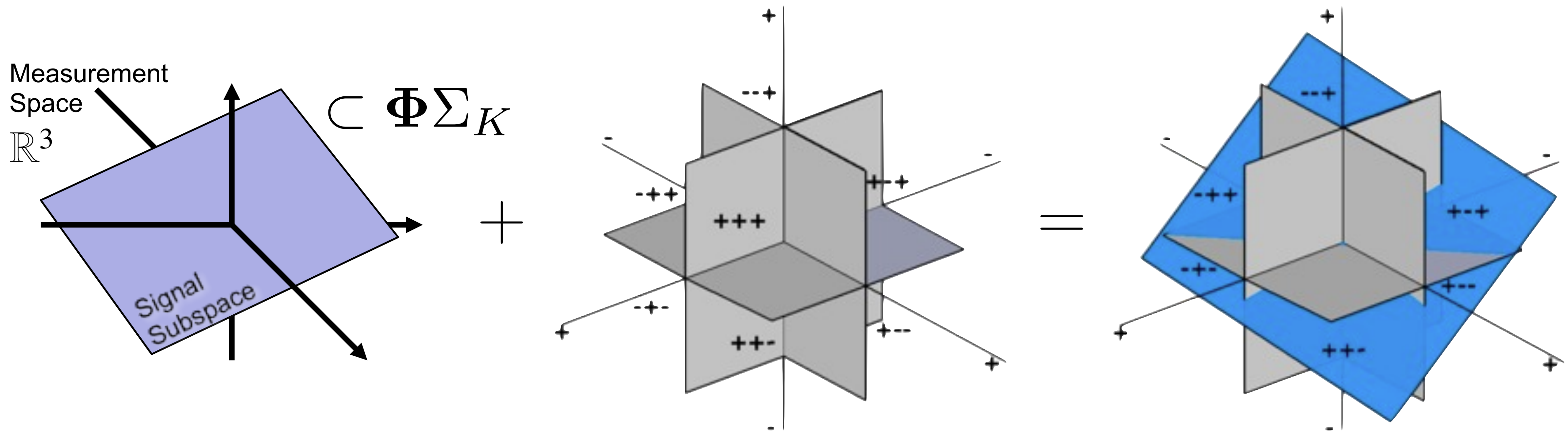
Lower bound: cell intersection viewpoint



Not all quantization cells intersected!

no more than $C = 2^K \binom{N}{K}\binom{M}{K}$

Lower bound: cell intersection viewpoint

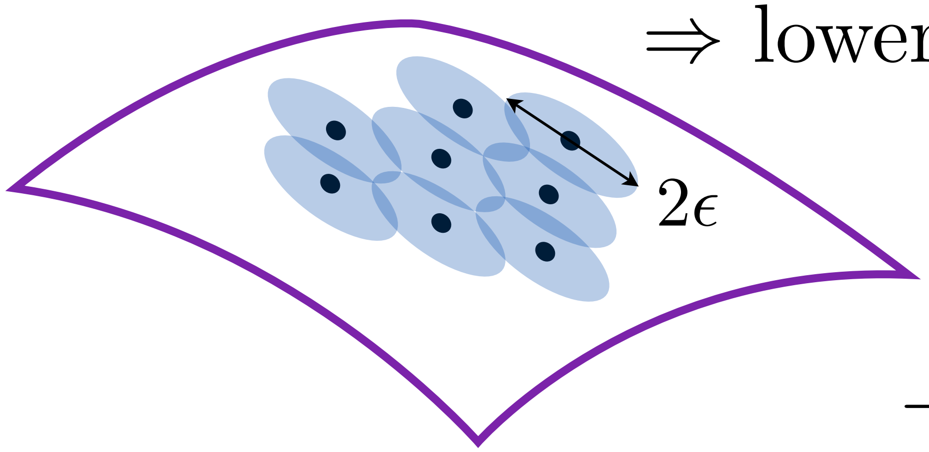


Not all quantization cells intersected!

no more than $C = 2^K \binom{N}{K}\binom{M}{K}$

Most efficient ϵ-covering of $S^{N-1} \cap \Sigma_K$ with ϵ-caps

$\Rightarrow$ lower bound on C by "vol$(S^{N-1} \cap \Sigma_K)$/vol(ϵ-cap)"



$$\boxed{\Rightarrow \epsilon = \Omega(K/M)}$$

$\rightarrow$ Lower bound on any 1-bit reconstruction error

Reaching this bound ?

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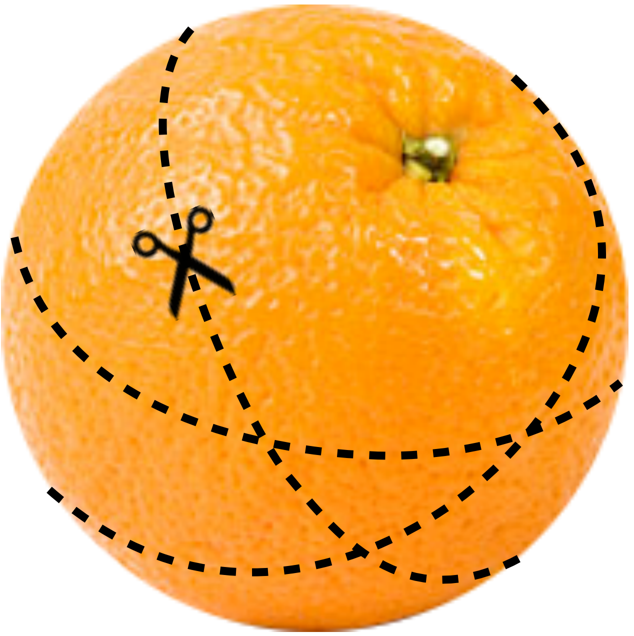
http://www.gaussfacts.com



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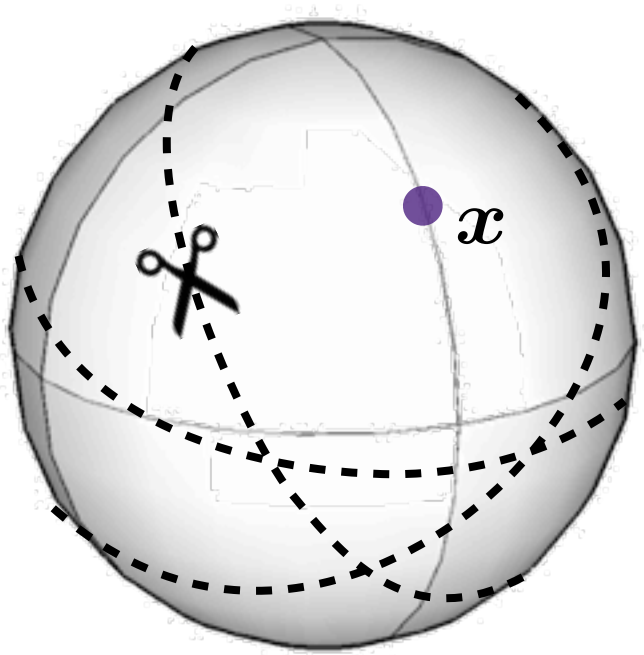
$\boldsymbol{x}$ on S^2

M vectors:

$\{\boldsymbol{\varphi}_i : 1 \leqslant i \leqslant M\}$

iid Gaussian



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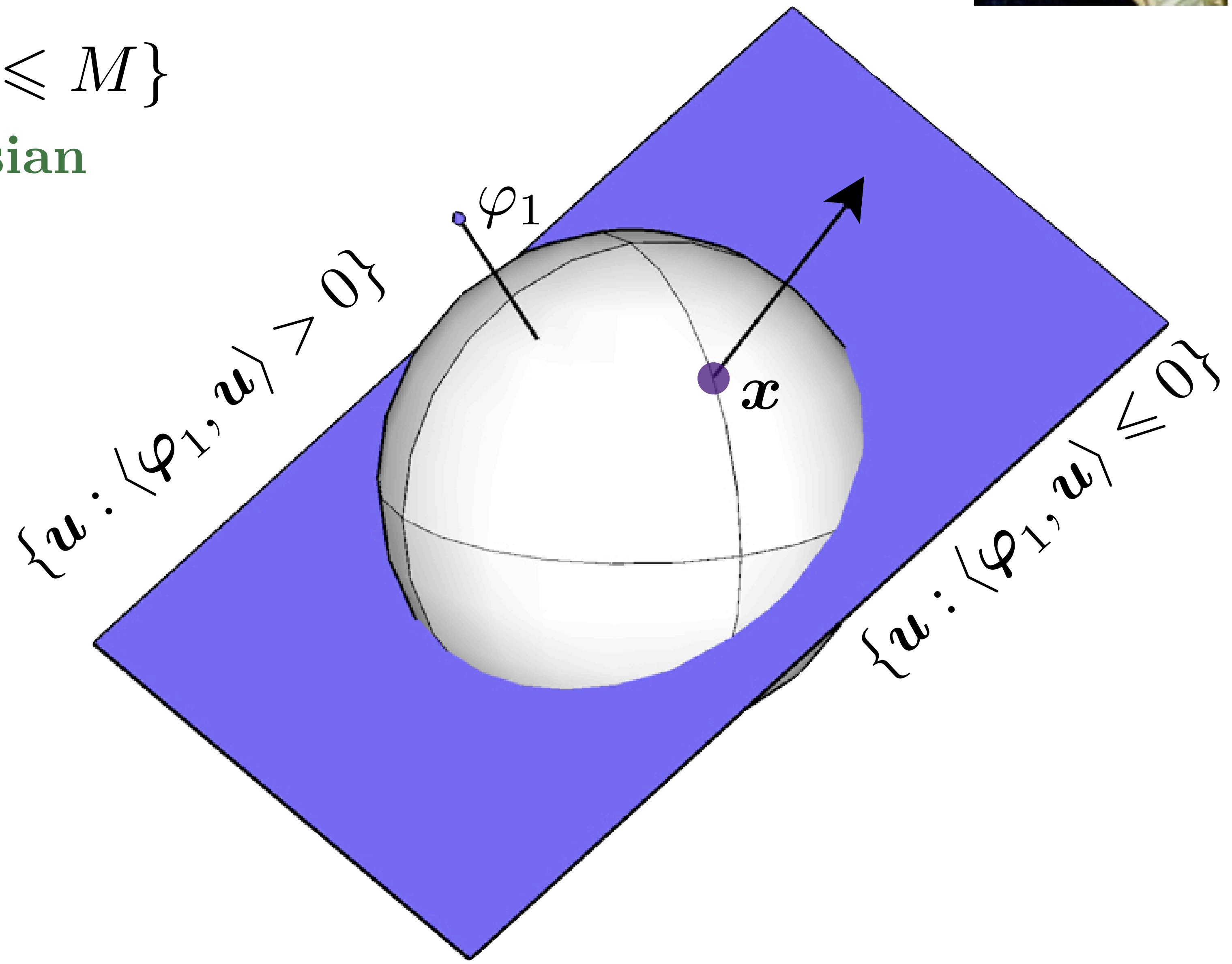
1-bit Measurements

$\langle \boldsymbol{\varphi}_1, \boldsymbol{x} \rangle > 0$



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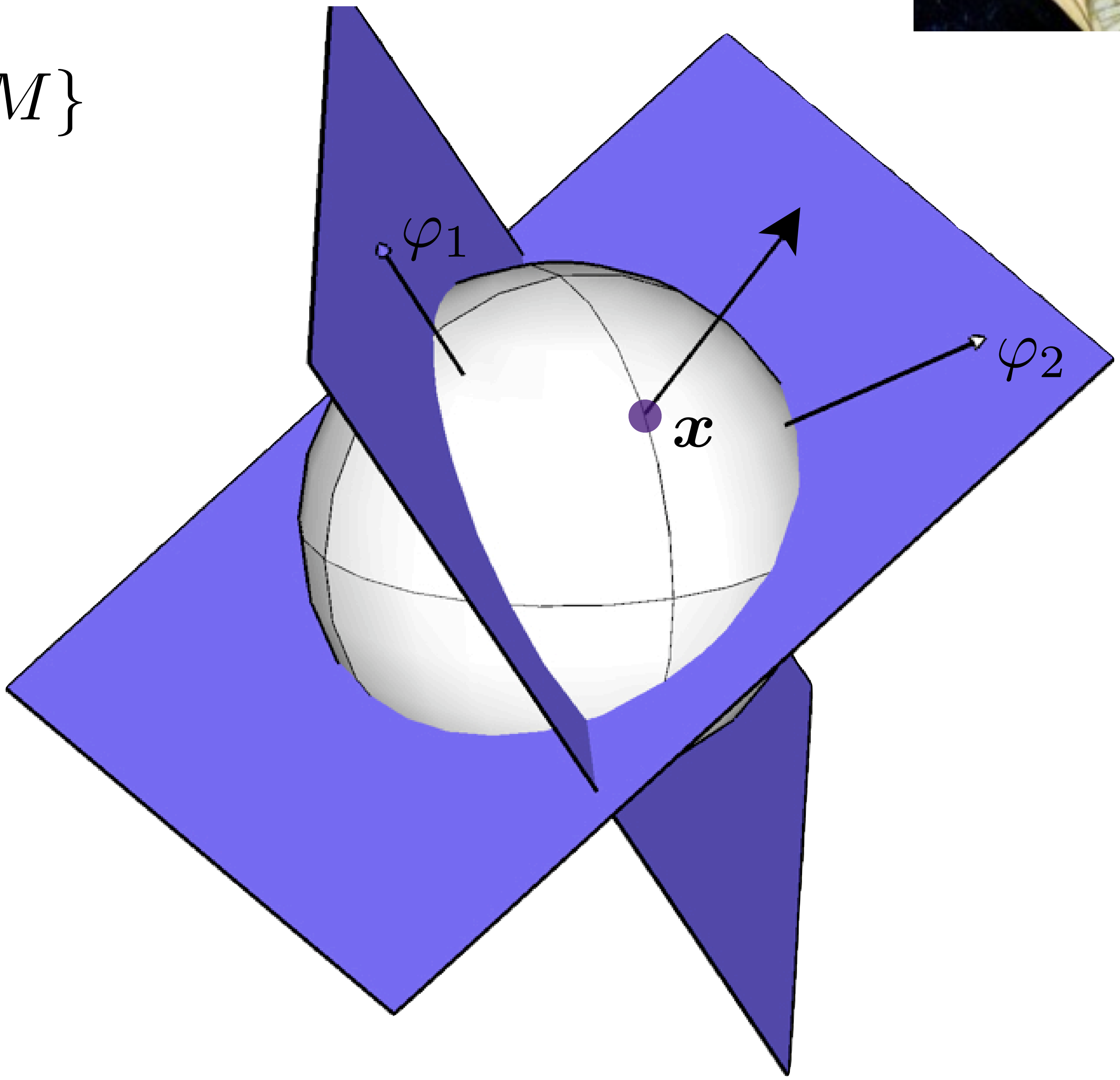

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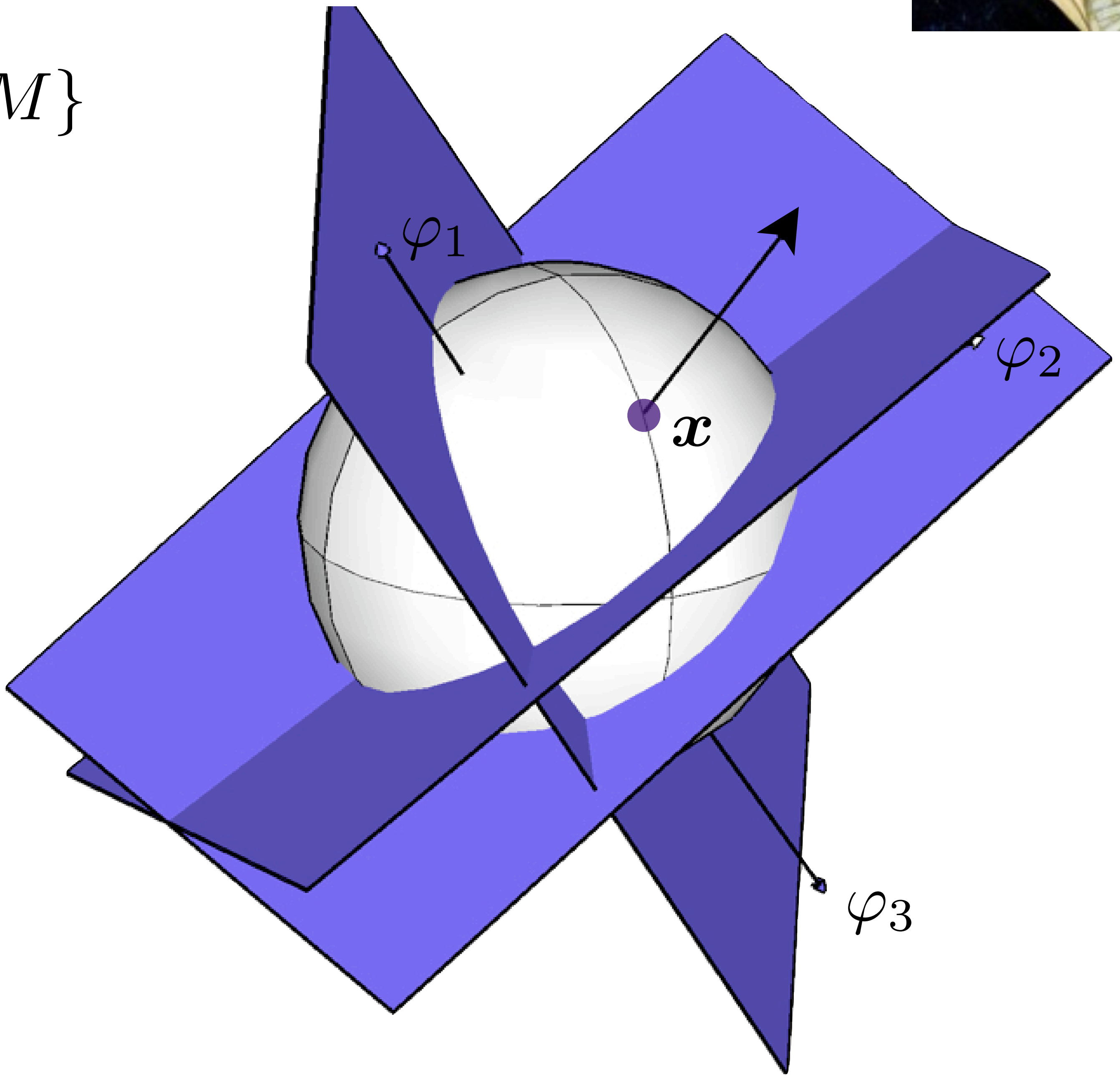

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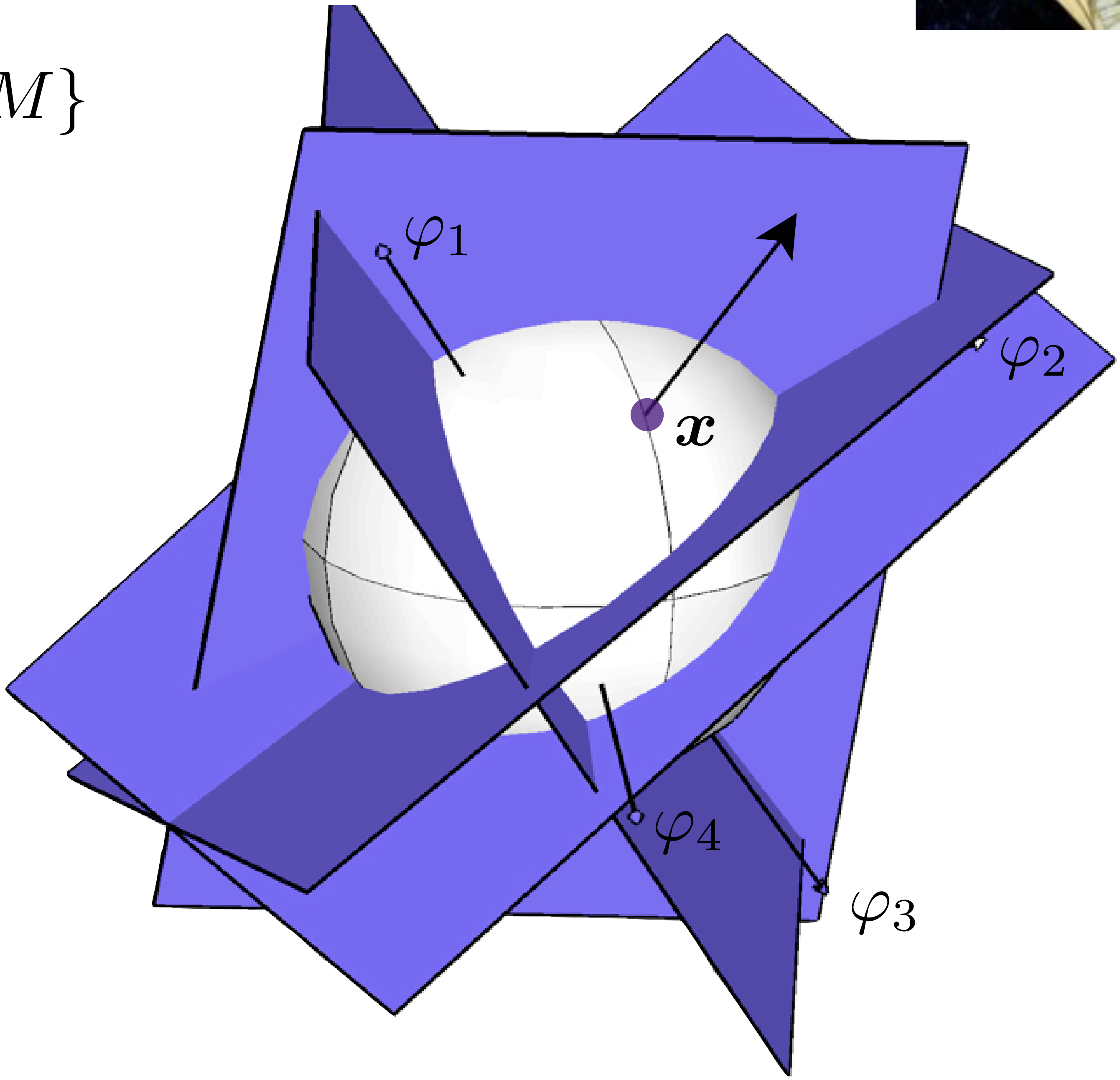

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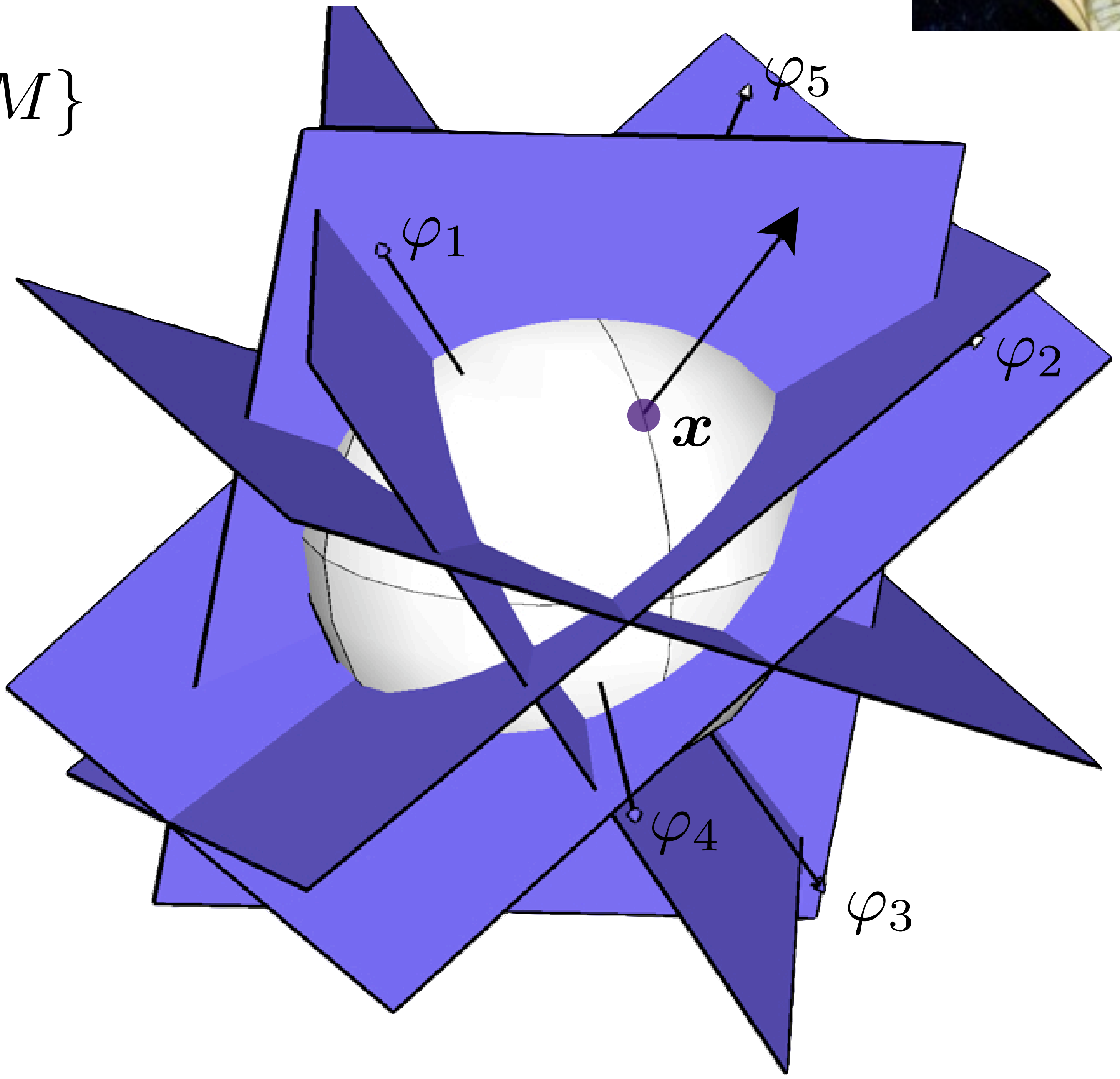

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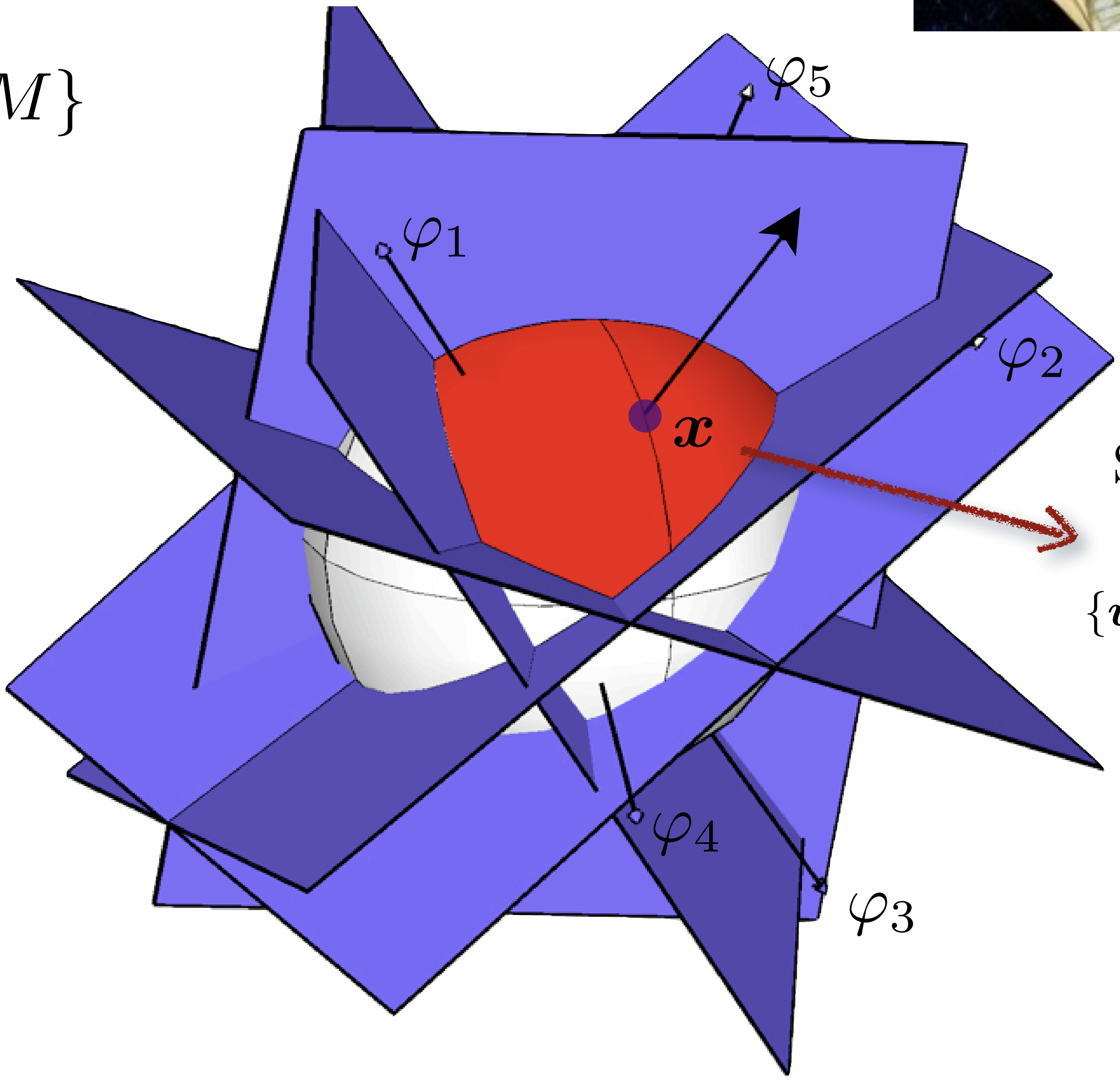

Smaller and smaller when M increases

$\{\boldsymbol{u} : \mathrm{sign}(\boldsymbol{\Phi u}) = \mathrm{sign}(\boldsymbol{\Phi x})\}$

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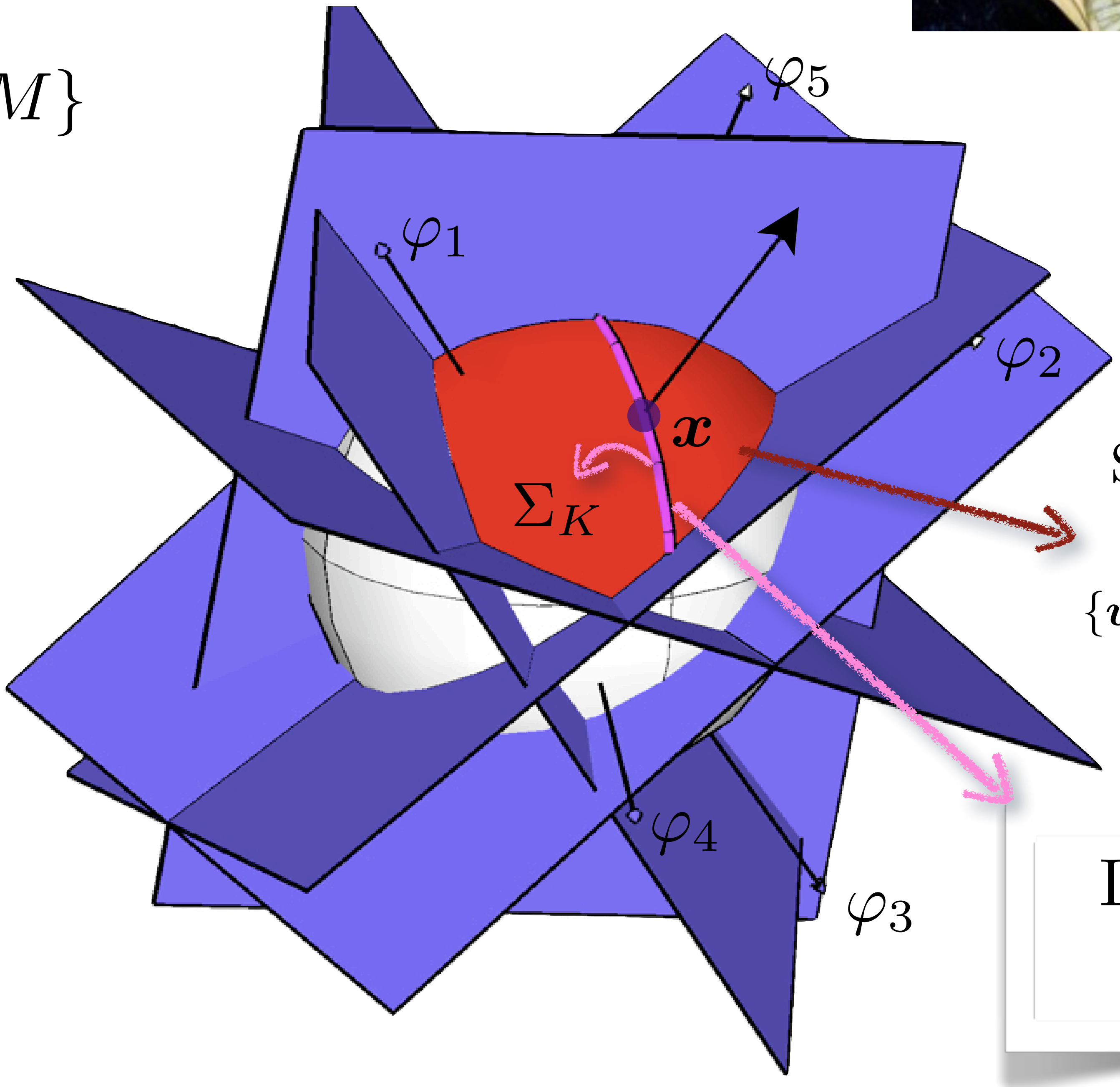

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Lower bound on this width?

Reaching this bound ?

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Let $A(\cdot) := \mathrm{sign}\,(\boldsymbol{\Phi}\,\cdot)$ with $\boldsymbol{\Phi} \sim \mathcal{N}^{M\times N}(0,1)$.

If $M = O(\epsilon^{-1} K \log N)$, then, w.h.p,

for any two unit K-sparse vectors $\boldsymbol{x}$ and $\boldsymbol{s}$,

$$A(\boldsymbol{x}) = A(\boldsymbol{s}) \quad \Rightarrow \quad \|\boldsymbol{x} - \boldsymbol{s}\| \leq \epsilon$$

$$\Leftrightarrow \epsilon = O\big(\tfrac{K}{M} \log \tfrac{MN}{K}\big)$$

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Note: You can even afford a small error, *i.e.*,

if only b bits are different between $A(\boldsymbol{x})$ and $A(\boldsymbol{s})$ $\Rightarrow \|\boldsymbol{x} - \boldsymbol{s}\| \leqslant \frac{K+b}{K}\,\epsilon$

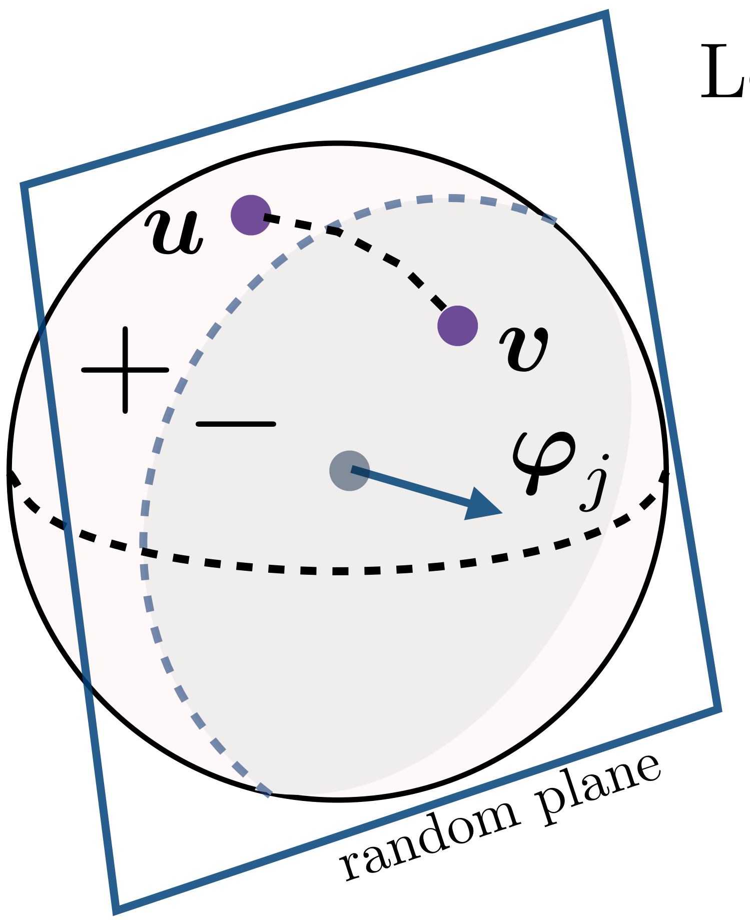
3. Stable embeddings: angles are preserved

What's known?

- Let's define

$$A(\boldsymbol{u}) := \operatorname{sign}(\boldsymbol{\Phi}\boldsymbol{u}) \;\Leftrightarrow\; A_j(\boldsymbol{u}) = \operatorname{sign}(\boldsymbol{\varphi}_j \cdot \boldsymbol{u}) \in \{\pm 1\}$$

j^{th} row of $\boldsymbol{\Phi}$



Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{N-1}$ (wlog)

$$\mathbb{P}[A_j(\boldsymbol{u}) \neq A_j(\boldsymbol{v})] \;=\; ?$$

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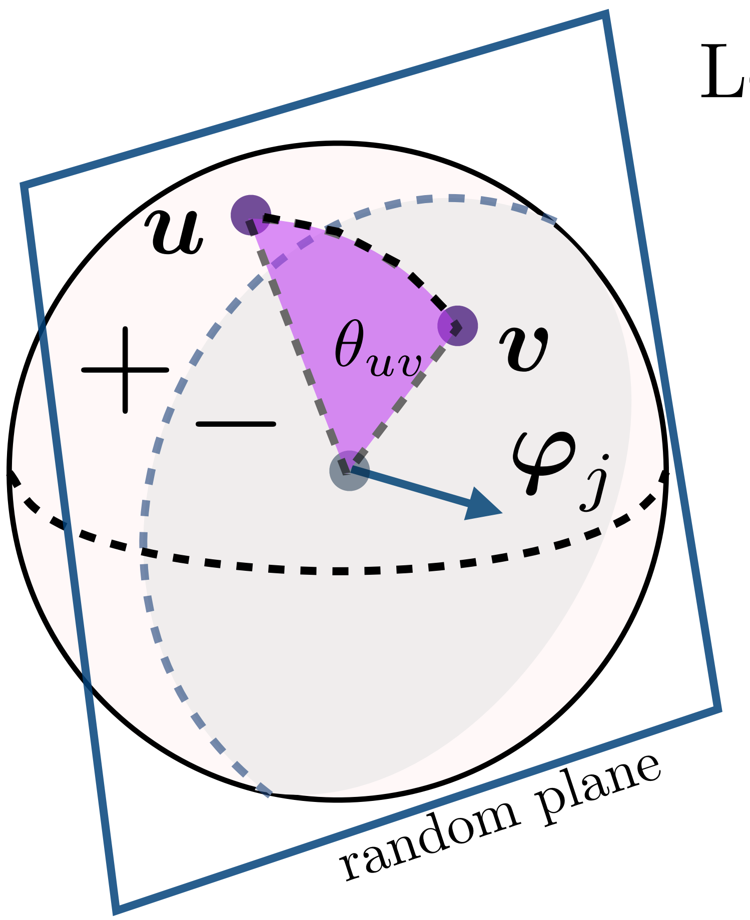

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{N-1}$ (wlog)

$$\begin{aligned}\mathbb{P}[A_j(\boldsymbol{u}) \neq A_j(\boldsymbol{v})] &= \tfrac{1}{\pi}\operatorname{angle}(\boldsymbol{u}, \boldsymbol{v}) \\ &= \tfrac{1}{\pi}\theta_{uv}\end{aligned}$$

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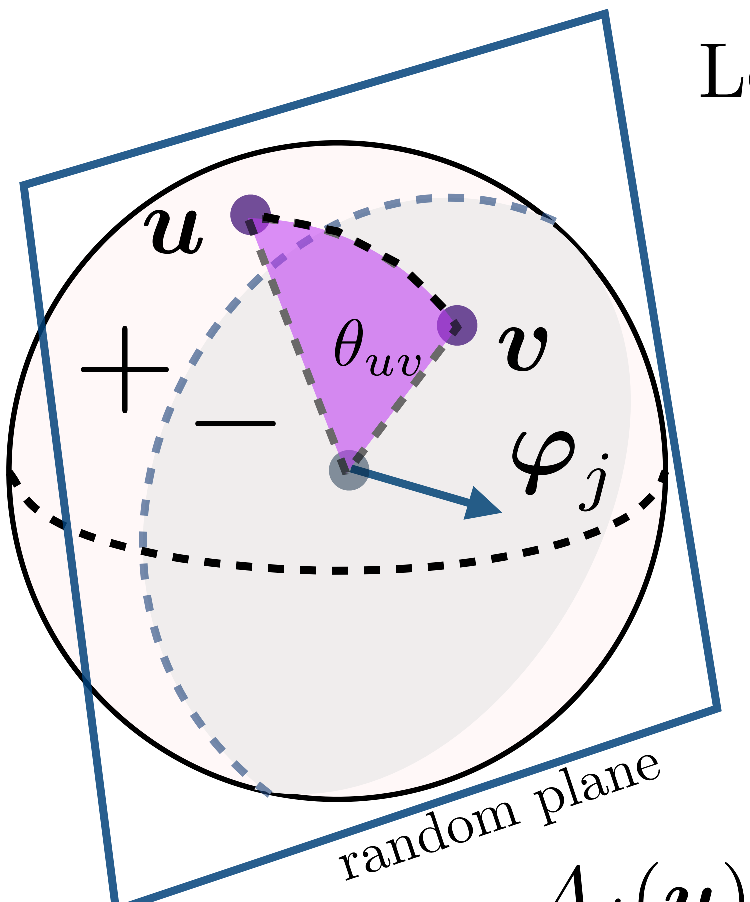

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$$\mathbb{P}[A_j(\boldsymbol{u}) \neq A_j(\boldsymbol{v})] = \tfrac{1}{\pi}\,\mathrm{angle}(\boldsymbol{u}, \boldsymbol{v}) = \tfrac{1}{\pi}\theta_{uv}$$

$$\Rightarrow \quad X_j = \tfrac{1}{2}\overbrace{|A_j(\boldsymbol{u}) - A_j(\boldsymbol{v})|}^{A_j(\boldsymbol{u}) \oplus A_j(\boldsymbol{v}) \;\text{(XOR)}} \underset{\text{iid}}{\sim} \mathrm{Bernoulli}(\tfrac{\theta_{uv}}{\pi}) \in \{0, 1\}$$

Starting point: Hamming/Angle Concentration

- Metrics of interest:

$$d_H(\boldsymbol{u}, \boldsymbol{v}) = \tfrac{1}{M} \textstyle\sum_i (u_i \oplus v_i) \quad \text{(norm. Hamming)}$$
$$d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s}) = \tfrac{1}{\pi} \arccos(\langle \boldsymbol{x}, \boldsymbol{s} \rangle) \quad \text{(norm. angle)}$$

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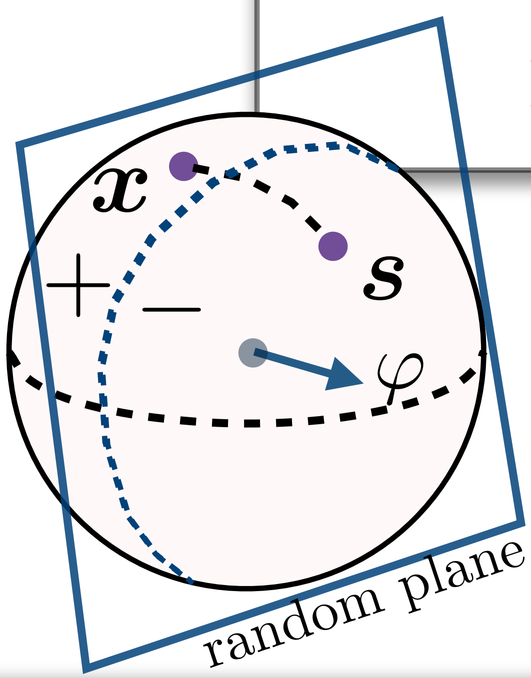
- Known fact: if $\boldsymbol{\Phi} \sim \mathcal{N}^{M\times N}(0,1)$ [*e.g.*, Goemans, Williamson 1995]

> Let $\boldsymbol{\Phi} \sim \mathcal{N}^{M\times N}(0,1)$, $A(\cdot) = \mathrm{sign}\,(\boldsymbol{\Phi}\ \cdot) \in \{-1,1\}^M$ and $\epsilon > 0$.
> For any $\boldsymbol{x},\boldsymbol{s} \in S^{N-1}$, we have
>
> $$\mathbb{P}_{\boldsymbol{\Phi}}\Big[\,\big|\underbrace{d_H\big(A(\boldsymbol{x}),A(\boldsymbol{s})\big)}_{\frac{1}{M}\sum_{i=1}^M X_i \,=\, \frac{1}{M}\sum_i A_i(\boldsymbol{x})\oplus A_i(\boldsymbol{s})} - d_{\rm ang}(\boldsymbol{x},\boldsymbol{s})\big| \leqslant \epsilon\,\Big] \geqslant 1 - 2\,e^{-2\epsilon^2 M}.$$



$\Rightarrow$ Thanks to $A(.)$, Hamming distance concentrates around vector angles!

Binary ϵ Stable Embedding (BϵSE)

A mapping $A : \mathbb{R}^N \to \{\pm 1\}^M$ is a **binary ϵ-stable embedding** (BϵSE) of order K for sparse vectors if

$$d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s}) - \epsilon \leq d_H(A(\boldsymbol{x}), A(\boldsymbol{s})) \leq d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s}) + \epsilon$$

for all $\boldsymbol{x}, \boldsymbol{s} \in S^{N-1}$ with $\boldsymbol{x} \pm \boldsymbol{s}$ K-sparse.

kind of "binary restricted (quasi) isometry"

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- *Corollary*: for any algorithm with output $\boldsymbol{x}^*$ jointly K-sparse and consistent (*i.e.*, $A(\boldsymbol{x}^*) = A(\boldsymbol{x})$),

$$d_{\rm ang}(\boldsymbol{x}, \boldsymbol{x}^*) \leqslant 2\epsilon!$$

- If limited binary noise, $d_{\rm ang}$ still bounded
- If not exactly sparse signals (but almost), $d_{\rm ang}$ still bounded

BϵSE existence? Yes!

Let $\mathbf{\Phi} \sim \mathcal{N}^{M \times N}(0,1)$, fix $0 \leqslant \eta \leqslant 1$ and $\epsilon > 0$. If

$$M \geqslant \tfrac{4}{\epsilon^2}\big(K \log(N) + 2K \log(\tfrac{50}{\epsilon}) + \log(\tfrac{2}{\eta})\big),$$

then $\mathbf{\Phi}$ is a BϵSE with $\Pr > 1 - \eta$.

$$\boxed{M = O(\epsilon^{-2} K \log N)}$$

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Proof sketch:

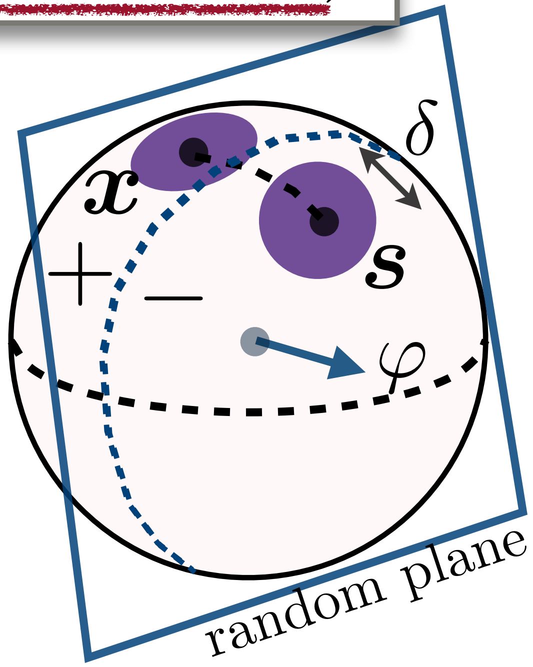
1) Generalize

$$\mathbb{P}_{\boldsymbol{\Phi}}\big[\,\big|\,d_H\big(A(\boldsymbol{x}), A(\boldsymbol{s})\big) \;-\; d_{\rm ang}(\boldsymbol{x},\boldsymbol{s})\big| \;\leqslant\; \epsilon\,\big] \;\geqslant\; 1 - 2\,e^{-2\epsilon^2 M}.$$

to

$$\mathbb{P}_{\boldsymbol{\Phi}}\Big[\,\big|\,d_H\big(A(\boldsymbol{u}), A(\boldsymbol{v})\big) \;-\; d_{\rm ang}(\boldsymbol{x},\boldsymbol{s})\big| \;\leqslant\; \epsilon + (\tfrac{\pi}{2}D)^{1/2}\,\delta\,\Big] \;\geqslant\; 1 - 2\,e^{-2\epsilon^2 M}.$$

for $\boldsymbol{u}, \boldsymbol{v}$ in a D-dimensional neighborhood of width δ around $\boldsymbol{x}$ and $\boldsymbol{s}$ resp.



2) Covers the space of "K-sparse signal pairs" in $\mathbb{R}^N$ by

$$O\big(\tbinom{N}{K}\delta^{-2K}\big) = O\big((\tfrac{eN}{K\delta^2})^K\big) \text{ neighborhoods.}$$

3) Apply Point 1 with union bound, and "stir until the proof thickens"

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BϵSE consistency "width":

$$\epsilon \;=\; O\Big(\big(\tfrac{K}{M}\log\tfrac{MN}{K}\big)^{1/2}\Big)$$

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not as optimal but stronger result!

$d_H \leftrightarrow d_{\mathrm{ang}}$

4. Generalized Embeddings

Beyond strict sparsity ... [Plan, Vershynin]

Let $\mathcal{K} \subset S^{N-1}$ (*e.g.*, compressible signals s.t. $\|\boldsymbol{x}\|_2/\|\boldsymbol{x}\|_1 \leqslant \sqrt{K}$)
$\neq \Sigma_K$

What can we say on $d_H(A(\boldsymbol{x}), A(\boldsymbol{s}))$ for $\boldsymbol{x}, \boldsymbol{s} \in \mathcal{K}$?

Y. Plan, R. Vershynin, "Dimension reduction by random hyperplane tessellations", 2011, arXiv:1111.4452
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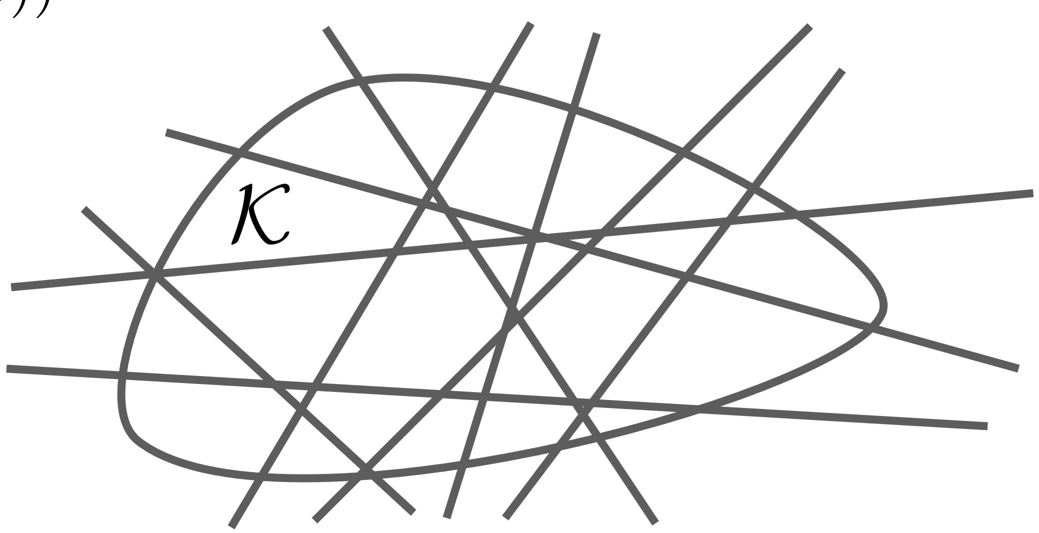
Uniform tesselation: [Plan, Vershynin, 11]

$$\mathrm{P}\big(\underbrace{\#\ \text{random hyperplanes btw } \boldsymbol{x}}_{d_H(A(\boldsymbol{x}), A(\boldsymbol{s}))} \text{ and } \boldsymbol{s} \propto d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s})\big)\ ?$$



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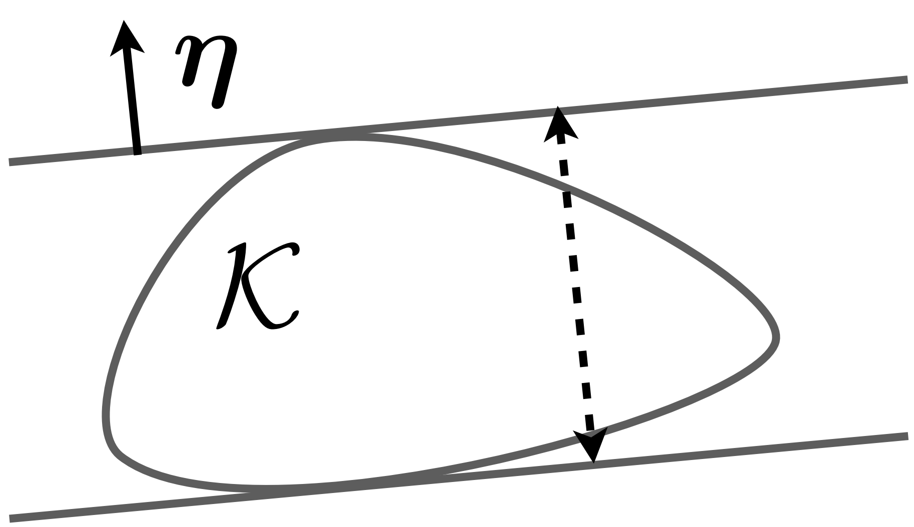
Measuring the "dimension" of $\mathcal{K}$ $\to$ Gaussian mean width:

$$\boxed{w(\mathcal{K}) := \mathbb{E} \sup_{\boldsymbol{u}\in\mathcal{K}-\mathcal{K}} \langle \boldsymbol{g}, \boldsymbol{u}\rangle, \text{ with } g_k \sim_{\text{iid}} \mathcal{N}(0,1)}$$



width in direction $\boldsymbol{\eta}$

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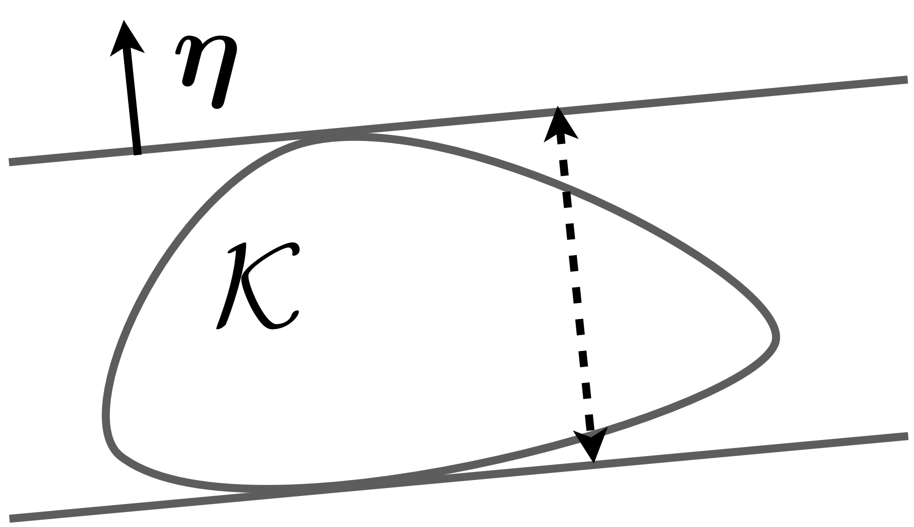

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Examples:

$w^2(\mathcal{S}^{N-1}) \leqslant 4N$

$w^2(\mathcal{K}) \leqslant C \log |\mathcal{K}|$ (for finite sets)

$w^2(\mathcal{K}) \leqslant L$ if subspace with $\dim \mathcal{K} = L$

$w^2(\Sigma_K) \simeq K \log(2N/K)$

$\vdots$

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then, with $Pr \geqslant 1 - e^{-c\epsilon^2 M}$*, we have*

$$d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s}) - \epsilon \ \leqslant \ d_H(A(\boldsymbol{x}), A(\boldsymbol{s})) \ \leqslant \ d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s}) - \epsilon, \quad \forall \boldsymbol{x}, \boldsymbol{s} \in \mathcal{K}.$$

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not as optimal but stronger result!

then, with $Pr \geqslant 1 - e^{-c\epsilon^2 M}$, *we have*

$$d_{\rm ang}(\boldsymbol{x},\boldsymbol{s}) - \epsilon \leqslant d_H(A(\boldsymbol{x}), A(\boldsymbol{s})) \leqslant d_{\rm ang}(\boldsymbol{x},\boldsymbol{s}) - \epsilon, \quad \forall \boldsymbol{x},\boldsymbol{s} \in \mathcal{K}.$$

Generalize BϵSE to more general sets.

In particular, to

$$\mathcal{C}_K = \{\boldsymbol{u} \in \mathbb{R}^N : \|\boldsymbol{u}\|_2/\|\boldsymbol{u}\|_1 \leqslant \sqrt{K}\} \supset \Sigma_K$$

with $w^2(\mathcal{C}_K) \leqslant cK \log N/K$.

Y. Plan, R. Vershynin, "Dimension reduction by random hyperplane tessellations", 2011, arXiv:1111.4452

Y. Plan, R. Vershynin, "Robust 1-bit compressed sensing and sparse logistic regression: a convex programming approach", IEEE TIT 2012, arXiv:1202.1212.

Beyond strict sparsity ... [Plan, Vershynin]

Proposition *Let* $\boldsymbol{\Phi} \sim \mathcal{N}^{M\times N}(0,1)$ *and* $\mathcal{K} \subset \mathbb{R}^N$*. Then, for some* $C, c > 0$*, if*

$$M \geqslant C\epsilon^{-6} w^2(\mathcal{K}),$$

not as optimal but stronger result!

then, with $Pr \geqslant 1 - e^{-c\epsilon^2 M}$*, we have*

$$d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s}) - \epsilon \leqslant d_H(A(\boldsymbol{x}), A(\boldsymbol{s})) \leqslant d_{\rm ang}(\boldsymbol{x}, \boldsymbol{s}) - \epsilon, \quad \forall \boldsymbol{x}, \boldsymbol{s} \in \mathcal{K}.$$

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with $w^2(\mathcal{C}_K) \leqslant cK \log N/K$.

$\Rightarrow$ Extension to "1-bit Matrix Completion" possible!

i.e., $w^2(r\text{-rank } N_1 \times N_2 \text{ matrix}) \leqslant c\, r(N_1 + N_2)$!

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5. 1-bit CS Reconstructions?

Dumbest 1-bit reconstruction

Fact: If $M = O(\epsilon^{-2} K \log N/K)$ (for $\boldsymbol{x} \in \Sigma_K$ fixed, $\forall\, \boldsymbol{s} \in \Sigma_K$)

or, if $M = O(\epsilon^{-6} K \log N/K)$ ($\forall\, \boldsymbol{x}, \boldsymbol{s} \in \Sigma_K$), then, w.h.p,

$$\Big|\tfrac{\sqrt{\pi}/2}{M}\langle \operatorname{sign}(\boldsymbol{\Phi x}), \boldsymbol{\Phi s}\rangle - \langle \boldsymbol{x}, \boldsymbol{s}\rangle\Big| \leq \epsilon \qquad \text{[Plan, Vershynin, 12]}$$

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LJ, K. Degraux, C. De Vleeschouwer, "Quantized Iterative Hard Thresholding: Bridging 1-bit and High-Resolution Quantized Compressed Sensing", SAMPTA2013

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$$|\tfrac{\sqrt{\pi/2}}{M}\langle \mathrm{sign}\,(\boldsymbol{\Phi x}), \boldsymbol{\Phi s}\rangle - \langle \boldsymbol{x}, \boldsymbol{s}\rangle| \leq \epsilon$$ [Plan, Vershynin, 12]

- Implication? [LJ, Degraux, De Vleeschouwer, 13]

Let $\boldsymbol{x} \in \Sigma_K \cap S^{N-1}$ and $\boldsymbol{q} = \mathrm{sign}(\boldsymbol{\Phi x})$.
Compute

$$\boxed{\hat{\boldsymbol{x}} = \tfrac{\pi}{2M}\,\mathcal{H}_K(\boldsymbol{\Phi}^* \boldsymbol{q})}$$

Then, if previous property holds,

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \leq 2\epsilon.$$

Non-uniform case ($\boldsymbol{x}$ given):
$\Rightarrow \epsilon \;=\; O\big((\frac{K}{M} \log \frac{MN}{K})^{1/2}\big)$
Uniform case:
$\Rightarrow \epsilon \;=\; O\big((\frac{K}{M} \log \frac{MN}{K})^{1/6}\big)$

Y. Plan, R. Vershynin, "Robust 1-bit compressed sensing and sparse logistic regression: a convex programming approach", IEEE TIT 2012, arXiv:1202.1212.
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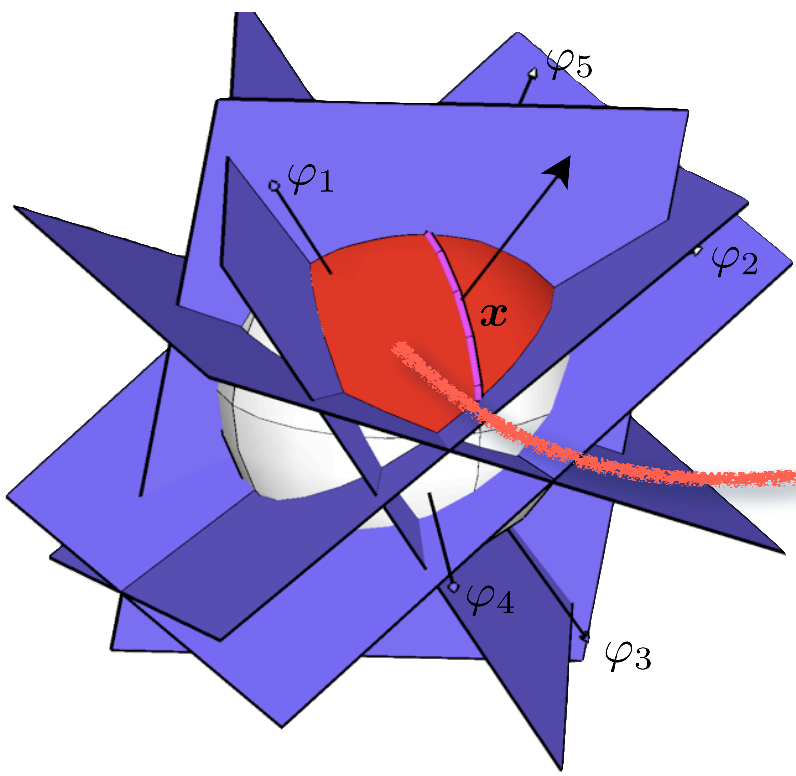
Initial approach

- Let $\boldsymbol{q} = \operatorname{sign}(\boldsymbol{\Phi x}) =: A(\boldsymbol{x})$
- Initially: [Boufounos, Baraniuk 2008]

$$\boxed{\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \operatorname{diag}(\boldsymbol{q})\, \boldsymbol{\Phi u} > 0 \quad \text{and} \quad \|\boldsymbol{u}\|_2 = 1}$$

Non-convex! 2 numerical choices :

1. relax + projection on $S^{N\text{-}1}$
2. *"trust region methods"*
   $\rightarrow$ *Restricted-Step Shrinkage (RSS)*



Consistency constraint:

$$\{\boldsymbol{u} \in \mathbb{R}^N \cap S^{N-1} : \boldsymbol{q} = A(\boldsymbol{u})\}$$

$$\Leftrightarrow \{\boldsymbol{u} \in \mathbb{R}^N \cap S^{N-1} : \operatorname{diag}(\boldsymbol{q})\boldsymbol{\Phi u} > 0\}$$

$$\ni \boldsymbol{x}$$

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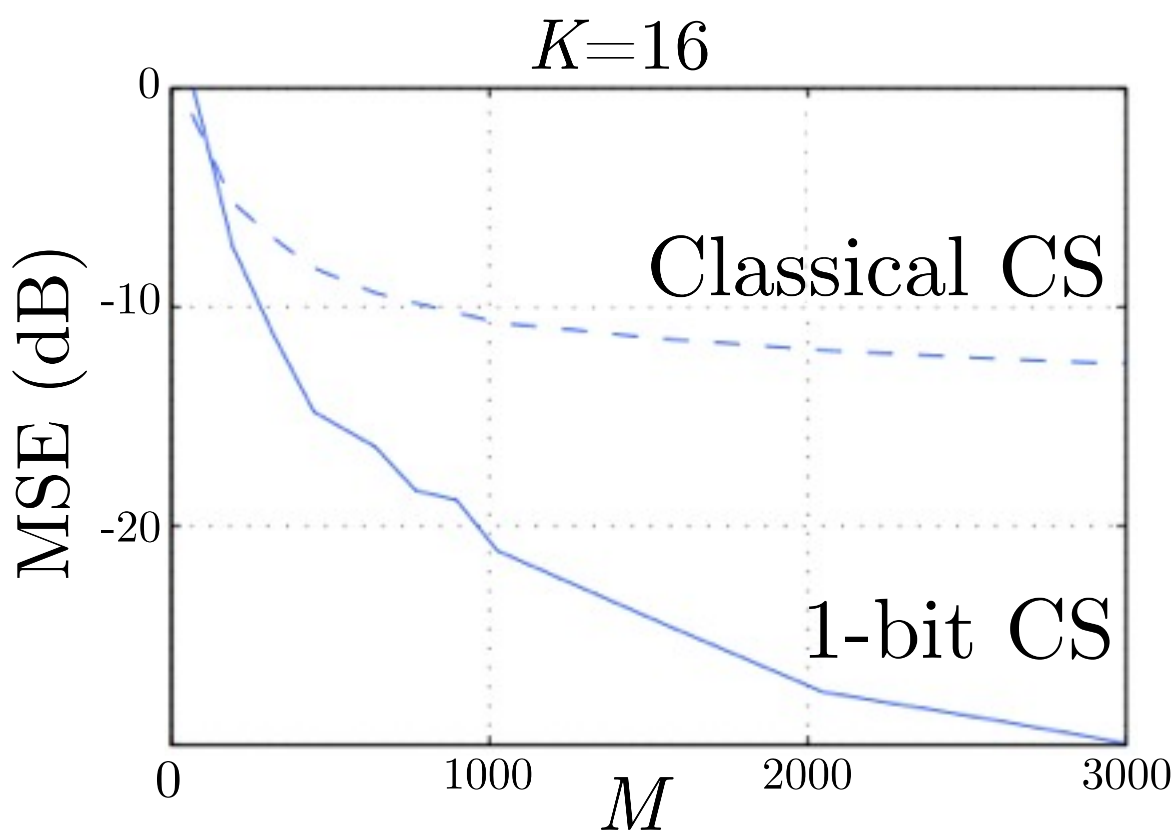
($e.g.$, take the 1st choice)

(relaxed) $\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 + \lambda \|(\mathrm{diag}(\boldsymbol{q})\,\boldsymbol{\Phi u})_-\|^2 \quad \text{s.t.} \quad \|\boldsymbol{u}\|_2 = 1$

$\rightarrow$ Solved by projected gradient descent



gain brought by (almost)consistency

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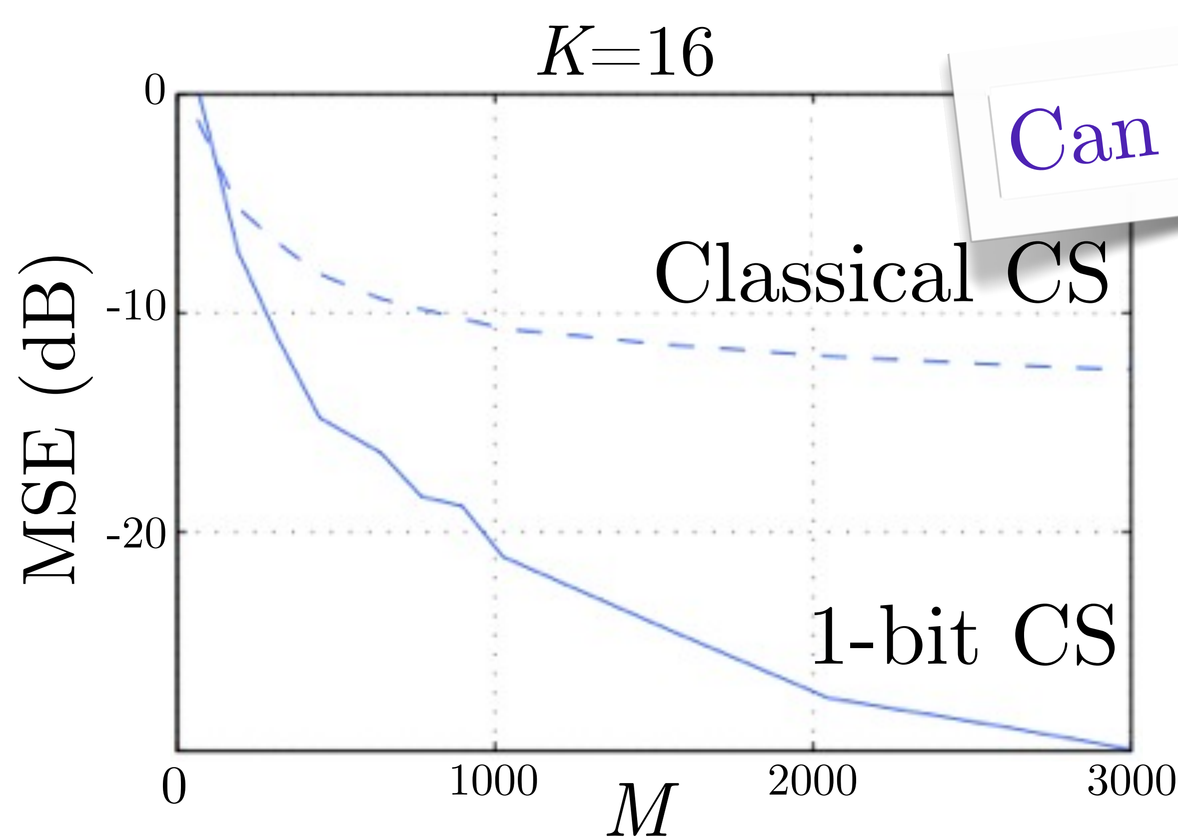


Can we do better?

gain brought by (almost)consistency

Other methods:

- ✓ Matching Sign Pursuit [Boufounos]
- Restricted-Step Shrinkage (RSS) [Laska, We, Yin, Baraniuk]
- ✓ Binary Iterative Hard Thresholding [Jacques, Laska, Boufounos, Baraniuk]
- ✓ Convex Optimization [Plan, Vershynin]
- ...

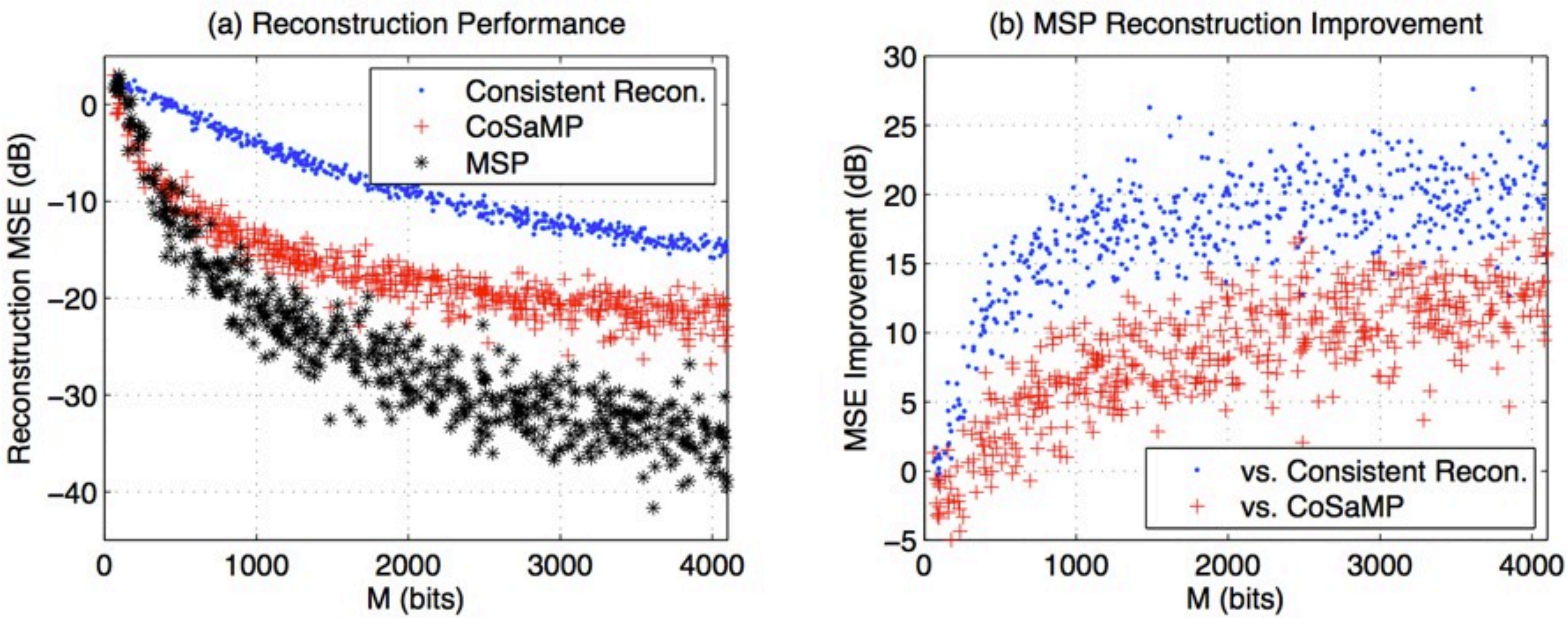
Matching Sign Pursuit (MSP)

- Iterative greedy algorithm, similar to CoSaMP [Needell, Tropp, 08]
- Maintains running signal estimate and its support T.
- <u>MSP iteration</u>:
  - Identify sign violations $\longrightarrow \boldsymbol{r} = (\mathrm{diag}(\boldsymbol{y})\, \boldsymbol{\Phi}\widehat{\boldsymbol{x}})_-$
  - Compute proxy $\longrightarrow \boldsymbol{p} = \boldsymbol{\Phi}^T \boldsymbol{r}$
  - Identify support $\longrightarrow \Omega = \mathrm{supp}\, \boldsymbol{p}|_{2K} \cup T$
  - Consistent Reconstruction over support estimate:

$$\boldsymbol{b}|_\Omega = \arg\min_{\boldsymbol{u}\in\mathbb{R}^N} \|(\mathrm{diag}(\boldsymbol{y})\boldsymbol{\Phi}\boldsymbol{u})_-\|_2^2 \text{ s.t } \|\boldsymbol{u}\|_2 = 1 \text{ and } \boldsymbol{u}|_{T^c} = 0$$

  - Truncate, normalize, and update estimate: $\widehat{\boldsymbol{x}} \leftarrow \boldsymbol{b}|_K \,/\, \|\boldsymbol{b}|_K\|_2$

Matching Sign Pursuit (MSP)



Boufounos, P. T. (2009, November). "Greedy sparse signal reconstruction from sign measurements".
In Signals, Systems and Computers, 2009 Conference Record of the Forty-Third Asilomar Conference on (pp. 1305-1309). IEEE.

Binary Iterative Hard Thresholding

Given $\boldsymbol{q} = A(\boldsymbol{x})$ and K, set $l = 0$, $\boldsymbol{x}^0 = 0$:

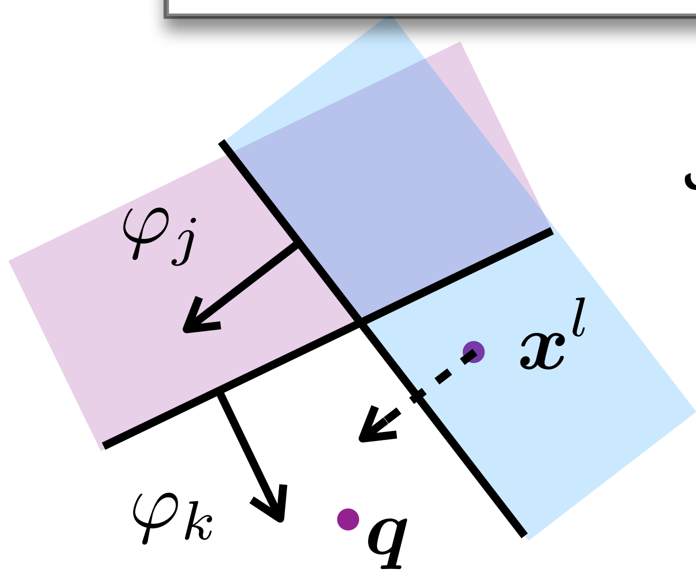
$$\boxed{\begin{aligned} \boldsymbol{a}^{l+1} &= \boldsymbol{x}^l + \tfrac{\tau}{2}\boldsymbol{\Phi}^T\big(\boldsymbol{q} - A(\boldsymbol{x}^l)\big), \\ \boldsymbol{x}^{l+1} &= \mathcal{H}_K(\boldsymbol{a}^{l+1}), \quad l \leftarrow l+1 \end{aligned}}$$

("gradient" towards consistency)
($\tau > 0$ controls gradient descent)
(proj. K-sparse signal set)

with $\mathcal{H}_K(\boldsymbol{u}) = K$-term hard thresholding

Stop when $d_H(\boldsymbol{q}, A(\boldsymbol{x}^{l+1})) = 0$ or $l =$ max. iter.

minimizes $\boxed{\mathcal{J}(\boldsymbol{x}') = \|[\,\mathrm{diag}(\boldsymbol{q})(\boldsymbol{\Phi}\boldsymbol{x}')]_-\|_1}$ with $(\lambda)_- = (\lambda - |\lambda|)/2$



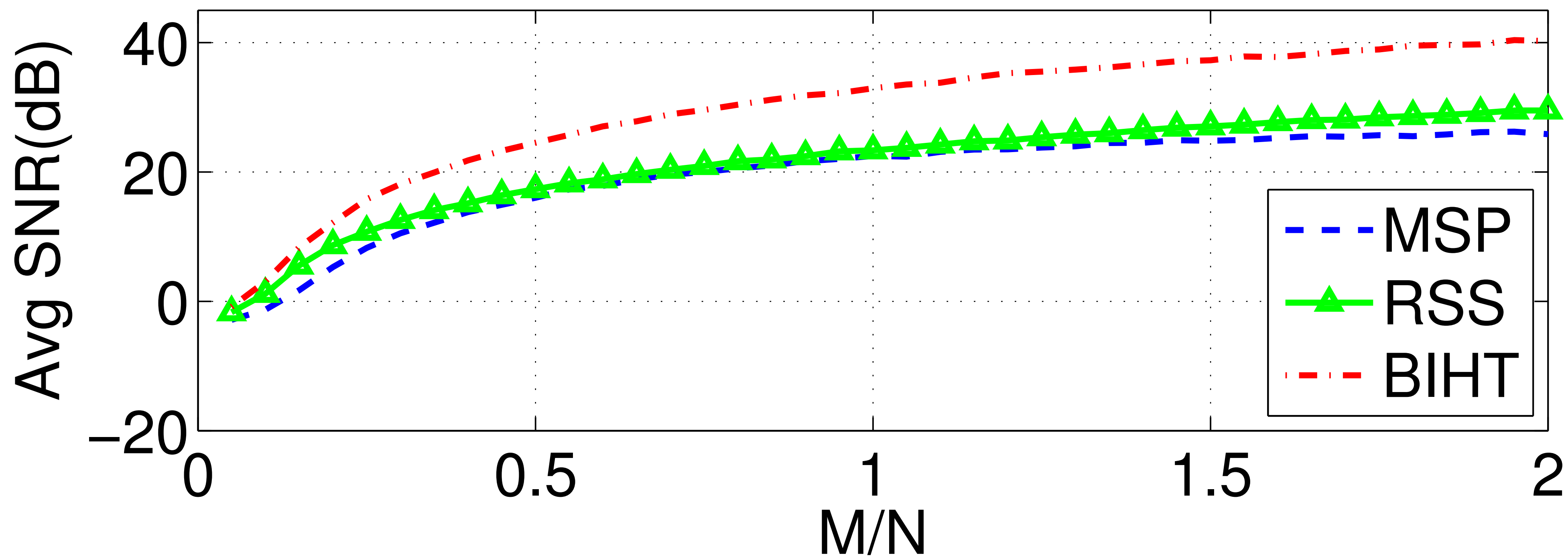
$$\mathcal{J}(\boldsymbol{x}') = \sum_{j=1}^{M} \Big|\big(\overbrace{\mathrm{sign}\,(\langle\boldsymbol{\varphi}_j, \boldsymbol{x}\rangle)}^{q_j}\,\langle\boldsymbol{\varphi}_j, \boldsymbol{x}'\rangle\big)_-\Big|$$

$$q_k - A(\boldsymbol{x}^l)_k = 0$$

$$q_j - A(\boldsymbol{x}^l)_j > 0$$

(connections with ML hinge loss, 1-bit classification)

Binary Iterative Hard Thresholding



$N = 1000, \; K = 10$

Bernoulli-Gaussian model

normalized signals

1000 trials

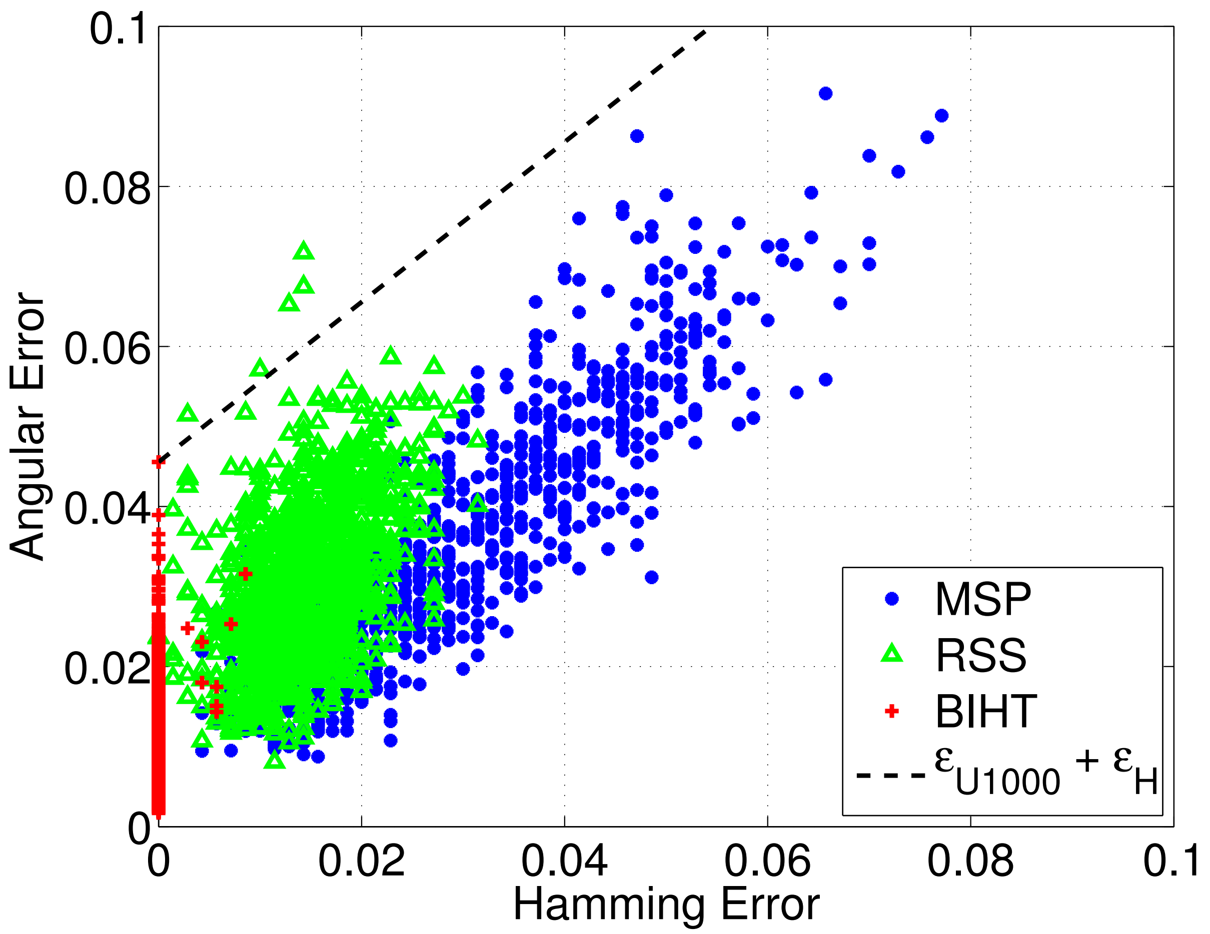
Matching Sign pursuit (MSP)

Restricted-Step Shrinkage (RSS)

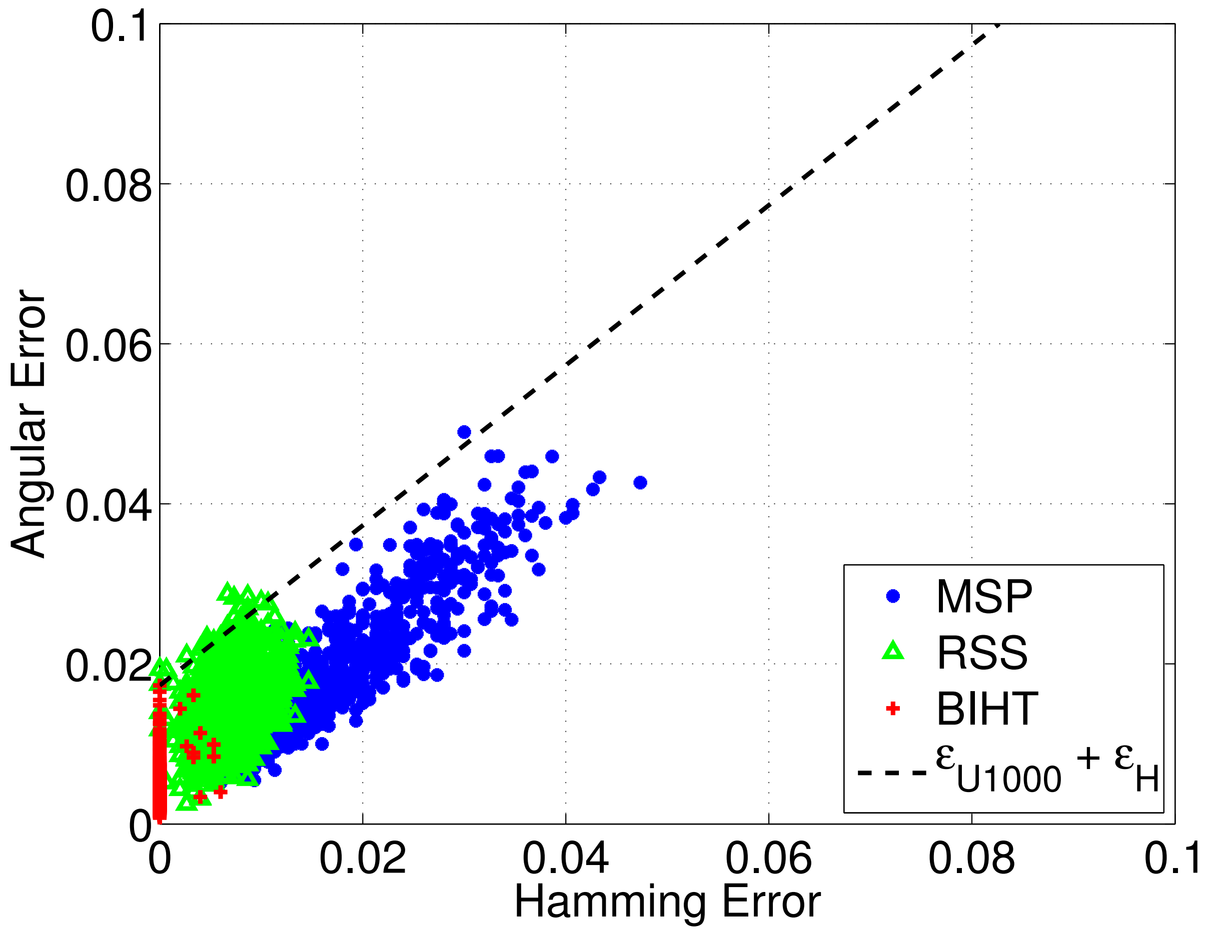
Binary Iterative Hard Thresholding (BIHT)

Binary Iterative Hard Thresholding

- Testing BϵSE: $d_{\rm ang}(\boldsymbol{x}, \boldsymbol{x}^*) \leqslant d_H(A(\boldsymbol{x}), A(\boldsymbol{x}^*)) + \epsilon(M)$

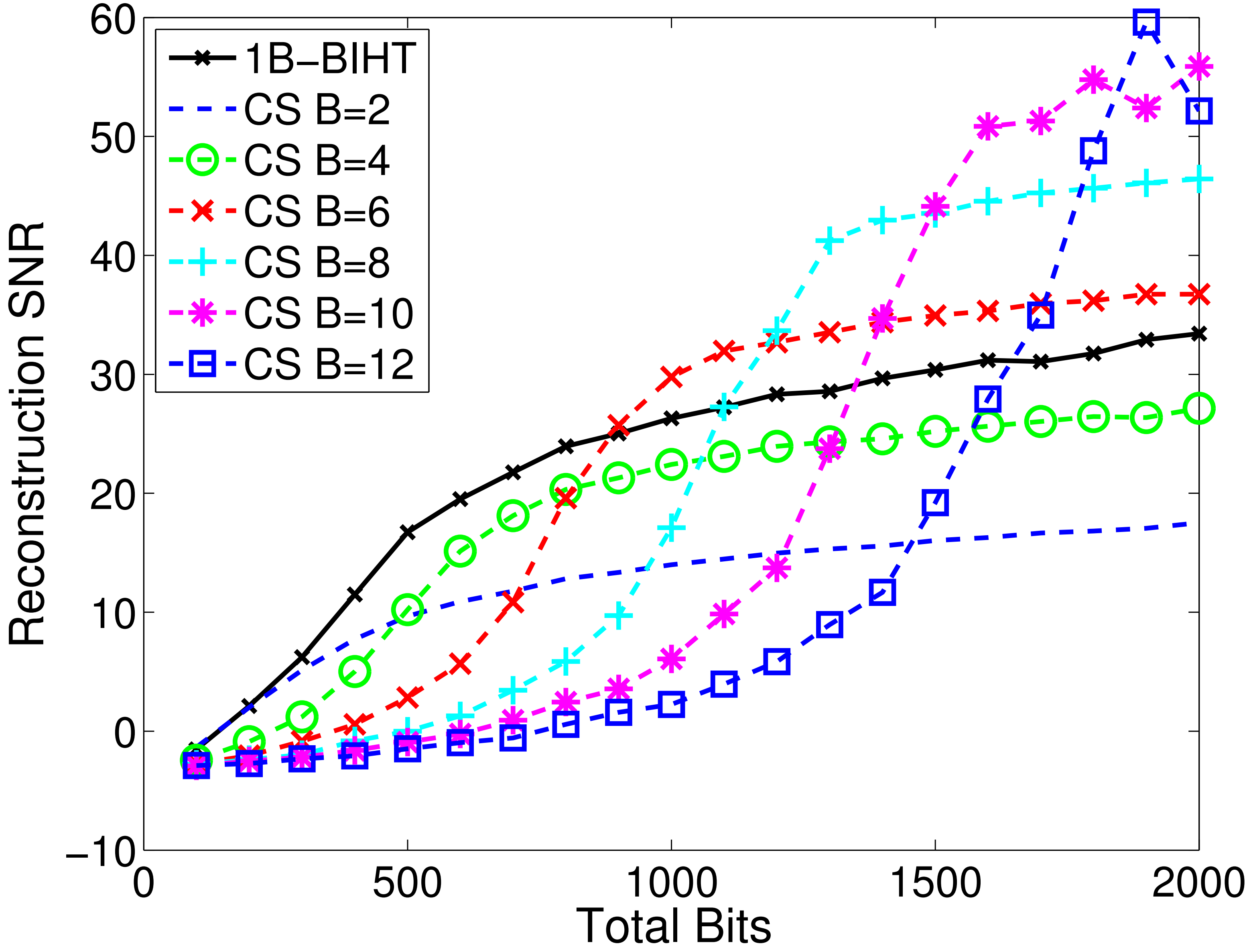


$M/N = 0.7$



$M/N = 1.5$

Remark: CS vs bits/meas.



$N = 2000,\ K = 20$

Bernoulli-Gaussian model

normalized signals

B bits/measurement

$B = 1, \ldots, 12$

$M = \text{Total Bits}/B$

1000 trials

Convex Optimization [Plan, Vershynin, 12]

Let $\boldsymbol{q} = \operatorname{sign}(\boldsymbol{\Phi x})$ for some signal $\boldsymbol{x} \in \mathcal{K} \subset B_2^N$

e.g., sparse, compressible, low-rank matrix

Compute $$\boxed{\hat{\boldsymbol{x}} = \arg\max_{\boldsymbol{u} \in \mathbb{R}^N} \boldsymbol{q}^T \boldsymbol{\Phi u} \quad \text{s.t.} \quad \boldsymbol{u} \in \mathcal{K}}$$

maximize consistency

Convex problem if $\mathcal{K}$ convex!
No ambiguous amplitude definition
($\boldsymbol{u} = 0$ avoided)

S. Bahmani, P.T. Boufounos, B. Raj, "Robust 1-bit Compressive Sensing via Gradient Support Pursuit", arxiv:1304.6626

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Convex problem if $\mathcal{K}$ convex!
No ambiguous amplitude definition
($\boldsymbol{u} = 0$ avoided)

Remark: (PV-L0 problem) [Bahmani, Boufounos, Raj, 13]

$$\hat{\boldsymbol{x}} = \frac{1}{\|\mathcal{H}_K(\boldsymbol{\Phi}^*\boldsymbol{q})\|}\mathcal{H}_K(\boldsymbol{\Phi}^*\boldsymbol{q}) \text{ if } \mathcal{K} = \Sigma_K \text{ !!}$$

S. Bahmani, P.T. Boufounos, B. Raj, "Robust 1-bit Compressive Sensing via Gradient Support Pursuit", arxiv:1304.6626

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maximize consistency

-2 if $\boldsymbol{x}$ is fixed

> **Proposition** *(assuming $\|\boldsymbol{x}\| = 1$) For some $C, c > 0$, if $M \geqslant C\epsilon^{-6} w^2(\mathcal{K})$, then, with $Pr \geqslant 1 - e^{-c\epsilon^2 M}$, we have $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|^2 \leqslant \sqrt{\frac{\pi}{2}}\, \epsilon$.*

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\+ Robust to noise:

noise (bit flip) noise power

Let $\boldsymbol{q}_{\mathrm{n}} = \mathrm{diag}(\boldsymbol{\eta})\,\boldsymbol{q}$ with $\eta_i \in \{\pm 1\}^M$, and assume $d_H(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{n}}) \leqslant p$

(under the same conditions)

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|^2 \leqslant \epsilon\sqrt{\log e/\epsilon} + 11\,p\sqrt{\log e/p}$$

Note: if $M = O(\epsilon^{-2}(p - 1/2)^{-2} K \log N/K)$
this term disappears if $\eta_i = \pm 1$ are iid RVs (with $P(\eta_i = 1) = p$)

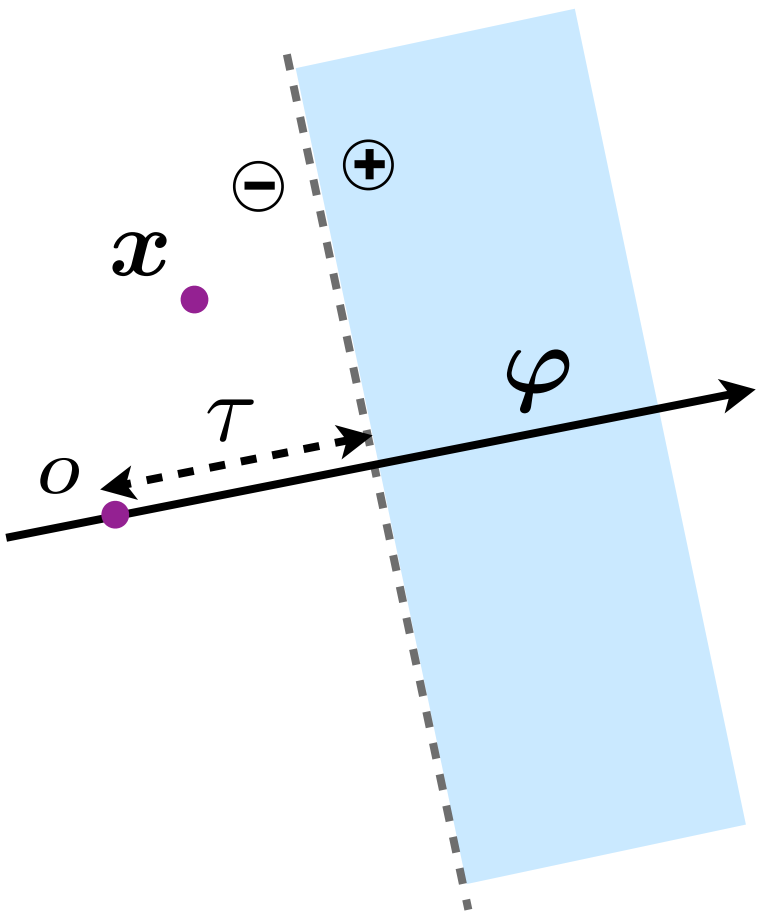
5. Playing with thresholds in 1-bit CS

Thresholds?

- Given $\boldsymbol{x} \in \mathbb{R}^N$ (*e.g.*, sparse)
  Is there an interest in sensing

$$\boxed{\operatorname{sign}\left(\langle \boldsymbol{\varphi}, \boldsymbol{x} \rangle - \tau\right)}$$

  for some (random) $\boldsymbol{\varphi}$ and $\tau \in \mathbb{R}$?



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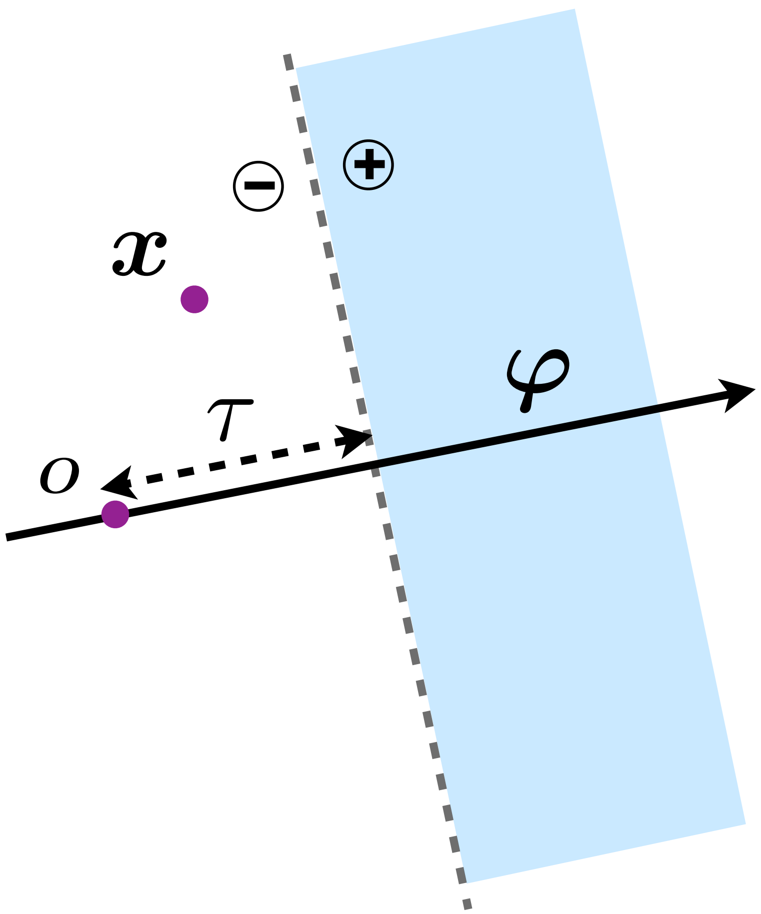
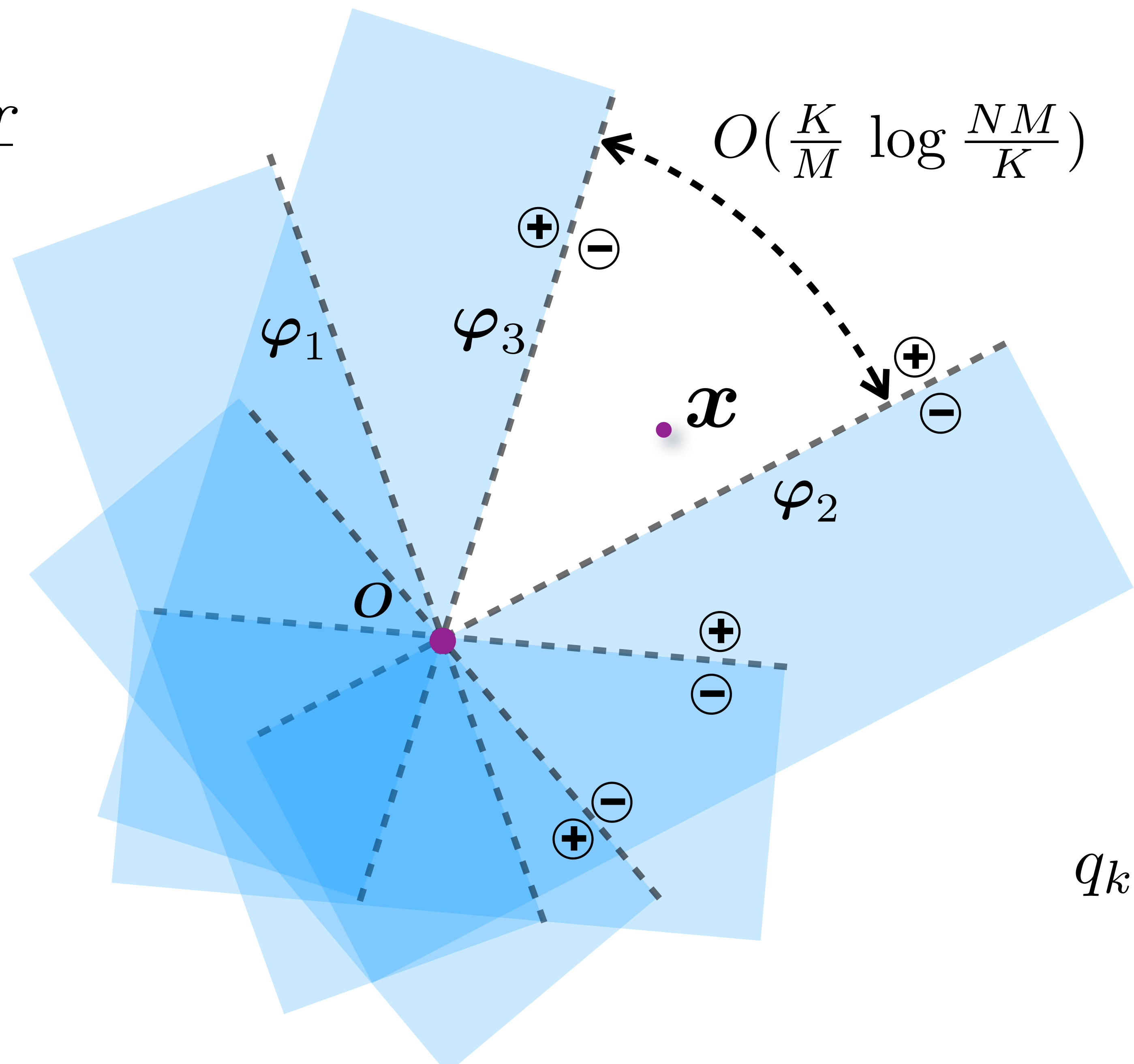


- Two recent applications:
  - adaptive thresholds [Kamilov, Bourquard, Amini, Unser, 12]
  - bridging 1-bit and B-bits QCS [LJ, Degraux, De Vleeschouwer, 13]

1-bit CS with adaptive thresholds

Non-adaptive 1-bit CS $(\tau = 0)$

<u>*Reminder*</u>



$$q_k = \operatorname{sign}\left(\langle \varphi_k, x \rangle\right)$$

1-bit CS with adaptive thresholds

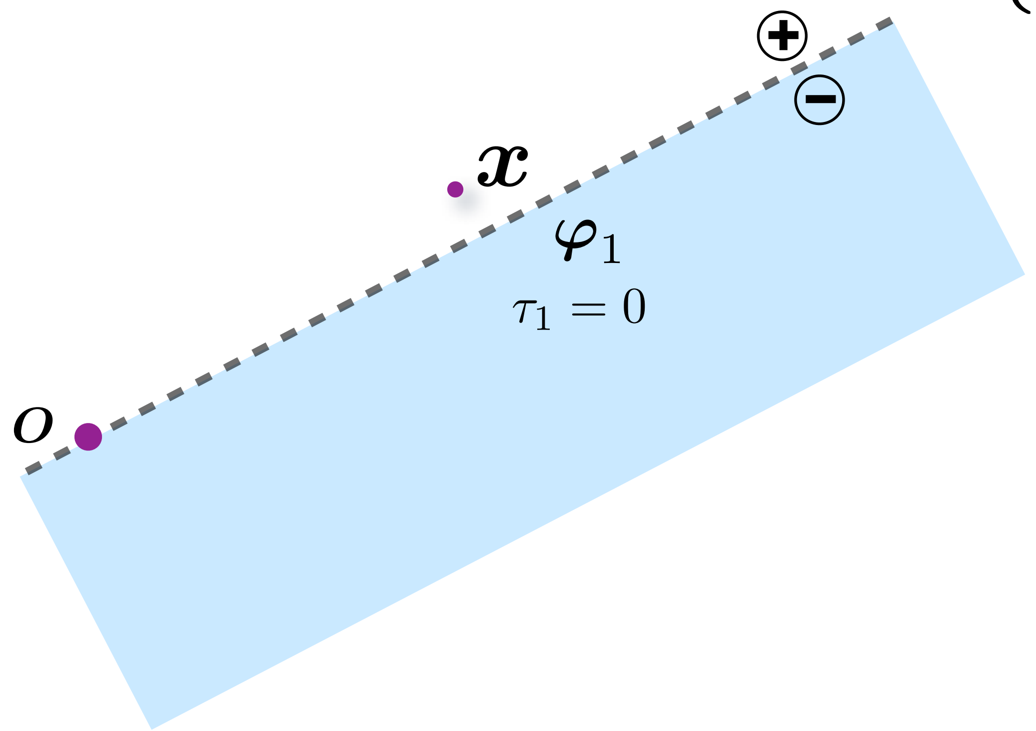
Adaptive 1-bit CS [Kamilov, Bourquard, Amini, Unser, 12]

Given a decoder Rec()

adapted from prev. meas.

$$q_k = \mathrm{sign}\left(\langle \boldsymbol{\varphi}_k, \boldsymbol{x}\rangle - \tau_k\right)$$

$$\begin{cases} \hat{\boldsymbol{x}}_k := \mathrm{Rec}(y_1, \cdots, y_k, \boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_k, \tau_1, \cdots, \tau_k) \\ \tau_{k+1} \text{ s.t. } \langle \boldsymbol{\varphi}_{k+1}, \hat{\boldsymbol{x}}_k\rangle - \tau_{k+1} = 0 \end{cases}$$



U.S. Kamilov, A. Bourquard, A. Amini, M. Unser,
"One-bit measurements with adaptive thresholds". Signal Processing Letters, IEEE, 19(10), 607-610.

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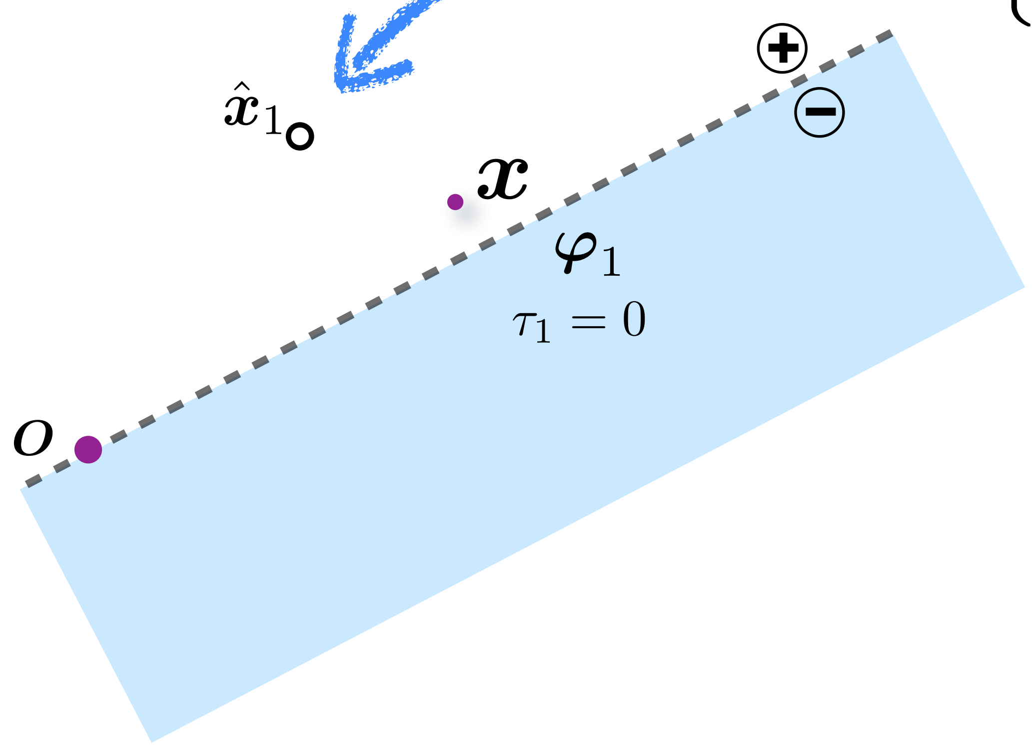

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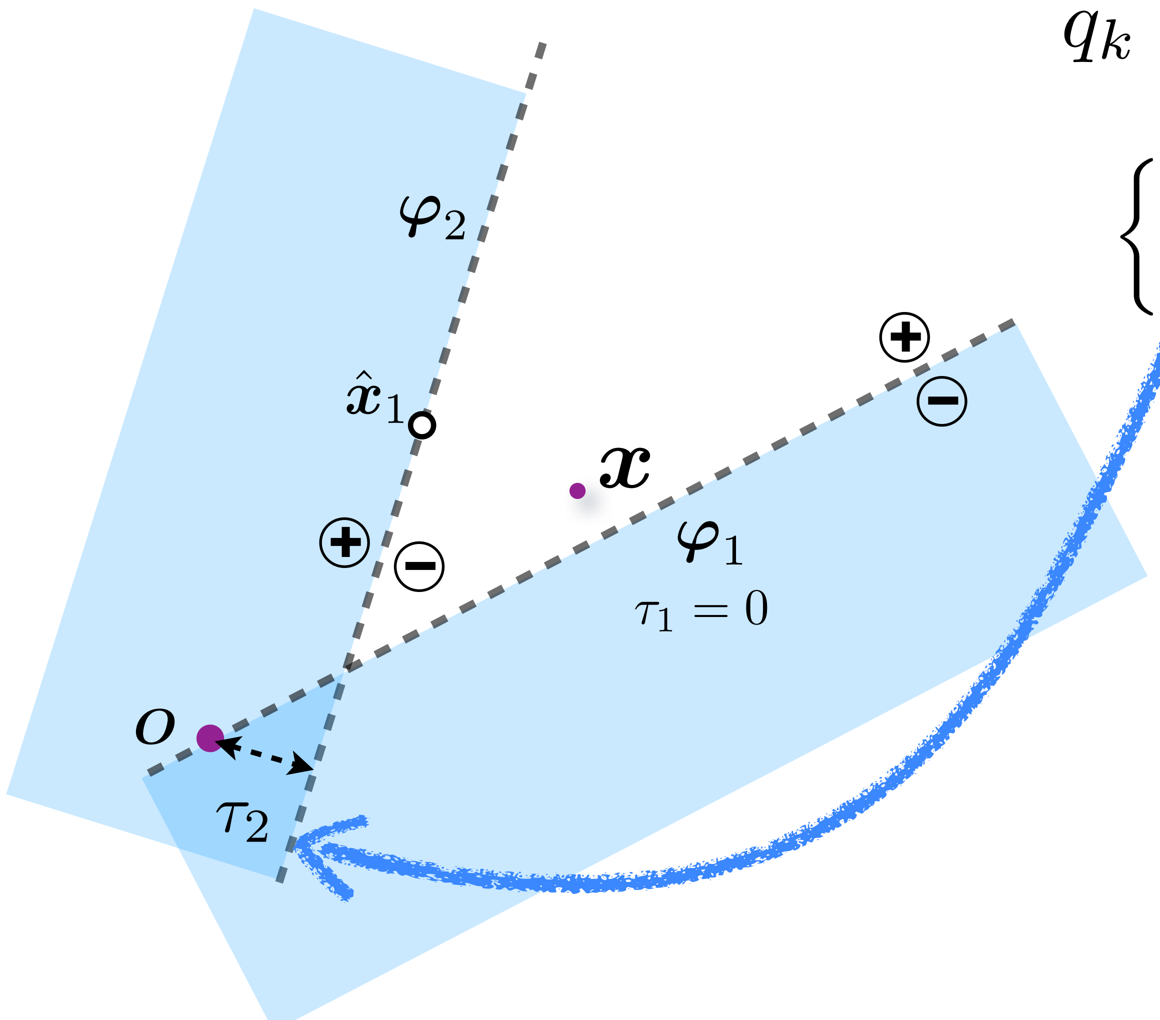

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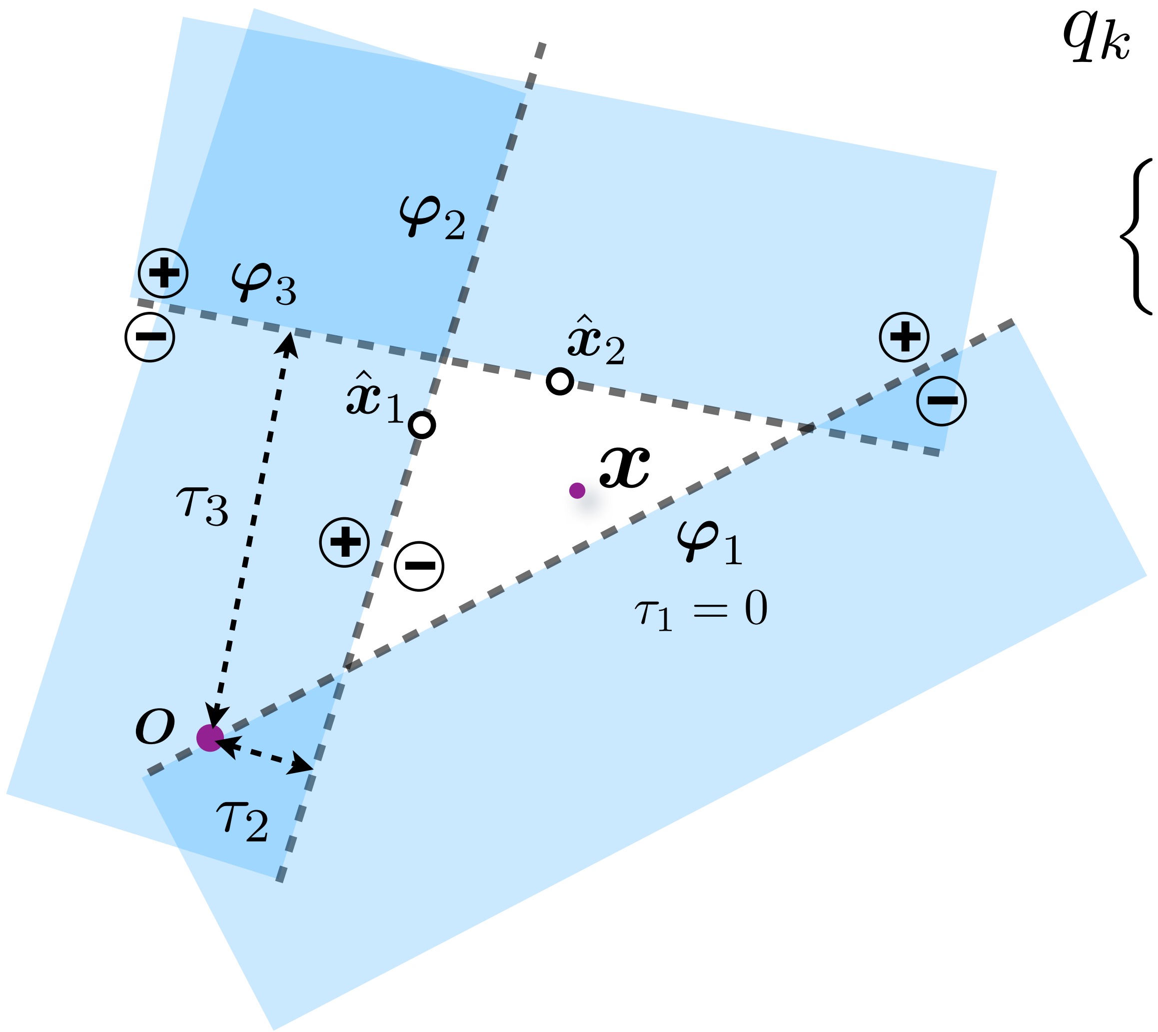

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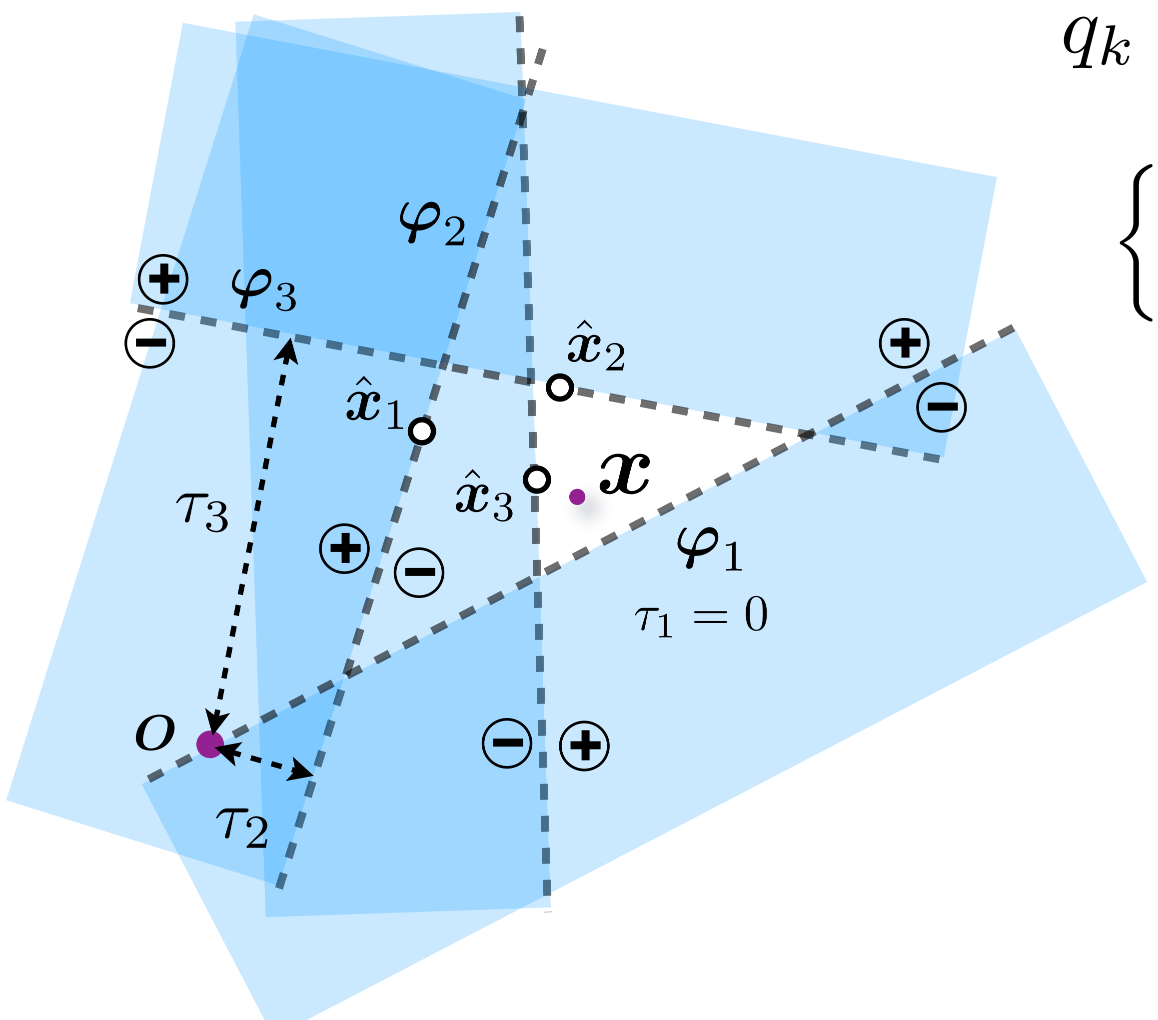

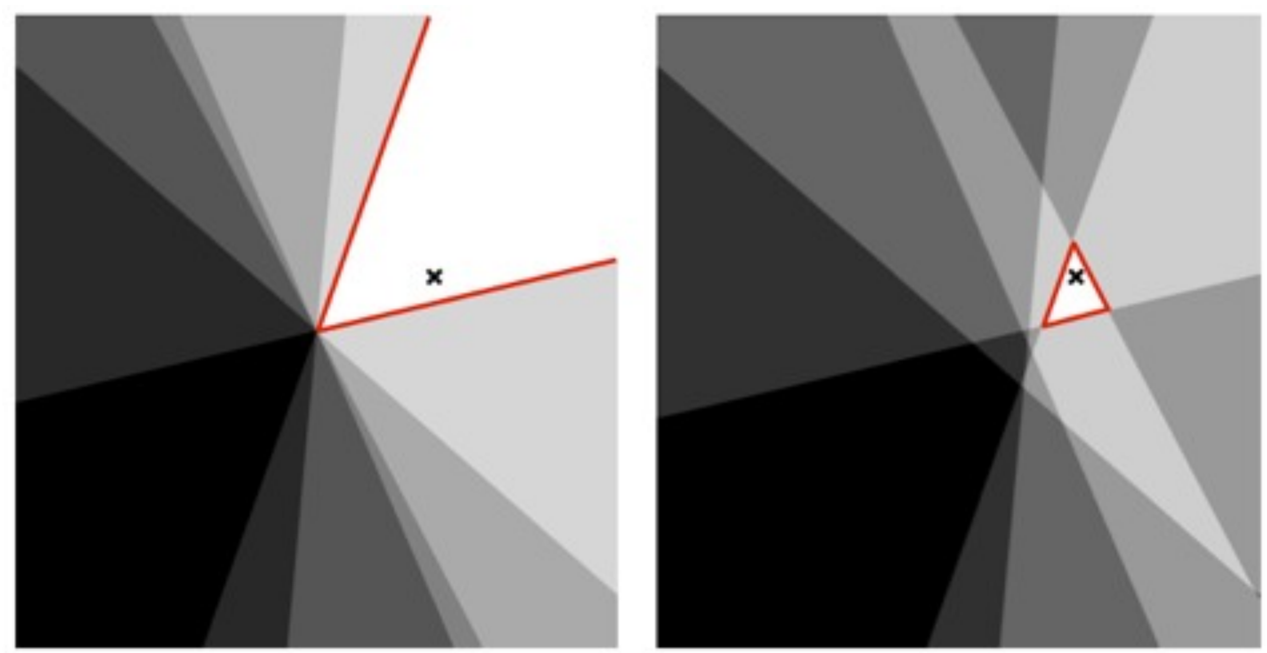

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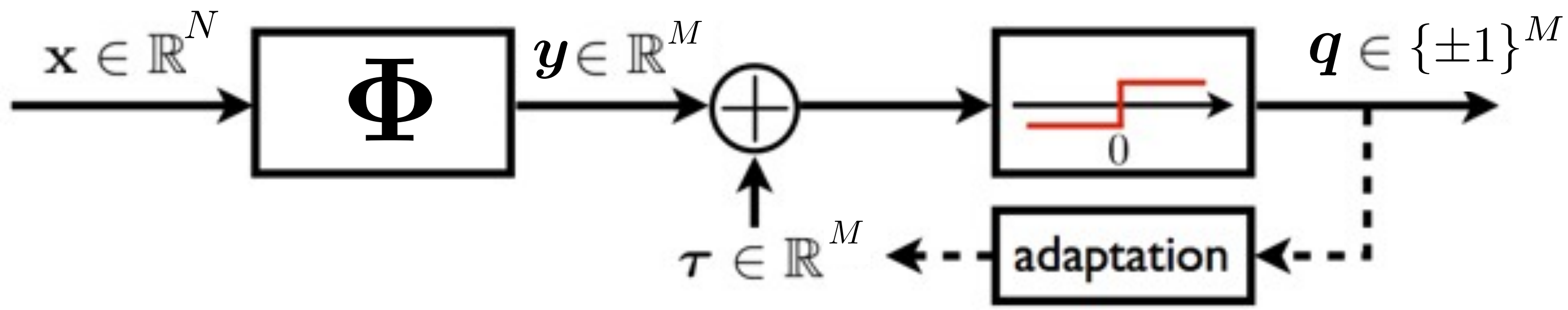
1-bit CS with adaptive thresholds

System view:

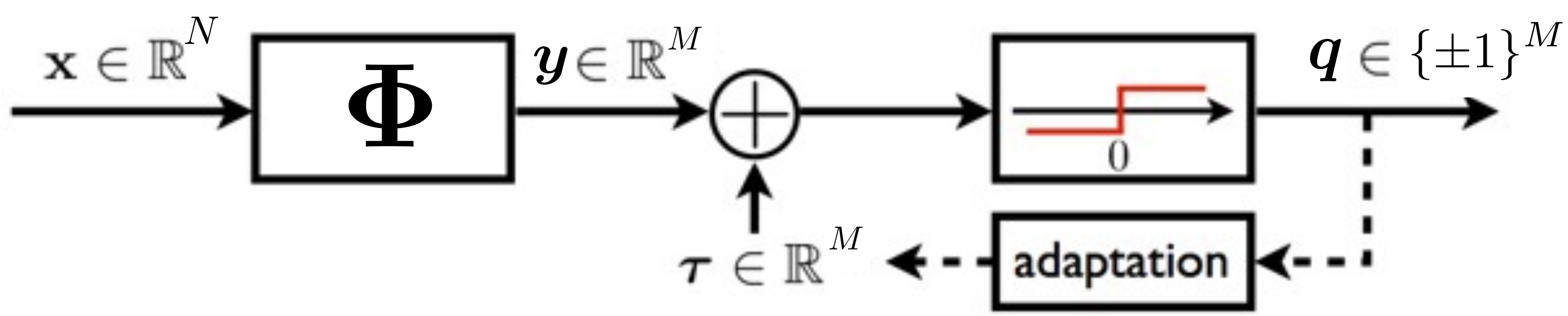


Kind of
$\Sigma\Delta$ loop

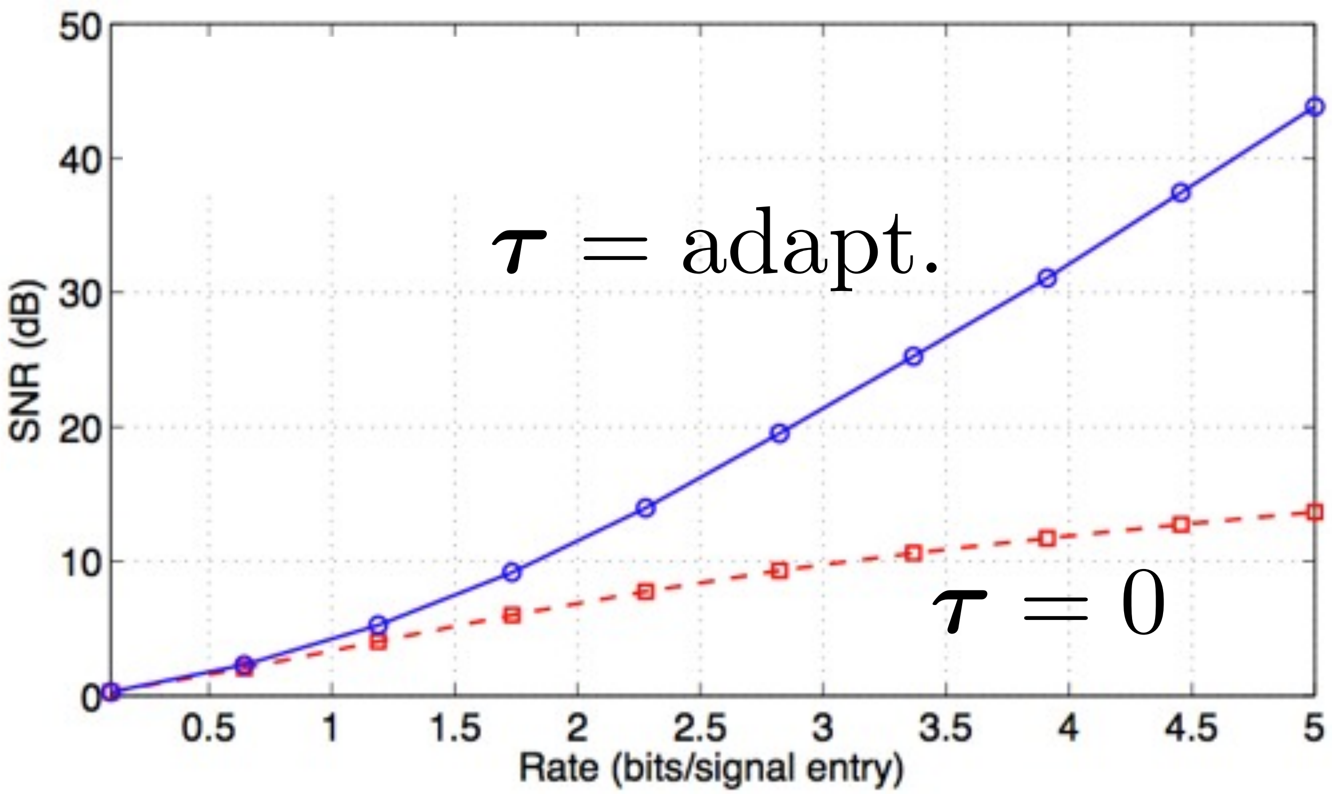
U.S. Kamilov, A. Bourquard, A. Amini, M. Unser,
"One-bit measurements with adaptive thresholds". Signal Processing Letters, IEEE, 19(10), 607-610.

1-bit CS with adaptive thresholds

System view:



Kind of $\Sigma\Delta$ loop



Rec() set to Generalized Approximate Message Passing

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Bridging 1-bit & B-bit CS?

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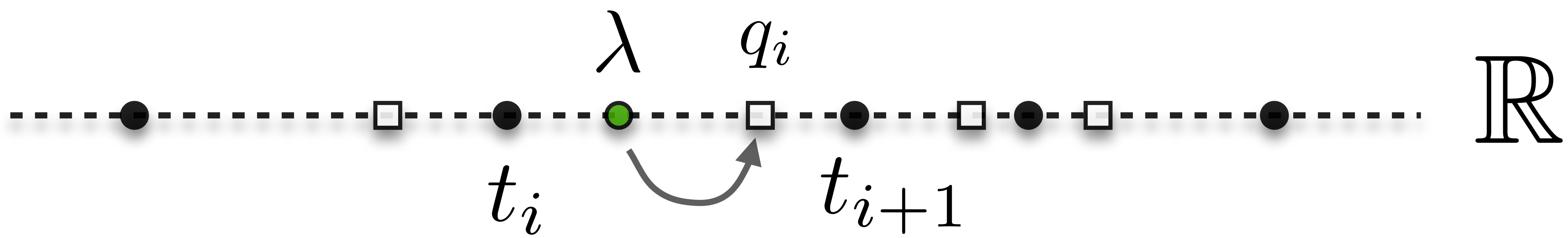
- B-bit quantizer defined with thresholds:



$$\lambda \in \mathcal{R}_i = [t_i, t_{i+1}) \Leftrightarrow \operatorname{sign}(\lambda - t_i) = +1 \ \& \ \operatorname{sign}(\lambda - t_{i+1}) = -1$$

- Can we combine multiple thresholds in 1-bit CS?

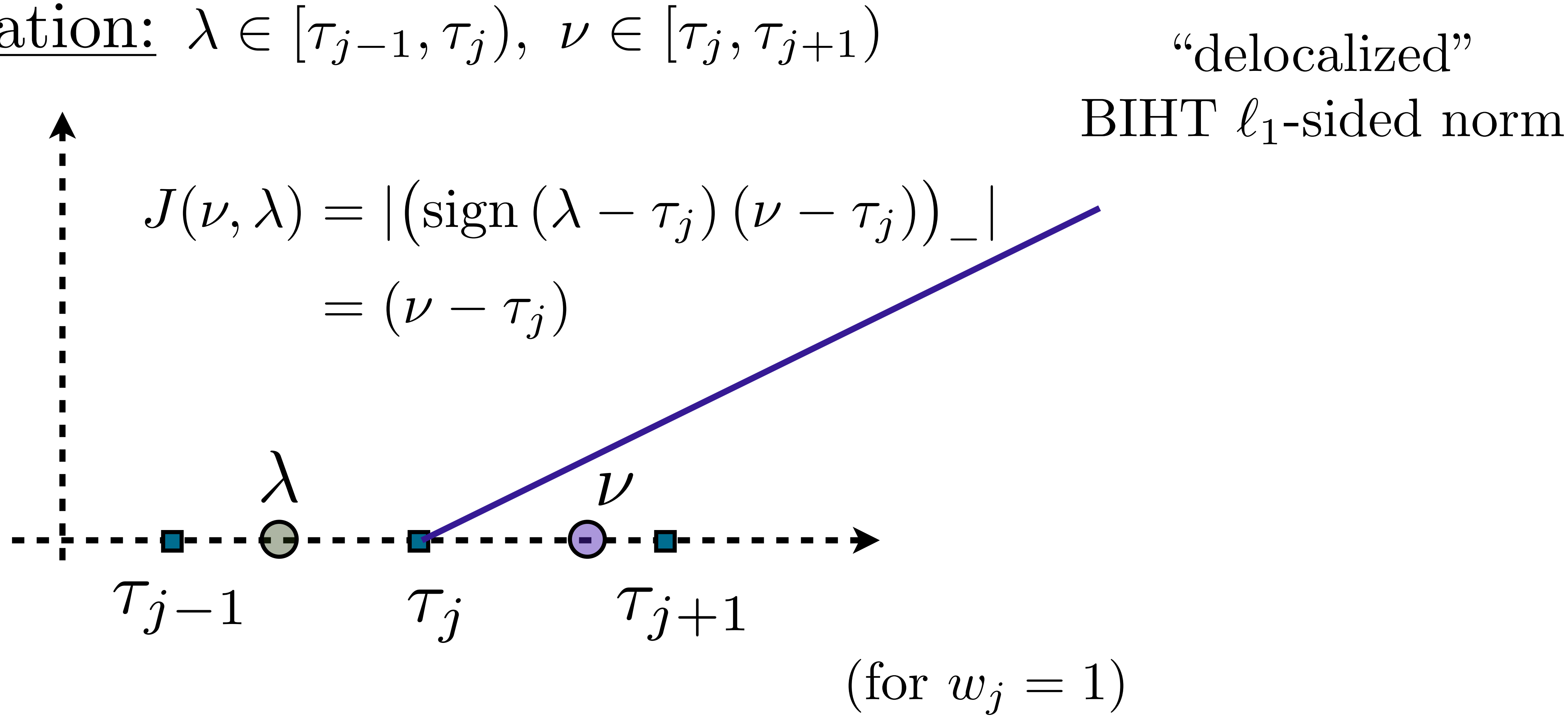
Bridging 1-bit & B-bit CS?

Given $\mathcal{T} = \{\tau_j\}$ and $\Omega = \{q_j\}$ ($|\mathcal{T}| = 2^B + 1 = |\Omega| + 1$), let's define

$$\boxed{J(\nu, \lambda) = \sum_{j=2}^{2^B} w_j \left|\left(\operatorname{sign}(\lambda - \tau_j)\,(\nu - \tau_j)\right)_-\right|,}$$

with $w_j = q_j - q_{j-1}$.

<u>Illustration:</u> $\lambda \in [\tau_{j-1}, \tau_j),\ \nu \in [\tau_j, \tau_{j+1})$



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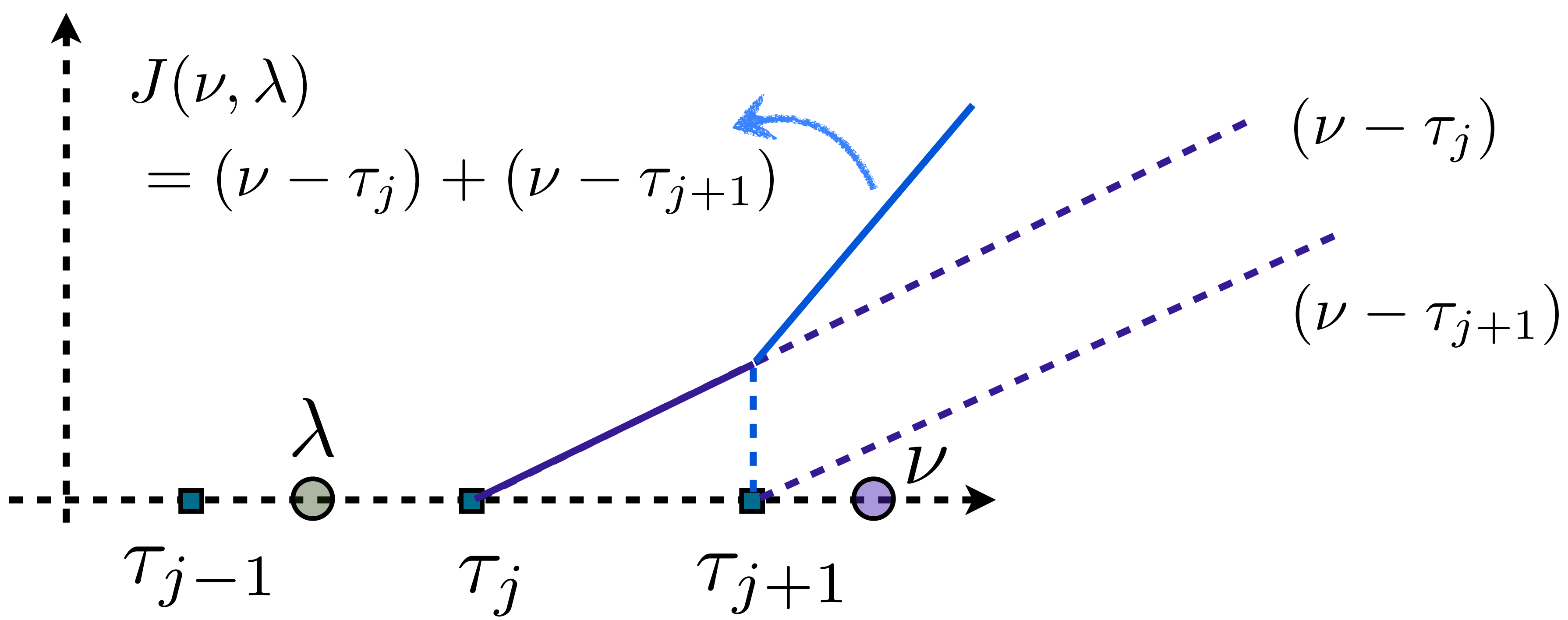
<u>Illustration:</u> $\lambda \in [\tau_{j-1}, \tau_j),\ \nu \in [\tau_{j+1}, \tau_{j+2})$



(for $w_j = 1$)

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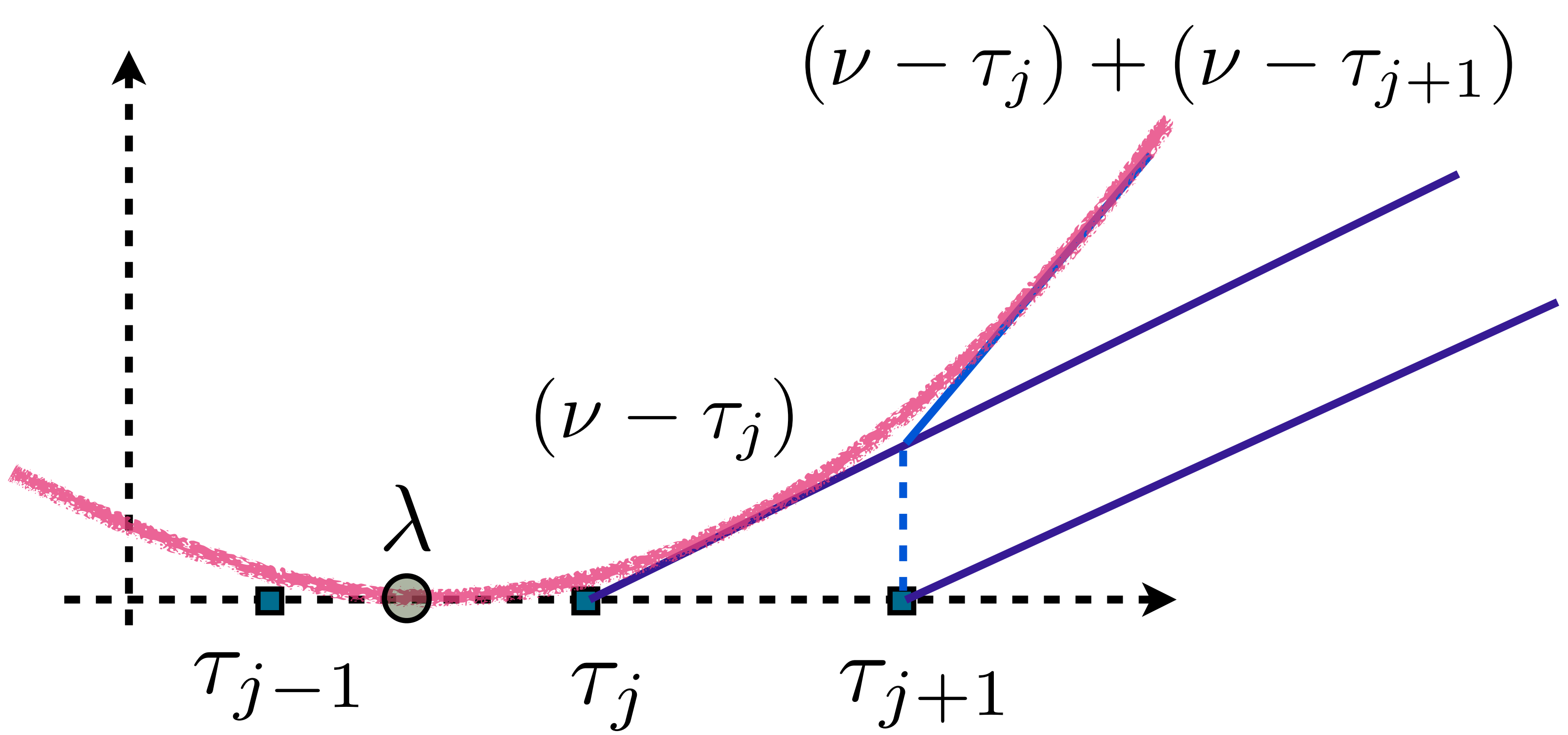

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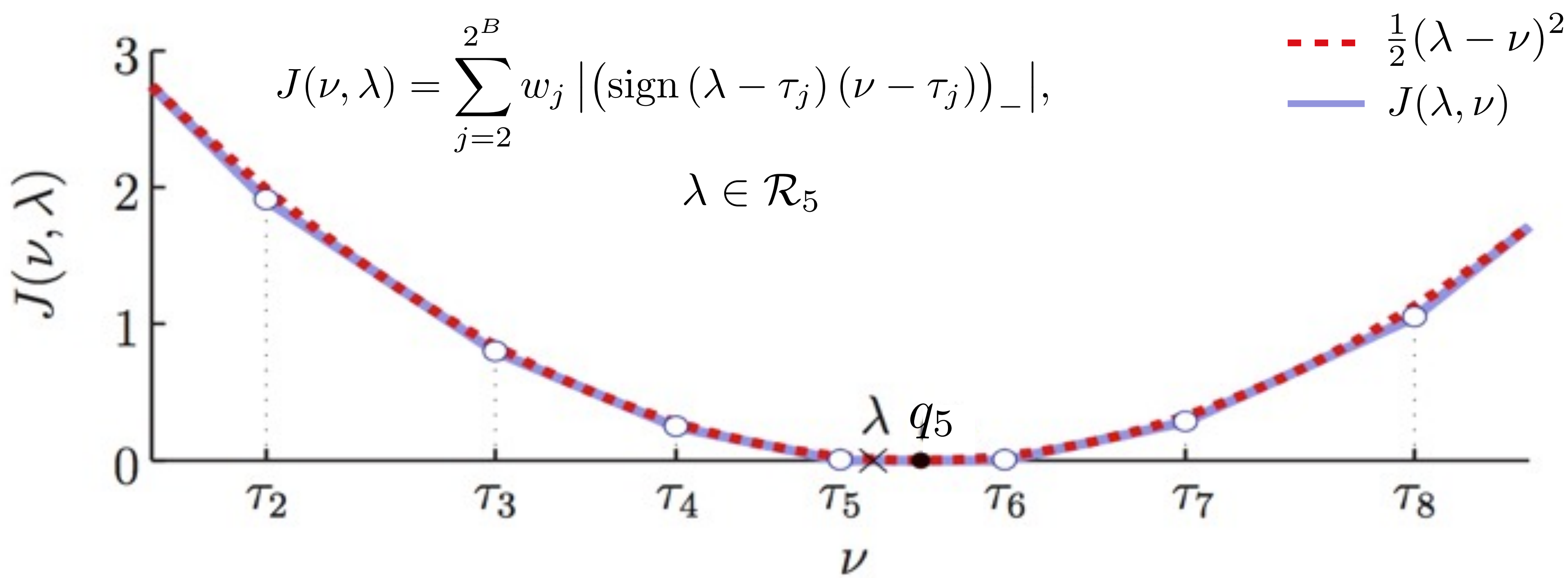
<u>Illustration:</u> *more bins*



Bridging 1-bit & B-bit CS?

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with $w_j = q_j - q_{j-1}$.

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^M$: $\boxed{\mathcal{J}(\boldsymbol{u}, \boldsymbol{v}) := \sum_{k=1}^M J(u_k, v_k)}$

Remarks:

- J is convex in ν
- For $B = 1$ ($j = 2$ only):

  $\mathcal{J}(\boldsymbol{u}, \boldsymbol{v}) \propto \|(\mathrm{sign}\,(\boldsymbol{v}) \odot \boldsymbol{u})_-\|_1 \to \ell_1$-sided 1-bit energy
- For $B \gg 1$:

  $J(\nu, \lambda) \to \frac{1}{2}(\nu - \lambda)^2$ and $\mathcal{J}(\boldsymbol{u}, \boldsymbol{v}) \to \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{v}\|^2$ (quadratic energy)

Bridging 1-bit & B-bit CS?

- Let's define an *inconsistency* energy:

$$\boxed{\mathcal{E}_B(\boldsymbol{u}) := \mathcal{J}(\boldsymbol{\Phi u}, \boldsymbol{q})} \text{ with } \boldsymbol{q} = \mathcal{Q}_B[\boldsymbol{\Phi x}] \text{ and } \mathcal{E}\_B(\boldsymbol{x}) = 0$$

- Idea: Minimize it in Σ_K (as for Iterative Hard Thresholding)

[Blumensath, Davies, 08]

$$\boxed{\min_{\boldsymbol{u}\in\mathbb{R}^N} \mathcal{E}_B(\boldsymbol{u}) \text{ s.t. } \|\boldsymbol{u}\|_0 \leqslant K,}$$

T. Blumensath, M.E. Davies, "Iterative thresholding for sparse approximations". *Journal of Fourier Analysis and Applications*, *14*(5-6), 629-654. (2008).

Bridging 1-bit & B-bit CS?

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- NP Hard but greedy solution (as for IHT):

$$\boxed{\boldsymbol{x}^{(n+1)} = \mathcal{H}_K[\boldsymbol{x}^{(n)} - \mu\, \underbrace{\partial\, \mathcal{E}_B(\boldsymbol{x}^{(n)})}_{\text{(sub) gradient}}] \text{ and } \boldsymbol{x}^{(0)} = 0.}$$

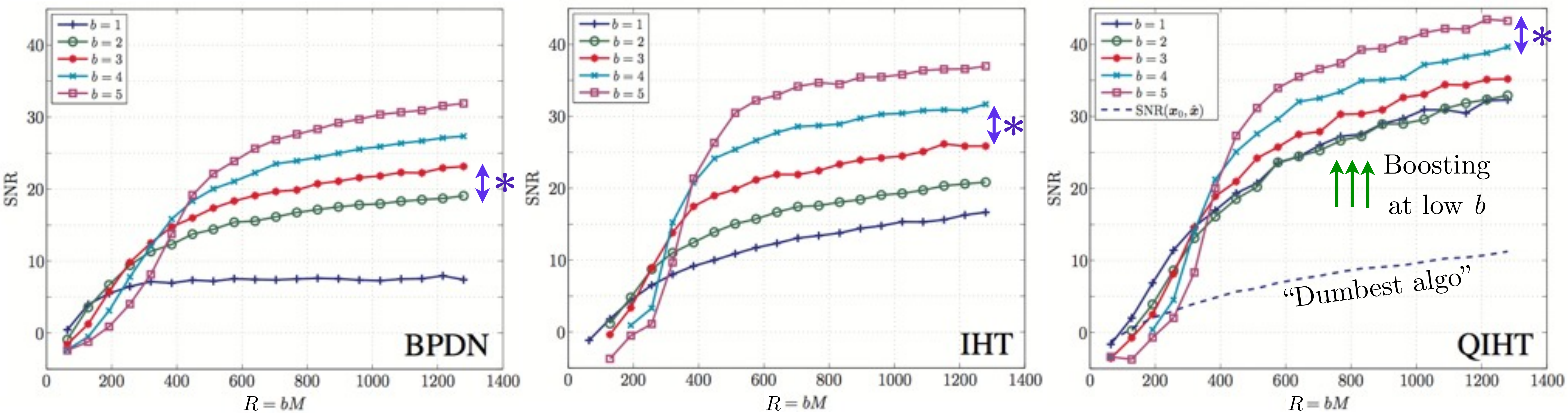
$$\boldsymbol{\Phi}^*(\operatorname{sign}(\boldsymbol{\Phi u}) - \operatorname{sign}(\boldsymbol{\Phi x})) \xleftarrow[B=1]{} \boxed{\partial\, \mathcal{E}_B(\boldsymbol{u}) = \boldsymbol{\Phi}^*(\mathcal{Q}_B(\boldsymbol{\Phi u}) - \boldsymbol{q})} \xrightarrow[B \gg 1]{} \boldsymbol{\Phi}^*(\boldsymbol{\Phi u} - \boldsymbol{q})$$

BIHT! | Quantized IHT (QIHT) | IHT!

T. Blumensath, M.E. Davies, "Iterative thresholding for sparse approximations". *Journal of Fourier Analysis and Applications*, *14*(5-6), 629-654. (2008).
LJ, K. Degraux, C. De Vleeschouwer, "Quantized Iterative Hard Thresholding: Bridging 1-bit and High-Resolution Quantized Compressed Sensing", SAMPTA2013

Bridging 1-bit & B-bit CS?

$N = 1024,\ K = 16,\ R = BM \in \{64, 128, \cdots, 1280\}$, 100 trials (+ Lloyd-Max Gauss. Q.)



R: total bit budget (BM)

*: almost "6dB per bit" gain

$\mu = \frac{1}{M}(1 - \sqrt{2K/M})$

Adjusted by limit case analysis: BIHT and IHT

Note: entropy could be computed instead of B (*e.g.*, for further efficient coding)

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Bridging 1-bit & B-bit CS?

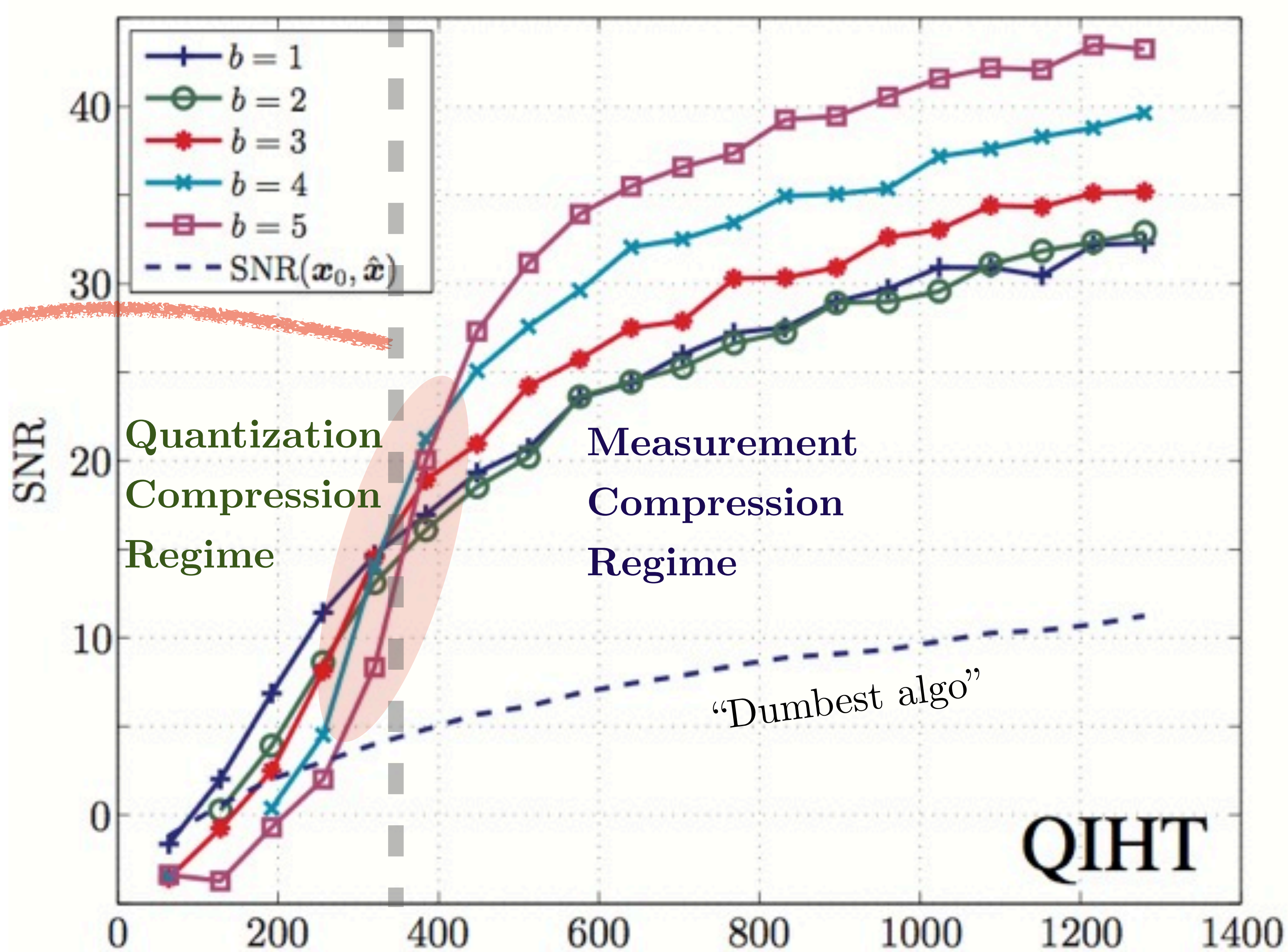
$N = 1024,\ K = 16,\ R = BM \in \{64, 128, \cdots, 1280\}, 100$ trials

Interesting transition at

$R_0 \simeq 375$

"**Regime Change**?" [Laska, Baraniuk, 12]

R_0 could increase with input noise power.



$R\!=\!BM$

J. N. Laska, R. G. Baraniuk, 'Regime change: Bit-depth versus measurement-rate in compressive sensing", *Signal Processing, IEEE Transactions on*, *60*(7), 3496-3505. (2012)

Further Reading

- T. Blumensath, M.E. Davies, "Iterative thresholding for sparse approximations". *Journal of Fourier Analysis and Applications*, 14(5-6), pp. 629-654, 2008
- P. T. Boufounos and R. G. Baraniuk, "1-Bit compressive sensing," *Proc. Conf. Inform. Science and Systems (CISS)*, Princeton, NJ, March 19-21, 2008.
- Boufounos, P. T. (2009, November). "Greedy sparse signal reconstruction from sign measurements". In *Conference Record of the Forty-Third Asilomar Conference on Signals, Systems and Computers*, 2009
- Y. Plan, R. Vershynin, "Dimension reduction by random hyperplane tessellations", arXiv:1111.4452, 2011.
- Y. Plan, R. Vershynin, "Robust 1-bit compressed sensing and sparse logistic regression: a convex programming approach", *IEEE Trans. Info. Theory*, arXiv:1202.1212, 2012.
- J. N. Laska, R. G. Baraniuk, 'Regime change: Bit-depth versus measurement-rate in compressive sensing", *IEEE Trans. Signal Processing*, 60(7), pp. 3496-3505, 2012.
- U.S. Kamilov, A. Bourquard, A. Amini, M. Unser, "One-bit measurements with adaptive thresholds". *IEEE Signal Processing Letters*, 19(10), pp. 607-610, 2012
- L. Jacques, J. N. Laska, P. T. Boufounos, and R. G. Baraniuk, "Robust 1-Bit Compressive Sensing via Binary Stable Embeddings of Sparse Vectors," *IEEE Trans. Info. Theory*, 59(4), 2013.
- L. Jacques, K. Degraux, C. De Vleeschouwer, "Quantized Iterative Hard Thresholding: Bridging 1-bit and High-Resolution Quantized Compressed Sensing", SAMPTA 2013, to appear.

Thank you!