Rank-one projections for compressive radio interferometric imaging

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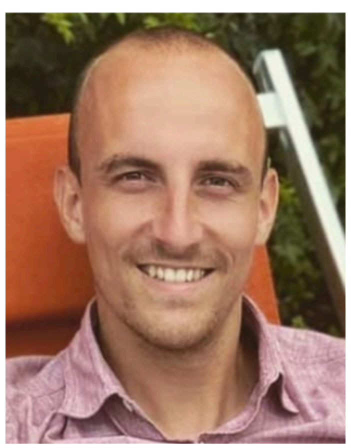

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Introduction to random projections and compressive sensing

The multiple use of random projections in “data science”

Object

“$\boldsymbol{x}$”

audio signal volume image distribution measure

$\boldsymbol{\Phi}$ →

Random projection

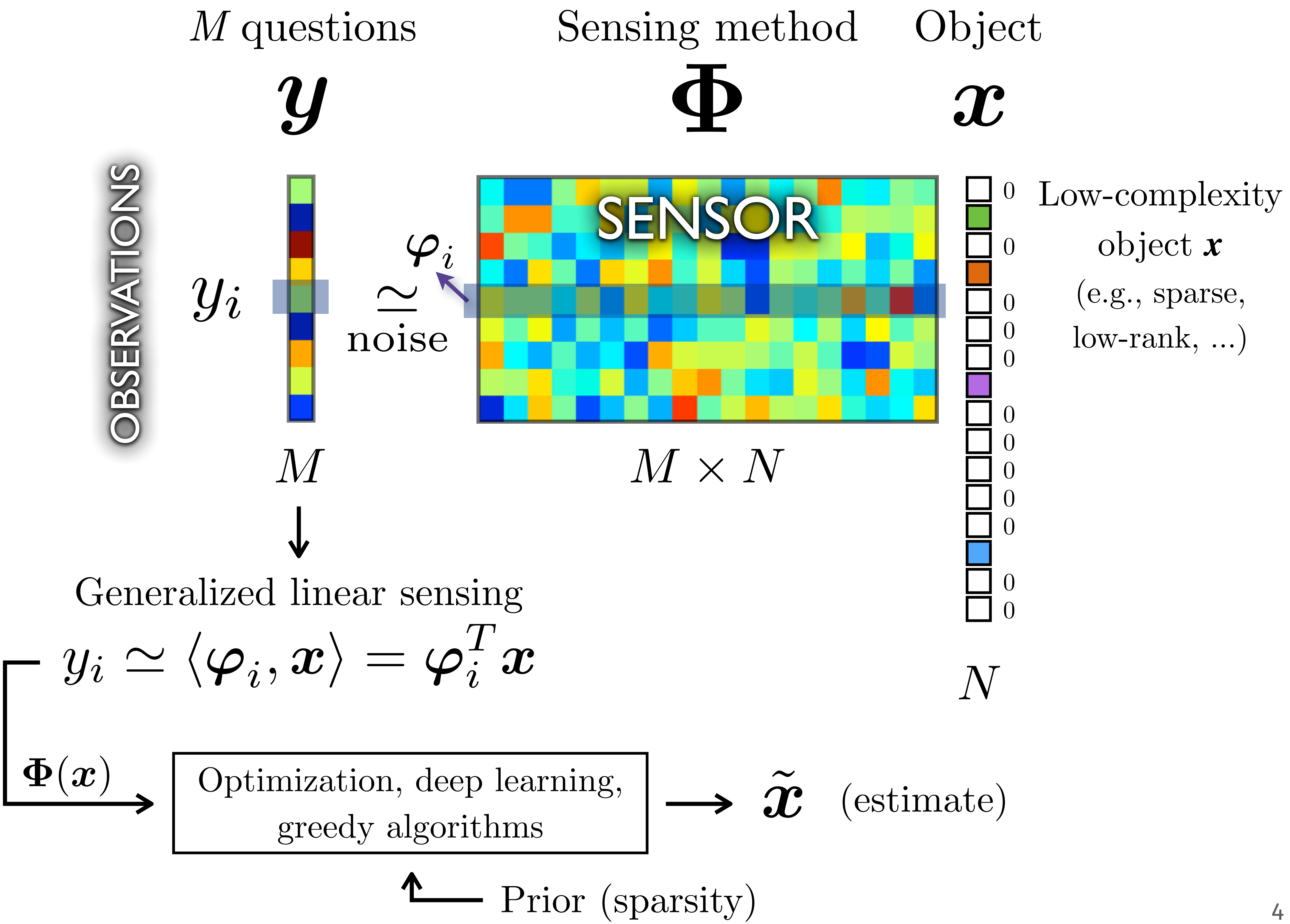
$\boldsymbol{y} = \boldsymbol{\Phi}(\boldsymbol{x})$

smaller dimension physics-driven distinctive compressed binarized

Random “projections" are ubiquitous in:

- Data mining & dimensionality reduction techniques
- Sensing and imaging methods (optics, astronomy, ...)
- Machine learning (sketching, explicit kernel, initialization, ...)
- Randomized numerical methods
- ...

Compressive sensing...



Embedding of sparse vectors / signals

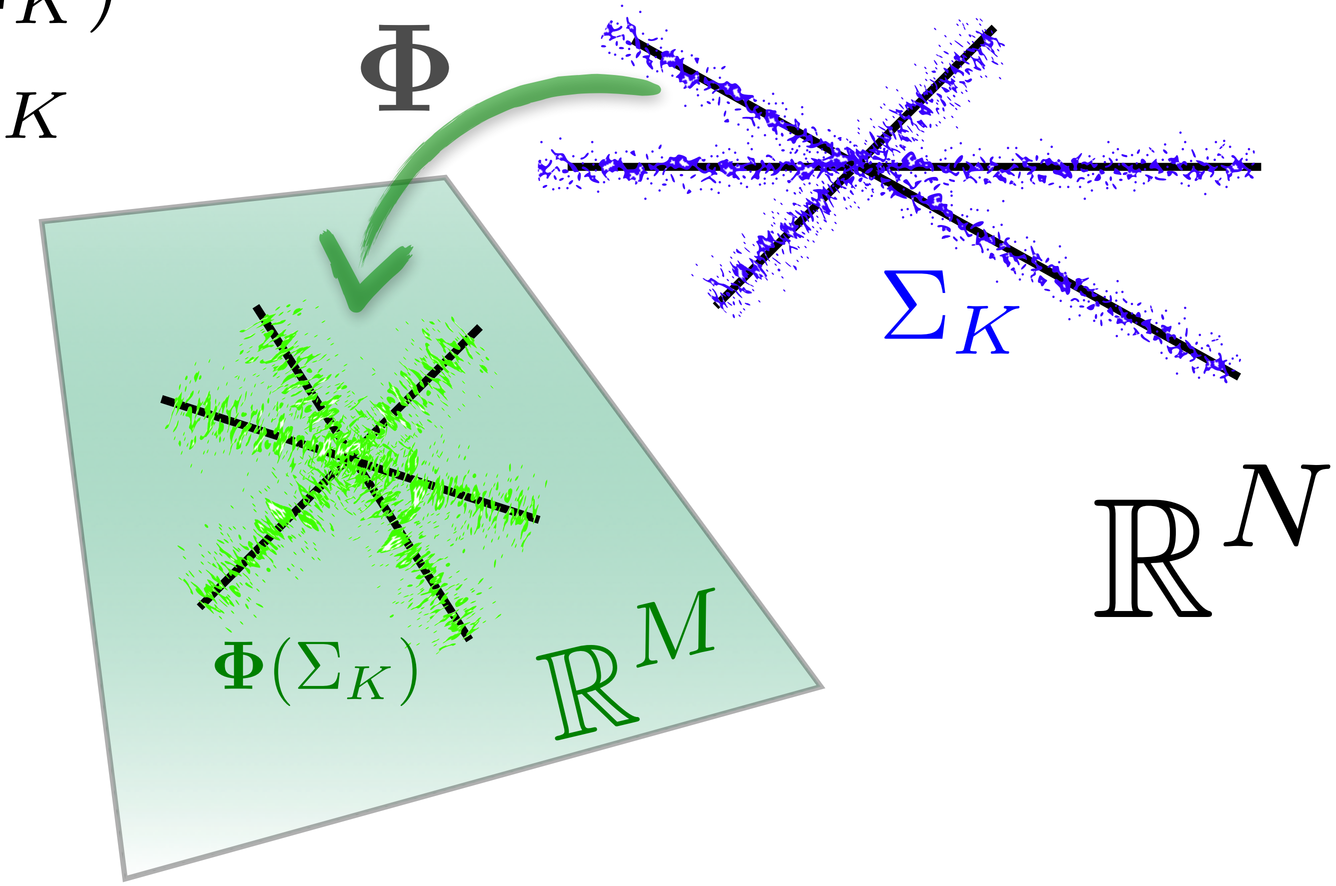
Two K-sparse signals $\boldsymbol{x}, \boldsymbol{x}' \in \Sigma_K := \{\boldsymbol{u} : \|\boldsymbol{u}\|_0 := |\operatorname{supp} \boldsymbol{u}| \leqslant K\}$
At most K non-zero elements

For many random $M \times N$ matrices $\boldsymbol{\Phi}$ (e.g., Gaussian, Bernoulli, structured) and "$M \gtrsim K \log(N/K)$", with high probability,

Geometry of $\boldsymbol{\Phi}(\Sigma_K)$
$\approx$ Geometry of Σ_K

$\boldsymbol{\Phi x} \approx \boldsymbol{\Phi x}' \Leftrightarrow \boldsymbol{x} \approx \boldsymbol{x}'$

observations — true signals



+ extension to other sparsity models, low-rankness, ...

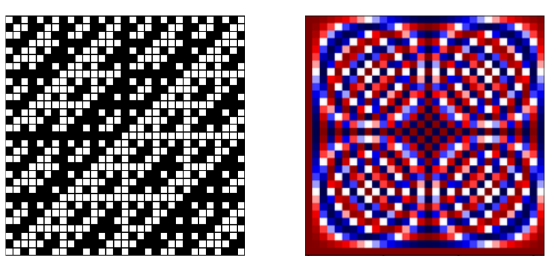
Structured random projections

Challenge: dense matrices $\boldsymbol{\Phi}$ not optimal for:

- memory and computational complexity
- physically friendly implementation
- sensing higher dimensional objects

Other solutions:

- Fourier (FFT) or Hadamard matrices



$\longrightarrow$ random subsampling & modulation $\longrightarrow$ $\boldsymbol{\Phi}$

- Rank-one projections (ROP) 

Object to project = symmetric $n \times n$ matrices $\boldsymbol{X} \in \mathbb{R}^{n \times n}$:

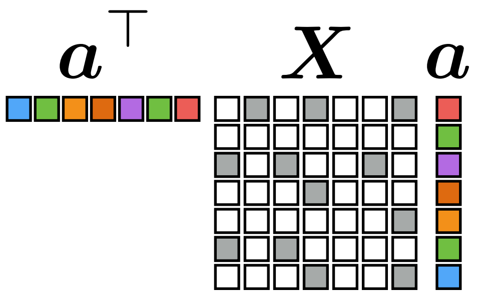
e.g., image, volume, covariance matrices, ...

Projection with m random vectors $\{\boldsymbol{a}_j \sim_{\text{iid}} \boldsymbol{a}\}_{j=1}^m \subset \mathbb{R}^n$

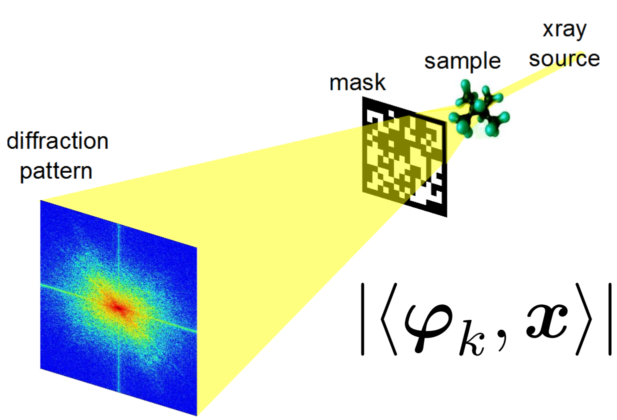
(e.g., Gaussian)

$$\boldsymbol{y} := \boldsymbol{\Phi}(\boldsymbol{X}) := \big(\underbrace{\boldsymbol{a}_j^\top \boldsymbol{X} \boldsymbol{a}_j}_{\langle \boldsymbol{a}_j \boldsymbol{a}_j^\top, \boldsymbol{X} \rangle_F} \big)_{j=1}^m \in \mathbb{R}^m$$

rank-one



Phase retrieval



$$|\langle \boldsymbol{\varphi}_k, \boldsymbol{x} \rangle|^2 = \boldsymbol{\varphi}_k^* (\boldsymbol{x}\boldsymbol{x}^*) \boldsymbol{\varphi}_k$$

Covariance matrix estimation

$$\mathcal{A}(\mathbb{E}\boldsymbol{x}\boldsymbol{x}^\top) \approx \mathcal{A}(\tfrac{1}{N} \textstyle\sum_k \boldsymbol{x}_k \boldsymbol{x}_k)$$
$$= \tfrac{1}{N} \textstyle\sum_k [(\boldsymbol{a}_j^\top \boldsymbol{x}_k)^2]_{j=1}^m$$

for $\boldsymbol{x}_k \sim_{\text{iid}} \boldsymbol{x}$

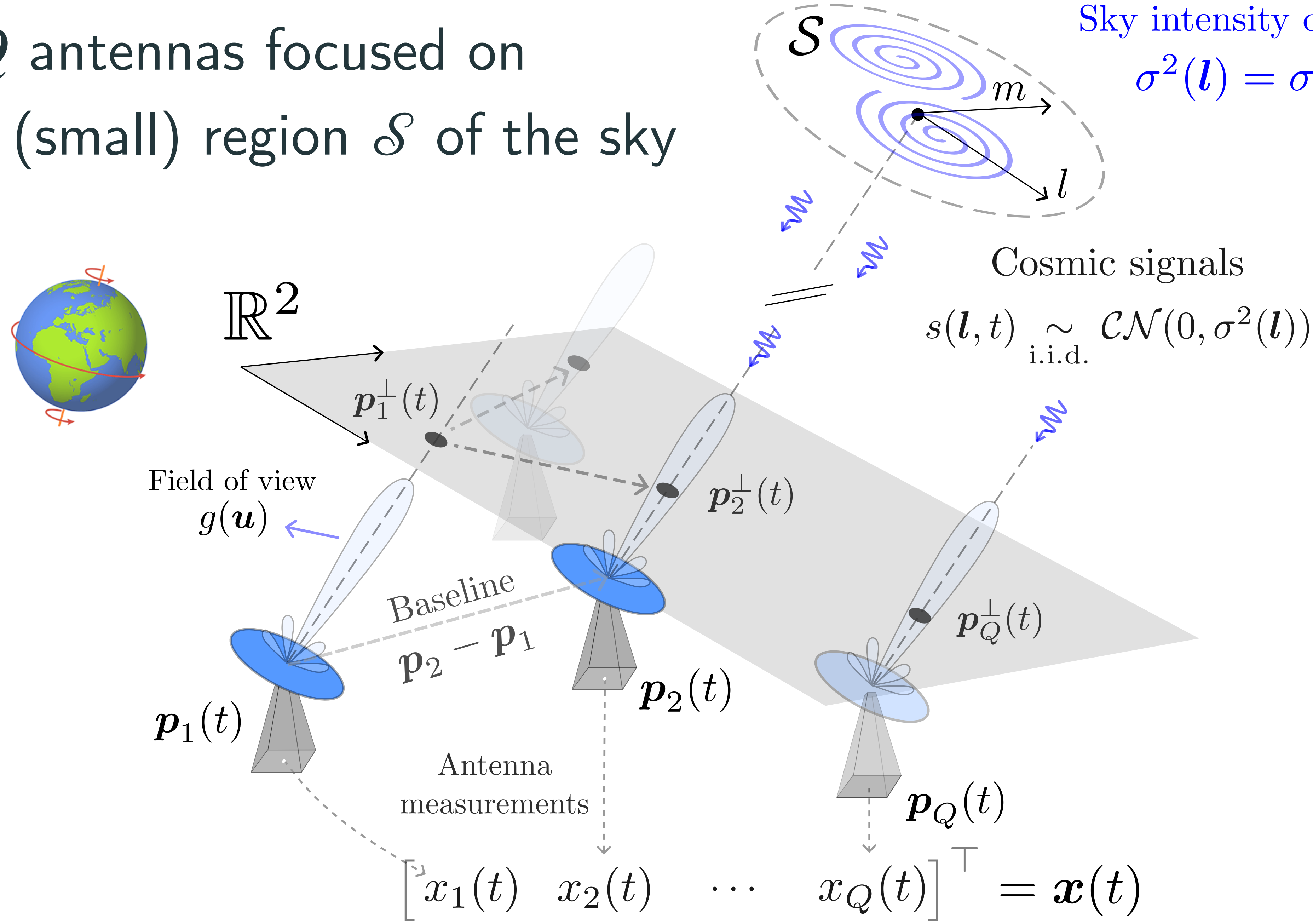
Chen & Goldsmith (2015) Cai & Zhang (2015)

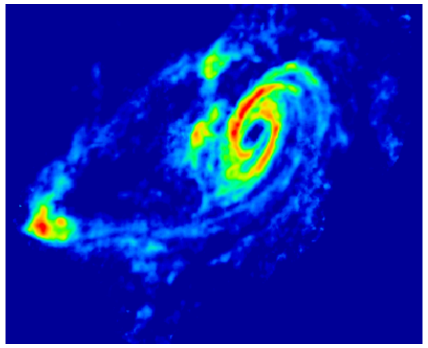
Acquisition and imaging models in radio astronomy

Radio interferometric sensing model

Q antennas focused on
a (small) region $\mathcal{S}$ of the sky

Sky intensity distribution
$\sigma^2(\boldsymbol{l}) = \sigma^2(l, m)$





$\sigma^2(\boldsymbol{l})$

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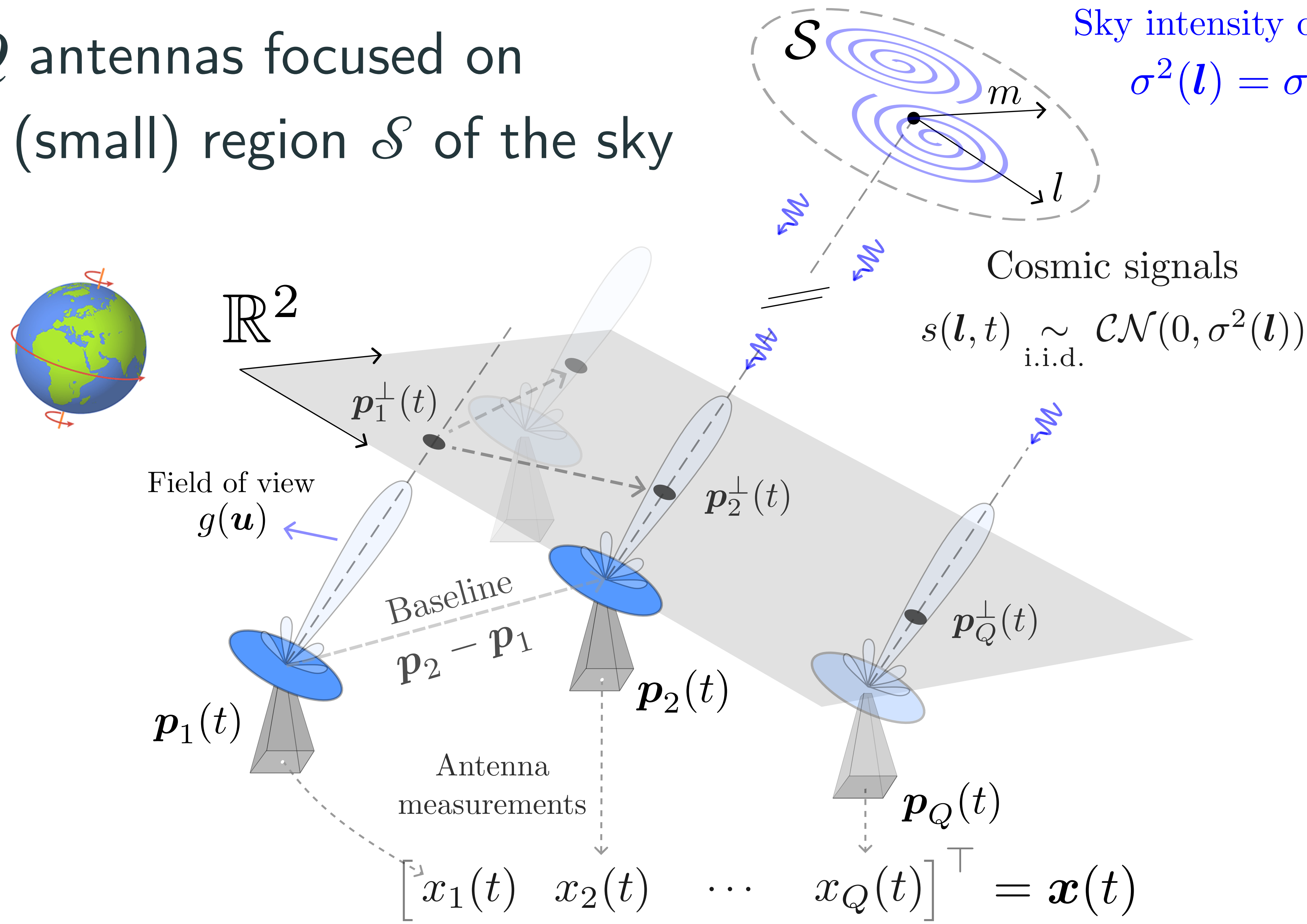
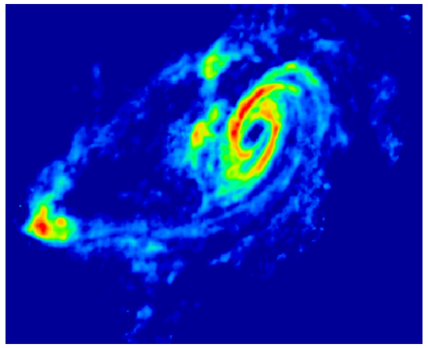




$\sigma^2(\boldsymbol{l})$

Sensing at q-th antenna signal:

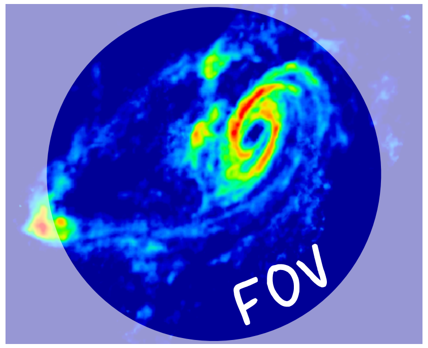
$$\underbrace{x_q(t)}_{\text{signal}} = \int_{\mathbb{R}^2} s(\boldsymbol{l}, t) \underbrace{g(\boldsymbol{l})}_{\text{FOV}} \exp\left(\tfrac{\mathrm{i}2\pi}{\lambda} \underbrace{\boldsymbol{p}_q^{\perp}(t)^\top \boldsymbol{l}}_{\text{geometric delay}} \right) \mathrm{d}\boldsymbol{l} + \underbrace{n_q(t)}_{\text{noise}}.$$

Radio interferometric sensing model

By the Van Cittert-Zernike theorem (VCZ)

$$\underbrace{\mathbb{E}_{\boldsymbol{s}}\mathbb{E}_{\boldsymbol{n}}[\overset{(x_1(t),\dots,x_Q(t))}{\boldsymbol{x}(t)}\boldsymbol{x}^*(t)]}_{\text{Short-Time Integration}} = \mathcal{I}_{\Omega(t)}[\sigma^\circ] + \overbrace{\underbrace{\boldsymbol{\Sigma}_{\mathrm{n}}}_{\text{cov. of } \boldsymbol{n}(t)}}^{\text{neglected}}$$

$\sigma^\circ(\boldsymbol{l})$



$g^2(\boldsymbol{l})\sigma^2(\boldsymbol{l})$

with

$$(\mathcal{I}_{\Omega(t)}(\sigma^\circ))_{jk} := \overset{\text{Fourier Tr.}}{\mathcal{F}[\sigma^\circ]}\Bigg(\underbrace{\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}}_{\in\mathcal{V}:=\lambda^{-1}(\Omega-\Omega)}\Bigg)$$

visibilities

$\Omega(t) = \{\boldsymbol{p}_q^\perp(t)\}_{q=1}^Q$

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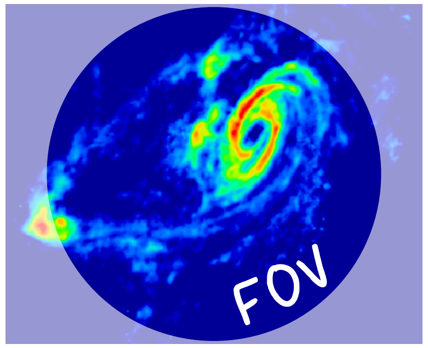

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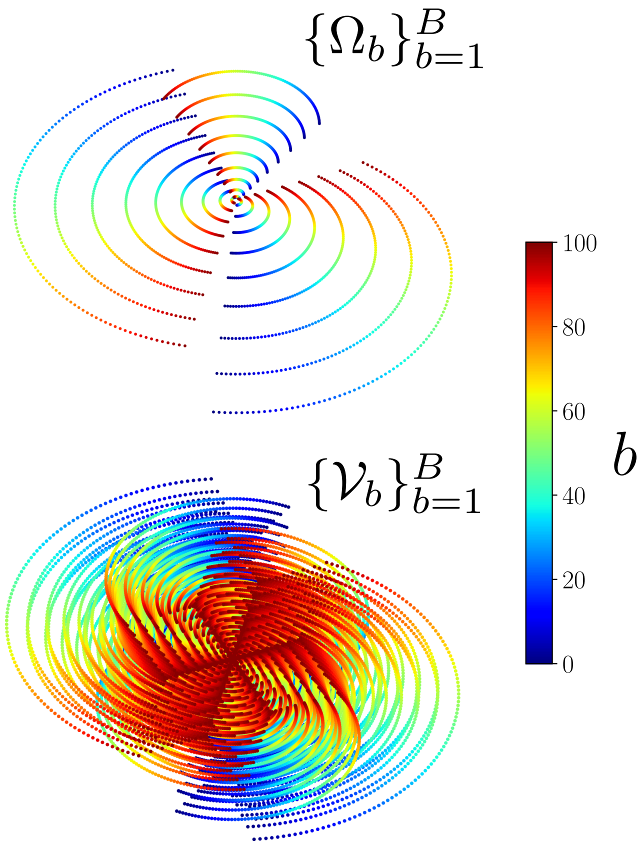
visibilities

Practically,

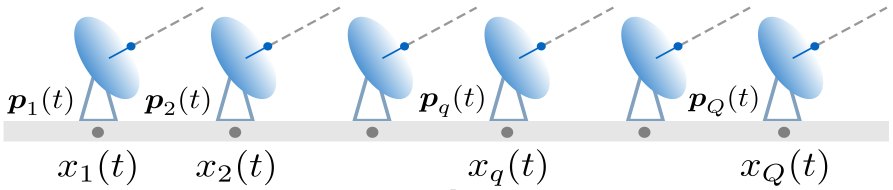
- B short-time integration intervals (STI) with I discrete time instants $\rightarrow \mathbb{E}(\,\cdot\,) \approx \langle\,\cdot\,\rangle_I$
- Approx: over each STI, visibilities are fixed



Very Large Array (VLA)

Radio interferometric sensing model

Summary: 2 sensing operators



(antennas level) $\boldsymbol{x}(t)$

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

(correlator)

Covariances estimations

$$\boldsymbol{C}_b(\mathcal{X}_b) := \langle \boldsymbol{x}_b[\cdot]\boldsymbol{x}_b^*[\cdot]\rangle_I := \frac{1}{I}\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]$$

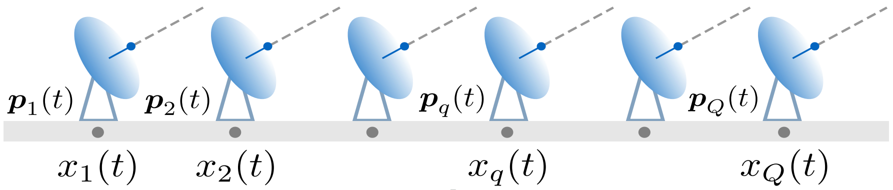
$(QBI \to Q^2B)$

Acquisition operator

$$\mathcal{X} := \cup_b \mathcal{X}_b \to \boldsymbol{\Psi}(\mathcal{X}) := \underbrace{\textstyle\bigcup_b \boldsymbol{C}_b(\mathcal{X}_b)}_{Q^2B \text{ values}}$$

Radio interferometric sensing model

Summary: 2 sensing operators



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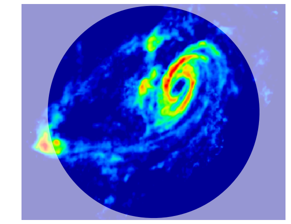
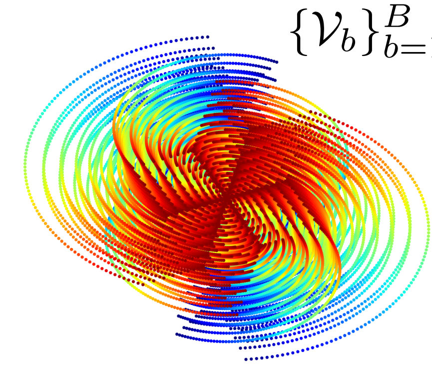


(image level) $\sigma^\circ(\boldsymbol{l})$

Sampling over a N-pixel grid

$$\boldsymbol{\sigma} \in \mathbb{R}^N$$

Given $\{\mathcal{V}_b\}_{b=1}^B =: \mathcal{V}$



Q^2B visibilities

(reconstruction level)

Imaging operator

$$\boldsymbol{\sigma} \xrightarrow[\text{NFFT}]{} \boldsymbol{\Phi}[\boldsymbol{\sigma}] = \underbrace{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V}) = \{\mathcal{I}_b(\boldsymbol{\sigma})\}_{b=1}^B}_{Q^2B \text{ values}}$$

equal in expectation

Challenges in radio-interferometry

Massive data stream:

- #*visibilities* $\mathscr{V} = \cup_{b=1}^{B} \mathscr{V}_b \to O(Q^2 B)$
  e.g., for the square-kilometer array (SKA)
  $Q = O(10^5)$, $B = O(100) \to$ Storing $O(10^7)$ visibilities
- Computing $\mathscr{F}[\sigma^\circ](\mathscr{V})$ via $\{C_b\}_{b=1}^{B} \to O(IB\,Q^2) = O(10^9 \cdot 10^5)$

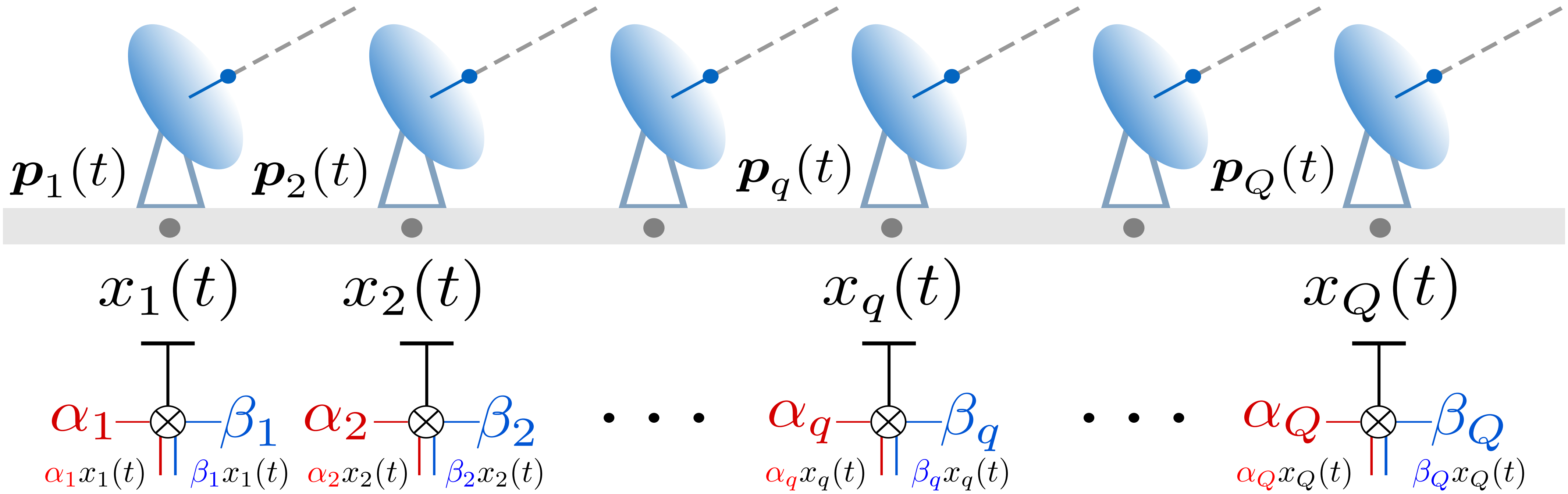
Solution: compressive radio-interferometric (RI) sensing scheme

- leveraging an old scheme, *beamforming*, in a new setup
- compressing measurements at antenna & reconstruction levels
- supported by theoretical guarantees (under a few simplifications).

Compressive radio astronomy
with rank-one projections and an old trick

Beamforming ≡ rank-one projections of covariance matrix

What if we create a ***virtual*** antenna? Let's do beamforming



Given Q complex weights α_q, β_q

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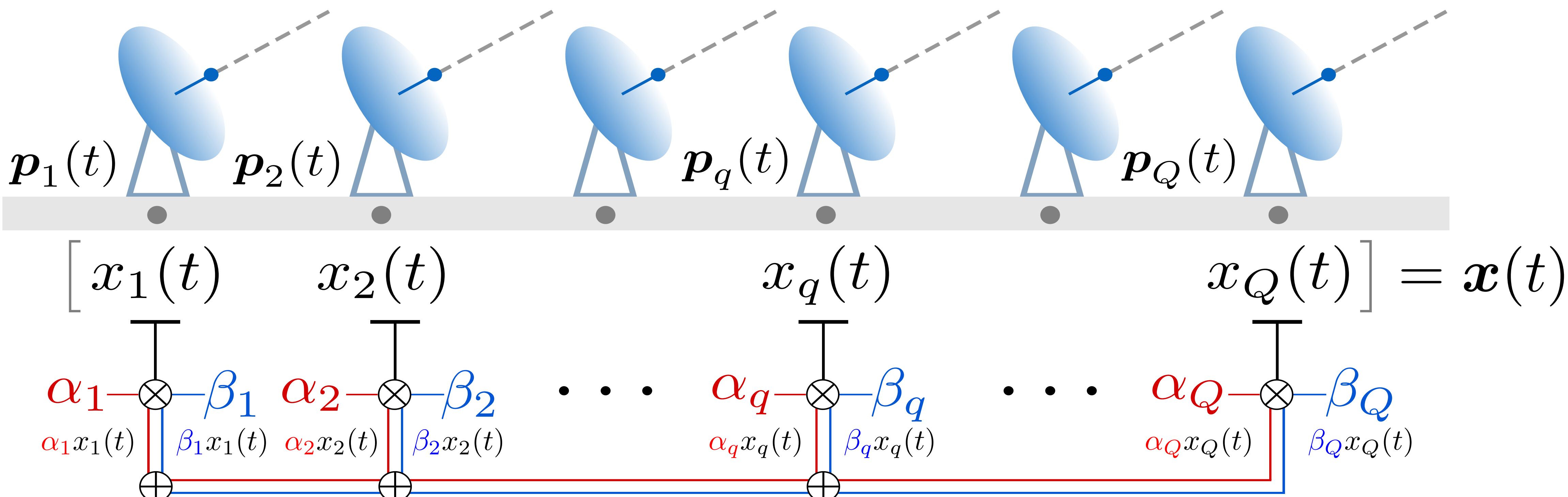


$$\mu(t) = \sum_{q=1}^{Q} \alpha_q^* x_q(t) = \langle \boldsymbol{\alpha}, \boldsymbol{x}(t) \rangle$$

$$\nu(t) = \sum_{q=1}^{Q} \beta_q^* x_q(t) = \langle \boldsymbol{\beta}, \boldsymbol{x}(t) \rangle$$

Van Cittert → Zernike

$$\mathbb{E}\mu\nu^* = \boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}_\Omega[\sigma^\circ] \boldsymbol{\beta}$$ ROP!! (asymmetric)

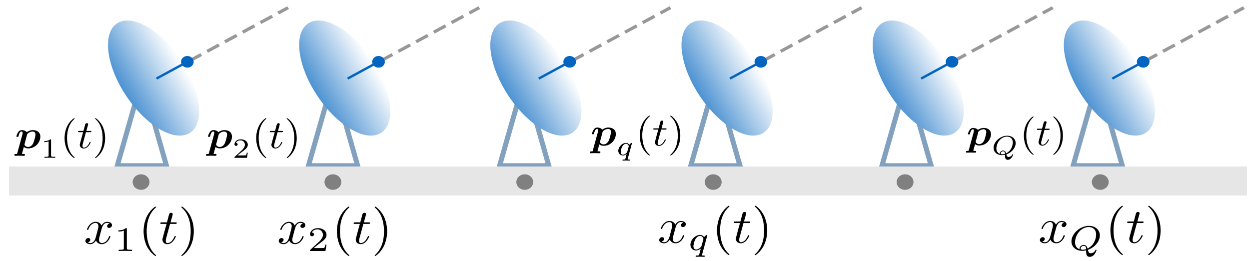
$$\text{with } (\boldsymbol{\mathcal{I}}_\Omega[\sigma^\circ])_{jk} = \mathcal{F}[\sigma^\circ]\left(\tfrac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}\right)$$

→ Let's compute N_p ROP per time/STI

Given Q complex weights α_q, β_q

We need new sensing operators

Acquisition operator Given $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}_{p=1,b=1}^{N_p,B} \subset \mathbb{C}^Q$, $\{\gamma_{mb}\}_{m=1,b=1}^{N_m,B} \subset \mathbb{C}^Q$ (Not specified yet)



$\boldsymbol{x}(t)$

(sampled antenna signals)

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

(B STI, I time samples per batch)

(1st compression @antennas level) $(QBI \to N_p B)$

Random beamforming: for $p \in [N_p]$ ROPs per b

$$\mu_{pb}[i] := \langle \boldsymbol{\alpha}_{pb}, \boldsymbol{x}_b[i] \rangle,\ \ \nu_{pb}[i] := \langle \boldsymbol{\beta}_{pb}, \boldsymbol{x}_b[i] \rangle$$

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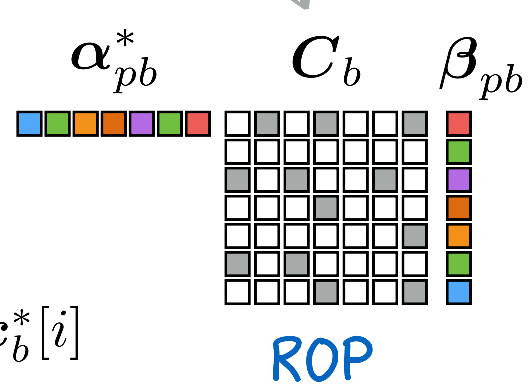

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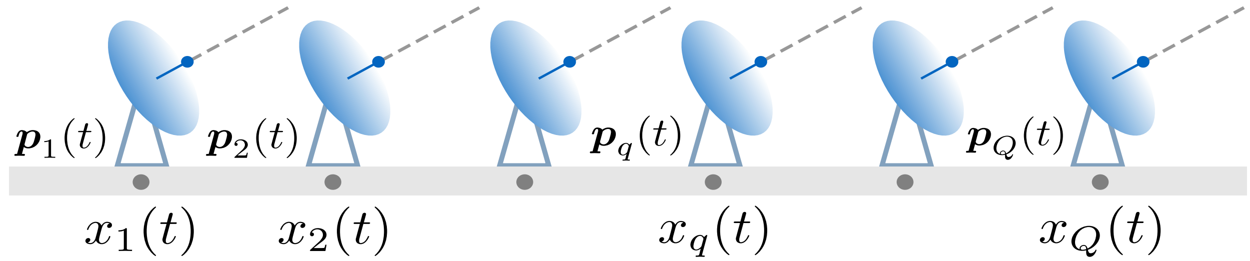

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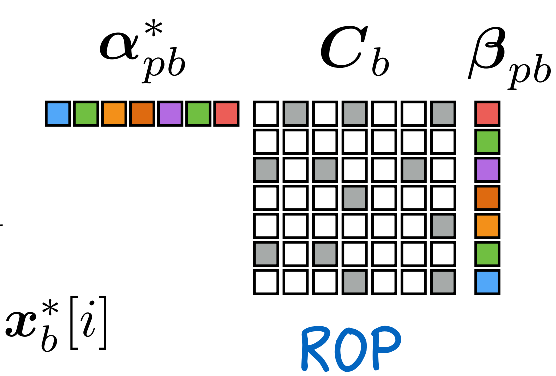

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(2nd compression) $(N_p B \to N_p N_m)$

Bernoulli modulations: for $m \in [N_m]$ modulations

$$\mathcal{X} \to \widetilde{\boldsymbol{\Psi}}(\mathcal{X}) = \Big\{ \boldsymbol{z}_m := \textstyle\sum_{b=1}^{B} \underbrace{\gamma_{mb}}_{\in\{\pm 1\}} \boldsymbol{y}_b \Big\}_{m=1}^{N_m}$$

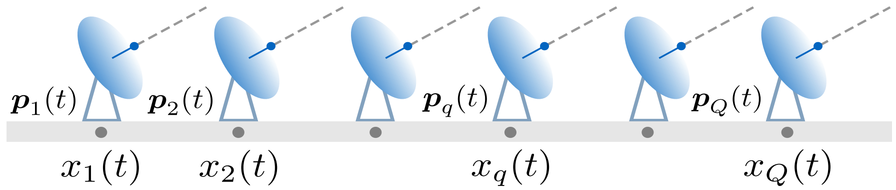
$N_p N_m$ values

$$\equiv \mathrm{ROP\ of\ } \boldsymbol{C} := \mathrm{bdiag}(\boldsymbol{C}_1, \ldots, \boldsymbol{C}_B)$$

We need new sensing operators

Acquisition operator

$\boldsymbol{x}(t)$



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$$(QBI \to N_p B)$$

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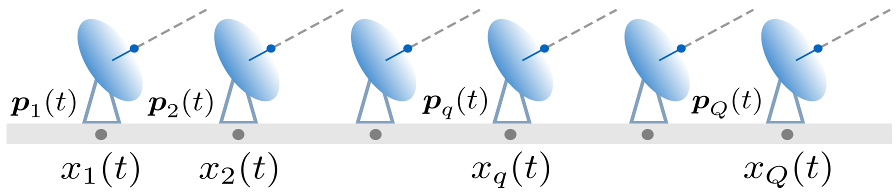

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$(QBI \rightarrow N_pB)$

(2nd compression) $(N_pB \rightarrow N_pN_m)$

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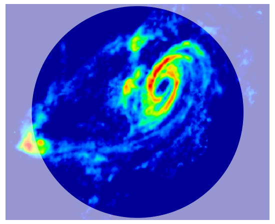
N_pN_m values

Imaging operator

$\sigma^{\circ}(\boldsymbol{l})$

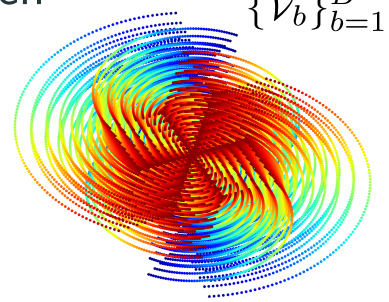


(image level)

Sampling over a N-pixel grid

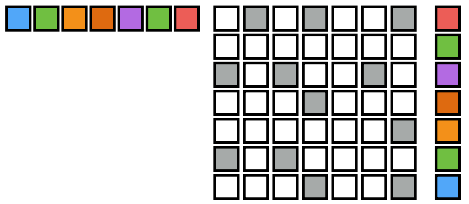
$$\boldsymbol{\sigma} \in \mathbb{R}^N$$

Given $\{\mathcal{V}_b\}_{b=1}^{B} =: \mathcal{V}$



Q^2B visibilities

$\boldsymbol{\alpha}_{pb}^{*}$ $\boldsymbol{\mathcal{I}}_b(\boldsymbol{\sigma})$ $\boldsymbol{\beta}_{pb}$



(compression for the reconstruction level)

$$\boldsymbol{\sigma} \xrightarrow{\text{ROP}} \Big\{ \boldsymbol{y}'_b = \big[\boldsymbol{\alpha}_{pb}^{*}\ \overbrace{\boldsymbol{\mathcal{I}}_b(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V}_b)}\ \boldsymbol{\beta}_{pb}\big)\big]_{p=1}^{N_p} \Big\}_{b=1}^{B}$$

$$\xrightarrow{\text{Mod.}} \widetilde{\boldsymbol{\Phi}}[\boldsymbol{\sigma}] = \Big\{ \sum_{b=1}^{B} \gamma_{mb} \boldsymbol{y}'_b \Big\}_{m=1}^{N_m}$$

N_pN_m values

equal in expectation

Reconstruction guarantees?

Questions:

- For which (distribution on) $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}, \gamma_{mb}\}$

  can we estimate the image $\boldsymbol{\sigma}$?

- What are the compression ratios?

Our answers:

1. **Theory**: ok if $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}$ are random and (sub)Gaussian

   but without modulations ($\gamma_{mb} = 1$, $N_m = 1$) and N_p large enough
2. **Experiments**: ok if $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}, \gamma_{mb}\}$ are random and (sub)Gaussian

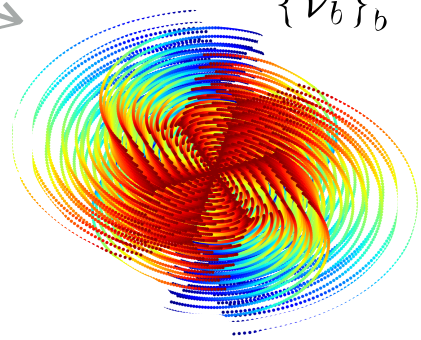
Reconstruction guarantees? Theory

Batched ROP model: with $\gamma_{mb} = 1$, $N_m = 1$, we find:

$$\widetilde{\boldsymbol{\Phi}}[\boldsymbol{\sigma}] = \textstyle\sum_{b=1}^{B} \big[\boldsymbol{\alpha}_{pb}^{*} \overbrace{\boldsymbol{\mathcal{I}}_b(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V}_b)} \boldsymbol{\beta}_{pb}\big]_{p=1}^{N_p} = \big[\boldsymbol{\alpha}_{p}^{*} \overbrace{\boldsymbol{\mathcal{I}}(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V})} \boldsymbol{\beta}_{p}\big]_{p=1}^{N_p}$$

with $\boldsymbol{\alpha}_p = [\boldsymbol{\alpha}_{pb}]_{b=1}^{B}$, $\boldsymbol{\beta}_p = [\boldsymbol{\beta}_{pb}]_{b=1}^{B}$, $\boldsymbol{\mathcal{I}} = \mathrm{bdiag}(\boldsymbol{\mathcal{I}}_1, \ldots, \boldsymbol{\mathcal{I}}_B)$.

$\{\mathcal{V}_b\}_{b=1}^{B} =: \mathcal{V}$



Batched ROP model: with $\gamma_{mb} = 1$, $N_m = 1$, we find:

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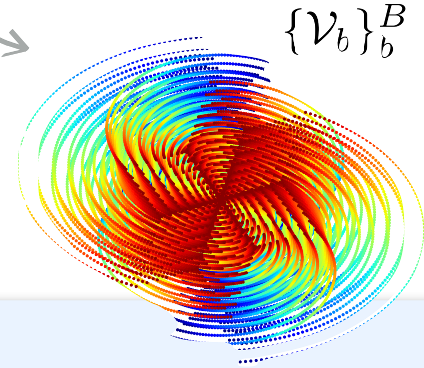


(under specific simplifying assumptions)

If $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}$ are (sub)Gaussian, given a sparsity level K

and provided $N_p = O(K)$ and $Q^2 B = O(K)$ (up to logs),

then, with high probability, given the observations $z = \tilde{\boldsymbol{\Phi}}[\boldsymbol{\sigma}] + \underbrace{\text{noise}}_{\|\cdot\|_1 \leqslant \epsilon}$,

an ℓ_1-minimization gives an estimate $\boldsymbol{\sigma}'$ with

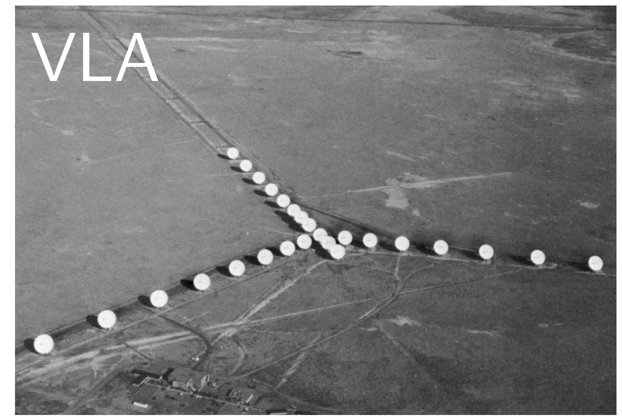
$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}'\|_2 \leqslant C \frac{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_K\|_1}{\sqrt{K}} + D \frac{\epsilon}{N_{\mathrm{p}}}$$

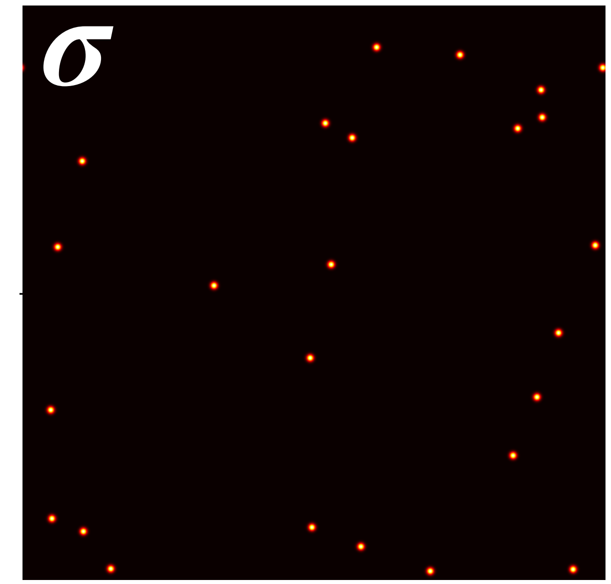
for some $C, D > 0$.

Reconstruction guarantees? Simulations

Modulated ROP model:

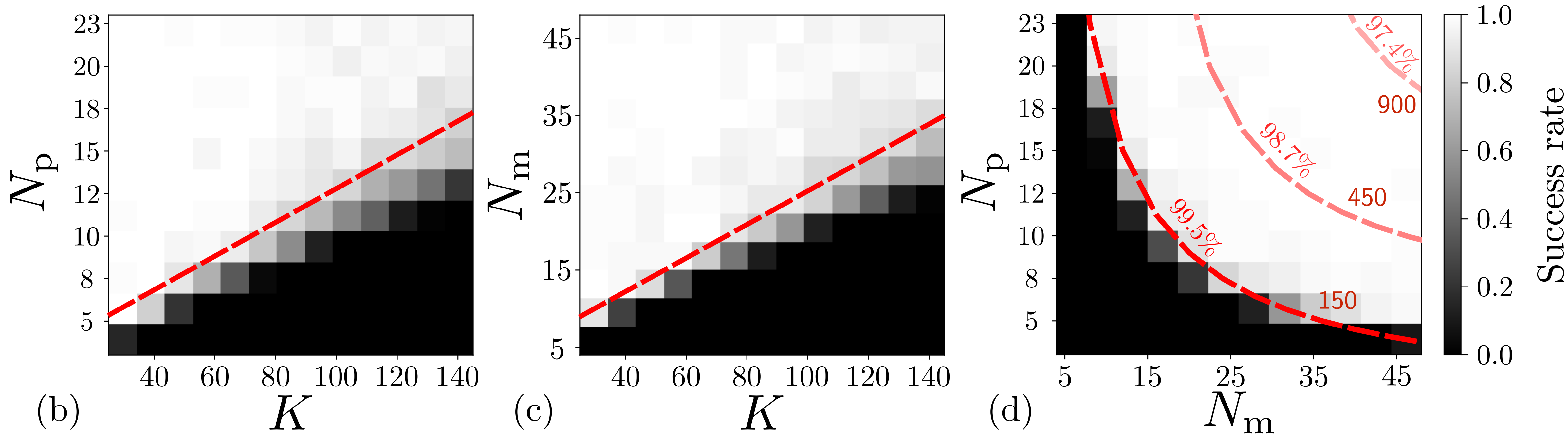
- Monte Carlo simulations
- $N = 10^4$, $B = 100$, $Q = 27$
- Various K, N_p, N_m
- Very Large Array (VLA) visibility/frequency coverage





$N = 100 \times 100$

Phase transition diagrams (success if SNR > 40 dB)



High reconstruction success as soon as $N_p N_m \geqslant CK$, with $C \simeq 5$.

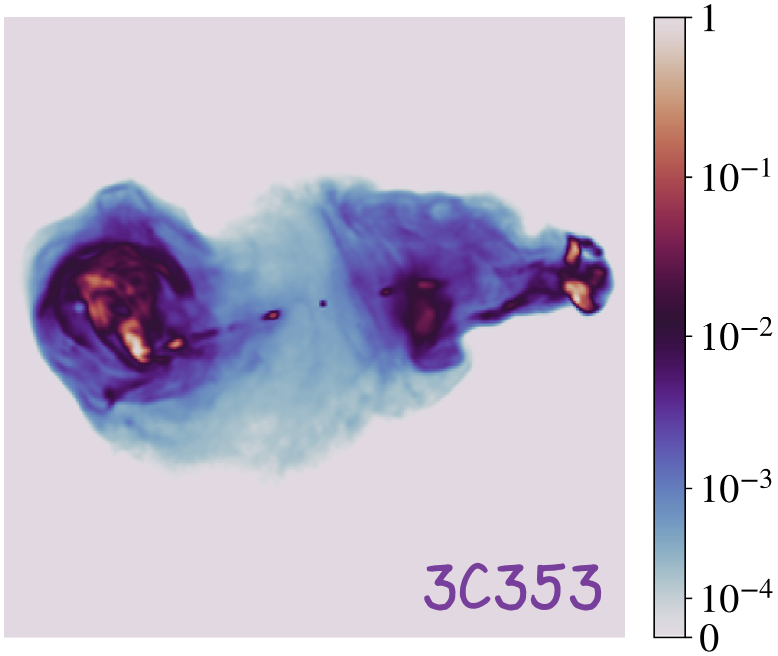
$$\rightarrow \; N_p N_m \ll \; Q^2 B = 70\,200$$

Radio Galaxy 3C353, uSARA reconstruction

Reconstruction results for

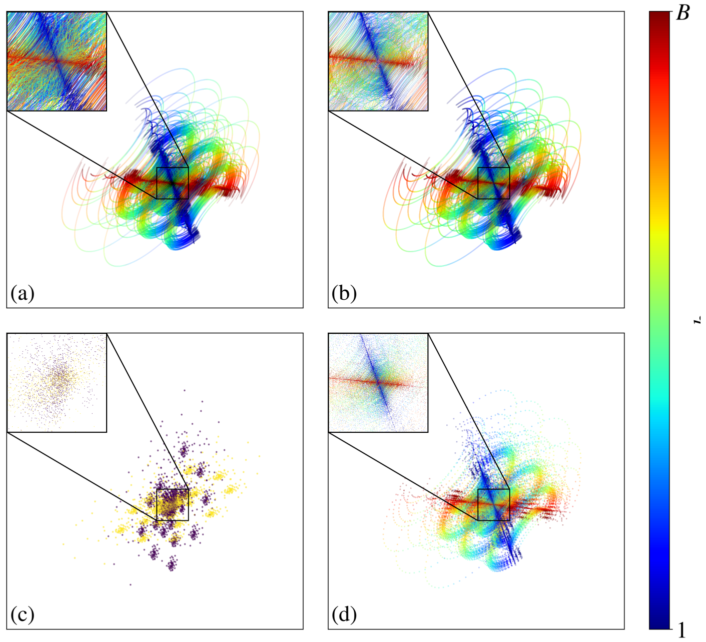
- the classical (all visibilities),
- subsampled visibilities ,
- Baseline dependent averaging (BDA),
- and MROP sensing models

using the image of the Radio Galaxy 3C353



Parameters:

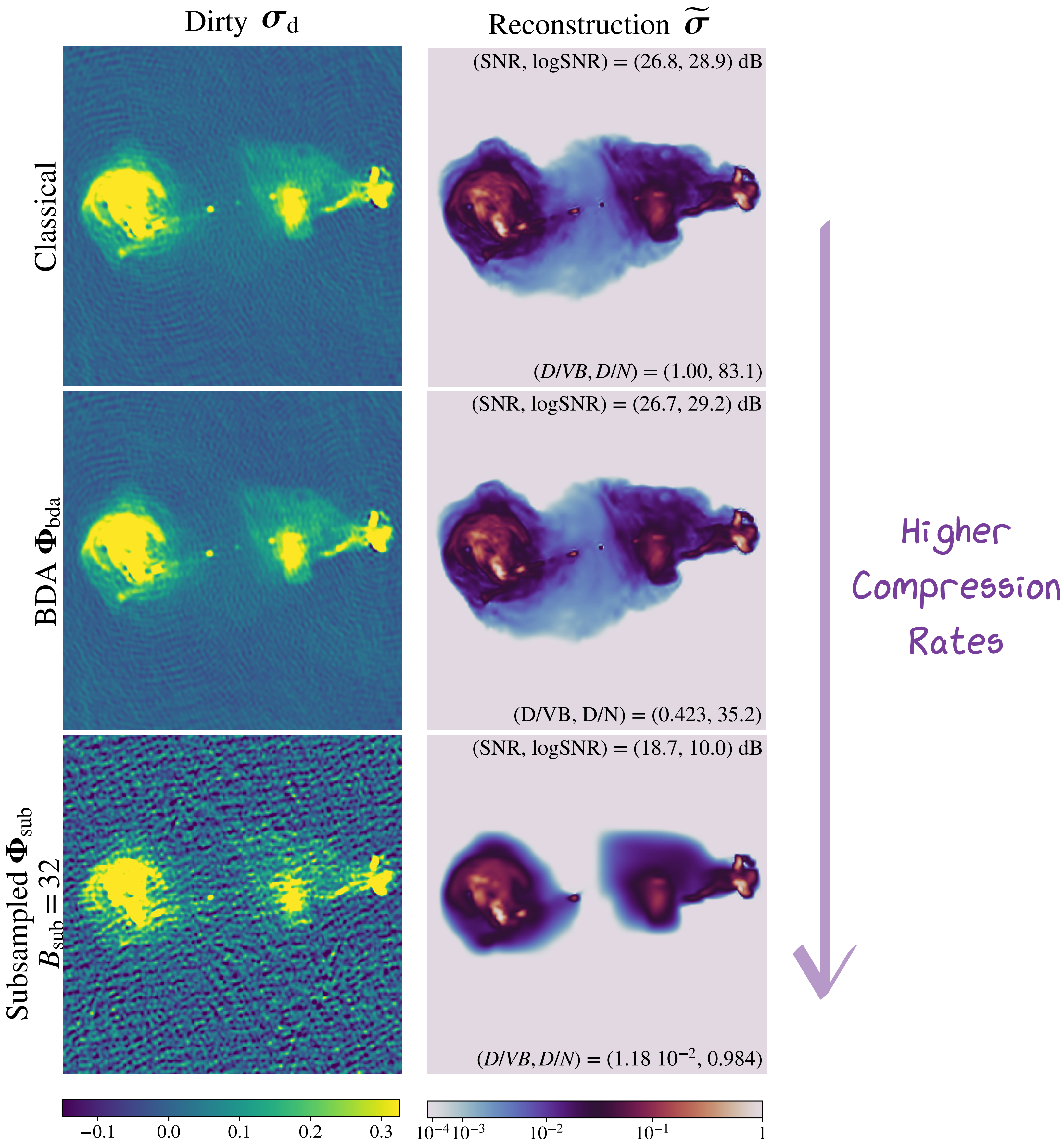
- #Visibilities (VB) / #Pixels (N) = 14.5
- ROP* compressions levels: D=PM/#Pixels $\in \{9.8\ 10^{-1}, 3.8\ 10^{-1}, 6.1\ 10^{-2}\}$
- Simulated MeerKat visibilities
- uSARA reconstruction (Terris et al, 2022) (inverse problem solving with "average sparsity" prior)



MeerKat visibilities

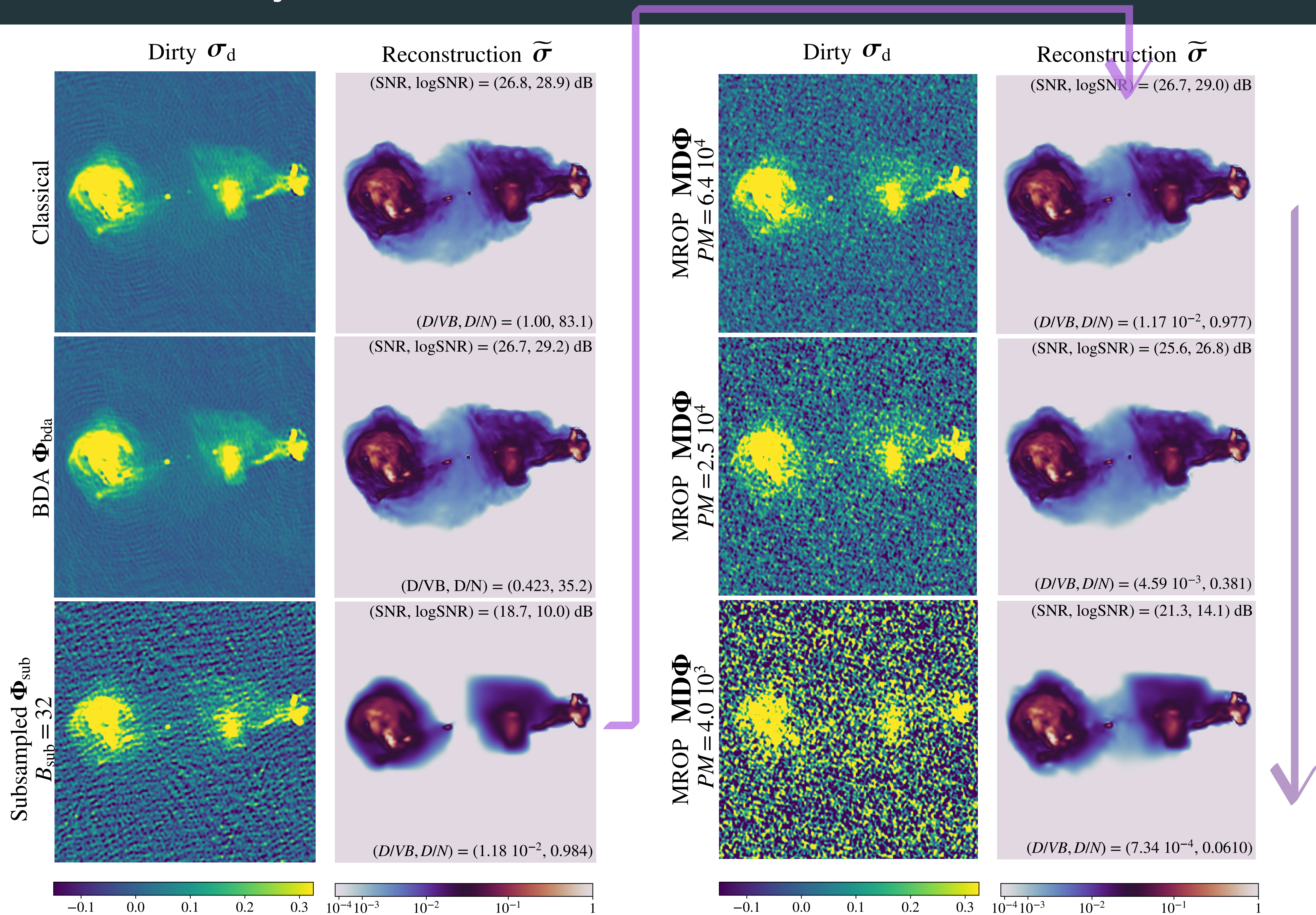
*: slightly different model, "visibility-based acquisition of MROP"

Radio Galaxy 3C353, uSARA reconstruction



Two compression metrics
D/VB and D/N

Radio Galaxy 3C353, uSARA reconstruction



Conclusions and perspectives

Summary:

- Interferometry and "*beamforming*" $\rightarrow$ ROP + Fourier
- Theory, experiments and simulations confirm
  the efficiency of such a compressive combination

Under review:

- Comprehensive analysis of ROP/BF schemes in RI

Open questions:

- Integrating frequency weighting?
- Faster ROP models?
- Calibration through beamforming sensing?

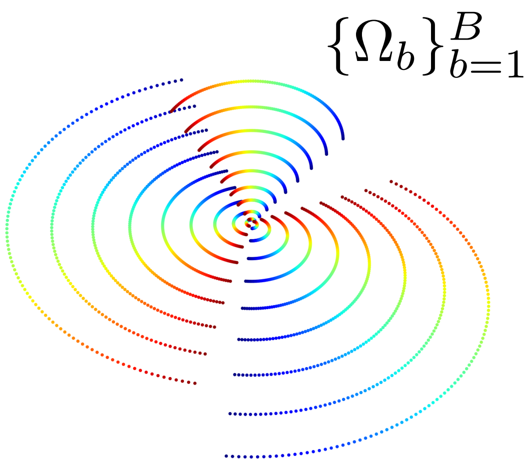
Thank you for your attention!

- **O. Leblanc, Y. Wiaux, L. Jacques, "Compressive radio-interferometric sensing with random beamforming as rank-one signal covariance projections", submitted to IEEE TCI** https://arxiv.org/abs/2409.15031
- **O. Leblanc, C. S. Chu, L. Jacques, Y. Wiaux, "MROP: Modulated Rank-One Projections for compressive radio interferometric imaging", submitted to MNRAS, 2025**
- **O. Leblanc, M. Hofer, S. Sivankutty, H. Rigneault, L. Jacques (2023). "Interferometric lensless imaging: rank-one projections of image frequencies with speckle illuminations". Submitted to IEEE TCI, arXiv:2306.12698.**

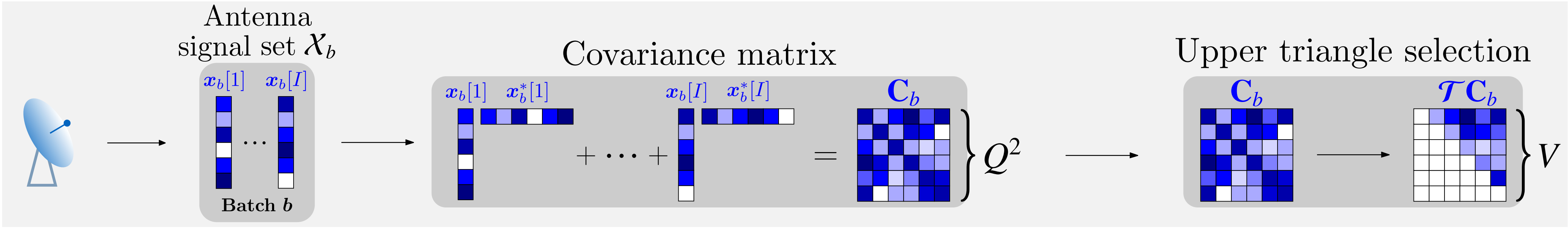
- S. Guérit, S. Sivankutty, J. A. Lee, H. Rigneault and L. Jacques, "Compressive Imaging through Optical Fiber with Partial Speckle Scanning," under review, 2021.
- E. R. Andresen, S. Sivankutty, V. Tsvirkun, et al., "Ultrathin endoscopes based on multicore fibers and adaptive optics: status and perspectives," Journal of Biomedical Optics, 2016.
- S. Sivankutty, V. Tsvirkun, O. Vanvincq, et al., "Nonlinear imaging through a fermat's golden spiral multicore fiber," Optics letters, 2018.
- Chen, Y., Chi, Y., & Goldsmith, A. J. (2015). Exact and stable covariance estimation from quadratic sampling via convex programming. *IEEE Transactions on Information Theory*, *61*(7), 4034-4059.
- Cai, T. T., & Zhang, A. (2015). ROP: Matrix recovery via rank-one projections. The Annals of Statistics, 43(1), 102-138.
- Saade, Alaa, et al. "Random projections through multiple optical scattering: Approximating kernels at the speed of light." 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 2016.
- M. Davenport, P. Boufounos, M. Wakin, R. Baraniuk, Signal Processing with Compressive Measurements, IEEE 2010.
- Terris M., Dabbech A., Tang C., Wiaux Y., 2022, Monthly Notices of the Royal Astronomical Society, 518, 604

Extra slides

Visibility-based acquisition of MROP

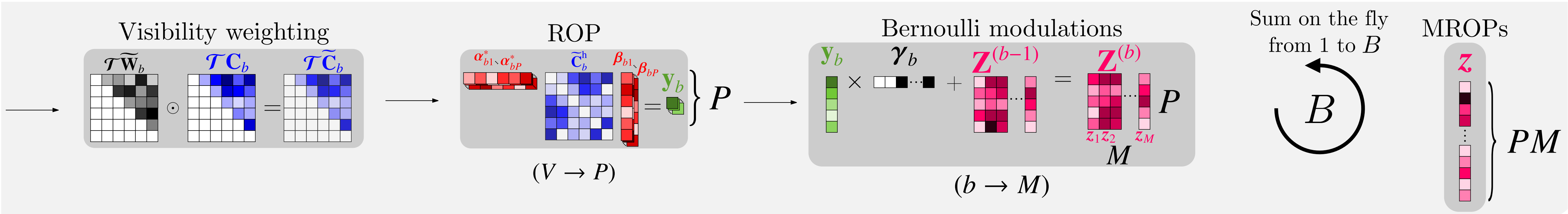


at each STI (b), compute the covariance matrix (classical)



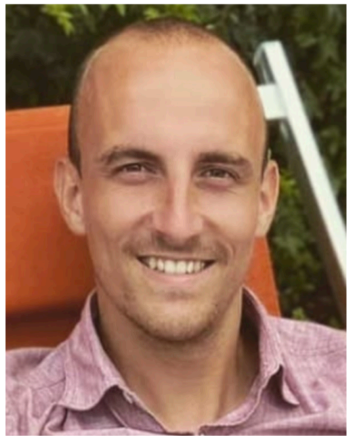
(visibility domain)

then do ROP across STIs after visibility weighting



(compressive interferometry #1)

Lensless interferometry & rank-one projections


O. Leblanc*


L. Jacques*

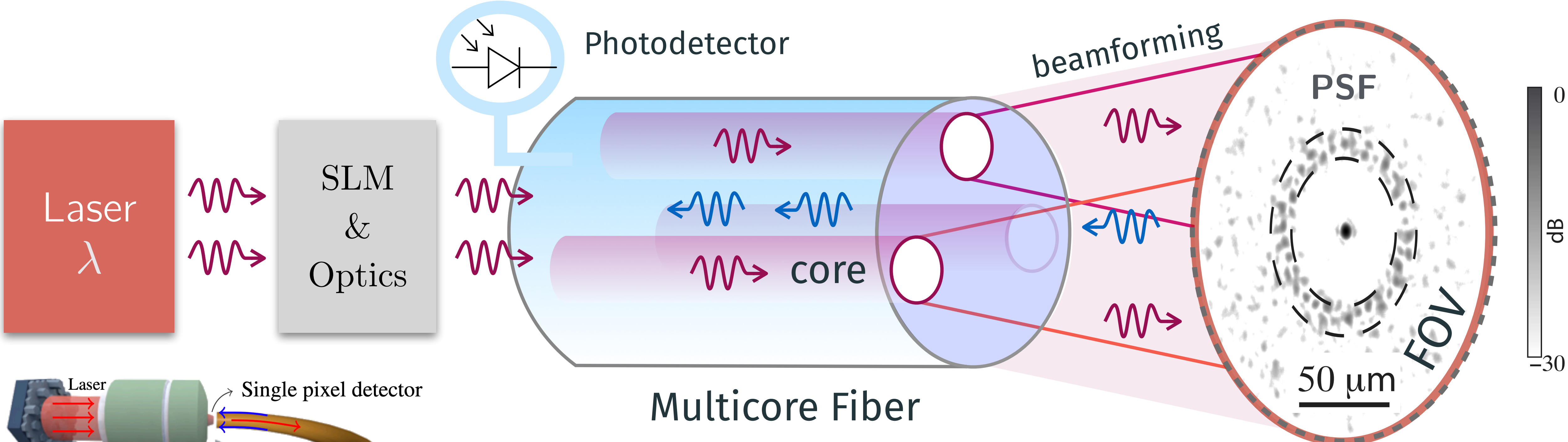

M. Hofer†

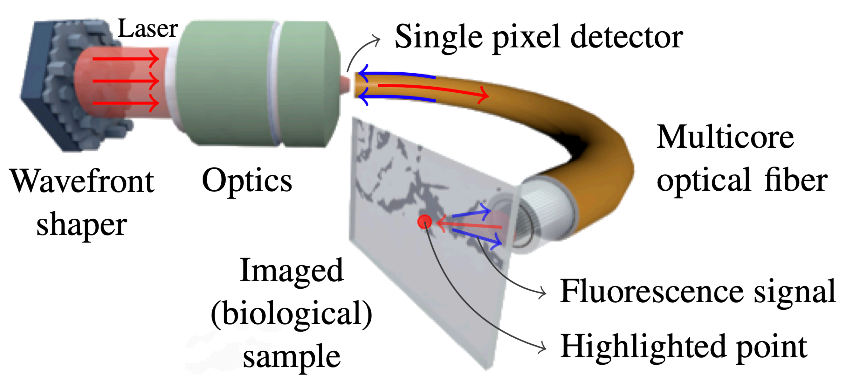

H. Rigneault†


S. Sivankutty‡

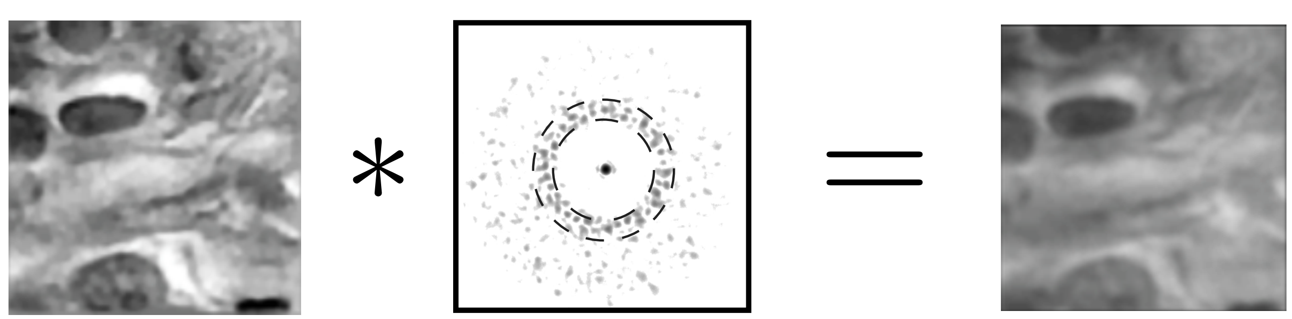
*: ISPGroup, INMA, UCLouvain, Belgium. †: Institut Fresnel, France. ‡: PhLAM, France.

Lensless endoscopy: focused mode

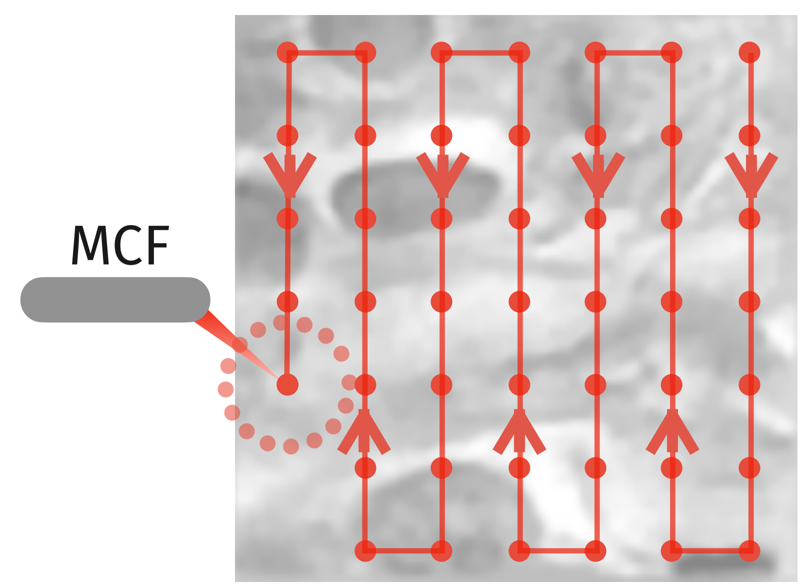




Sensing model



Raster Scanning

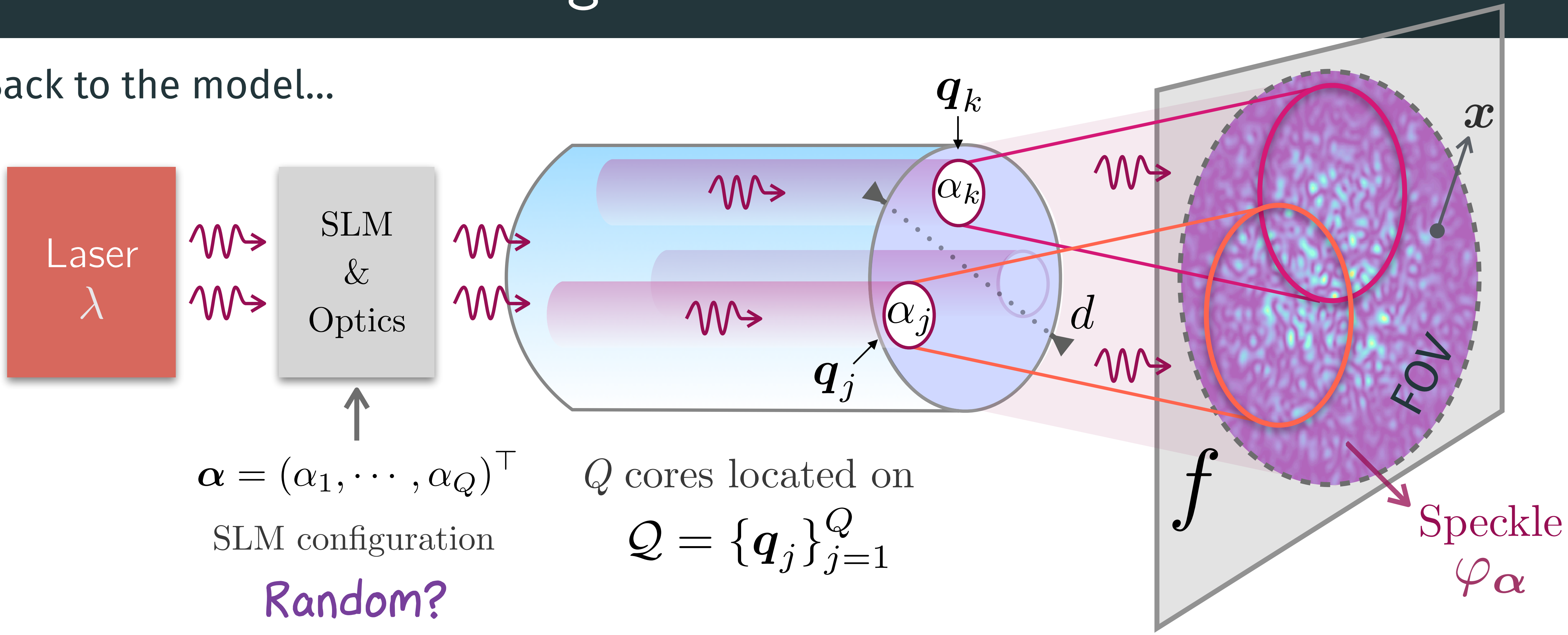


Biological sample

Andresen et al., 2016. Sivankutty et al., 2018.

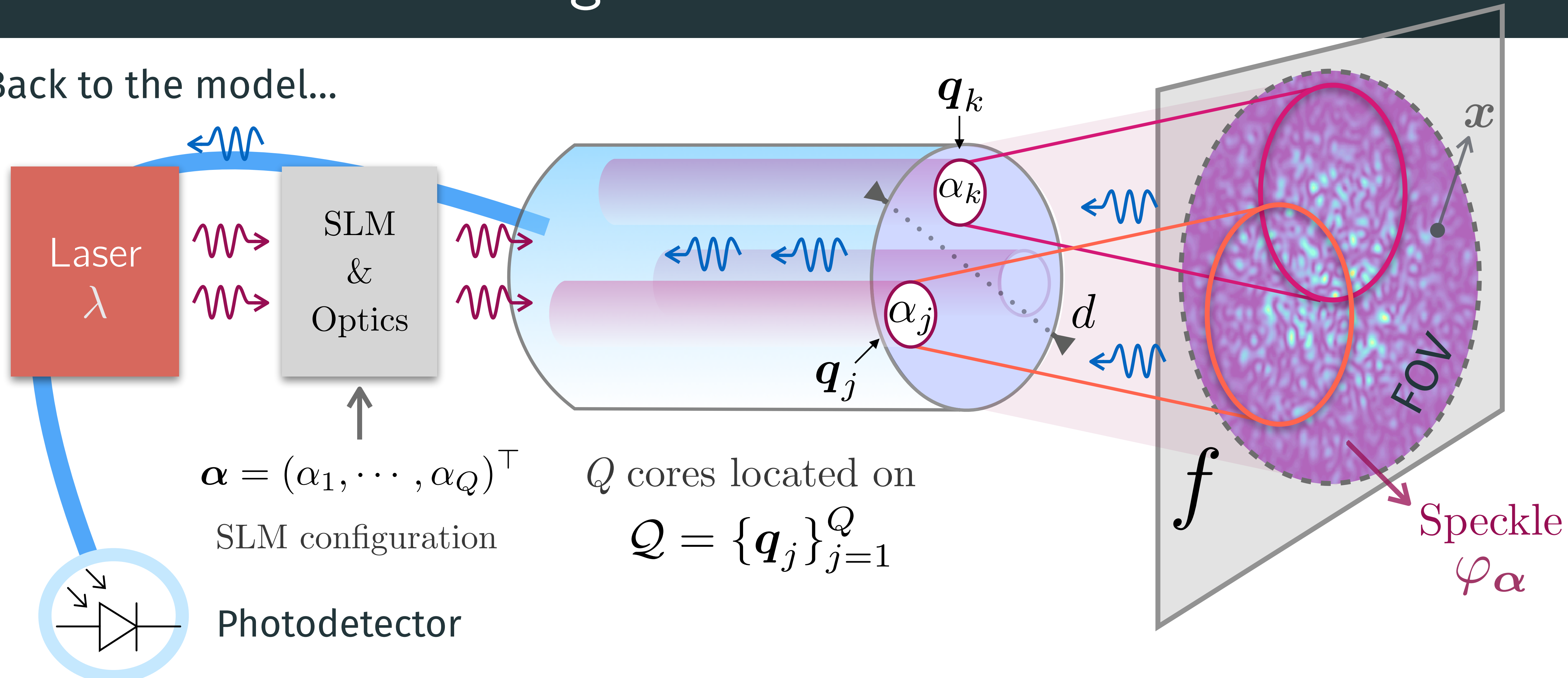
A closer look to sensing model

Back to the model...



A closer look to sensing model

Back to the model...

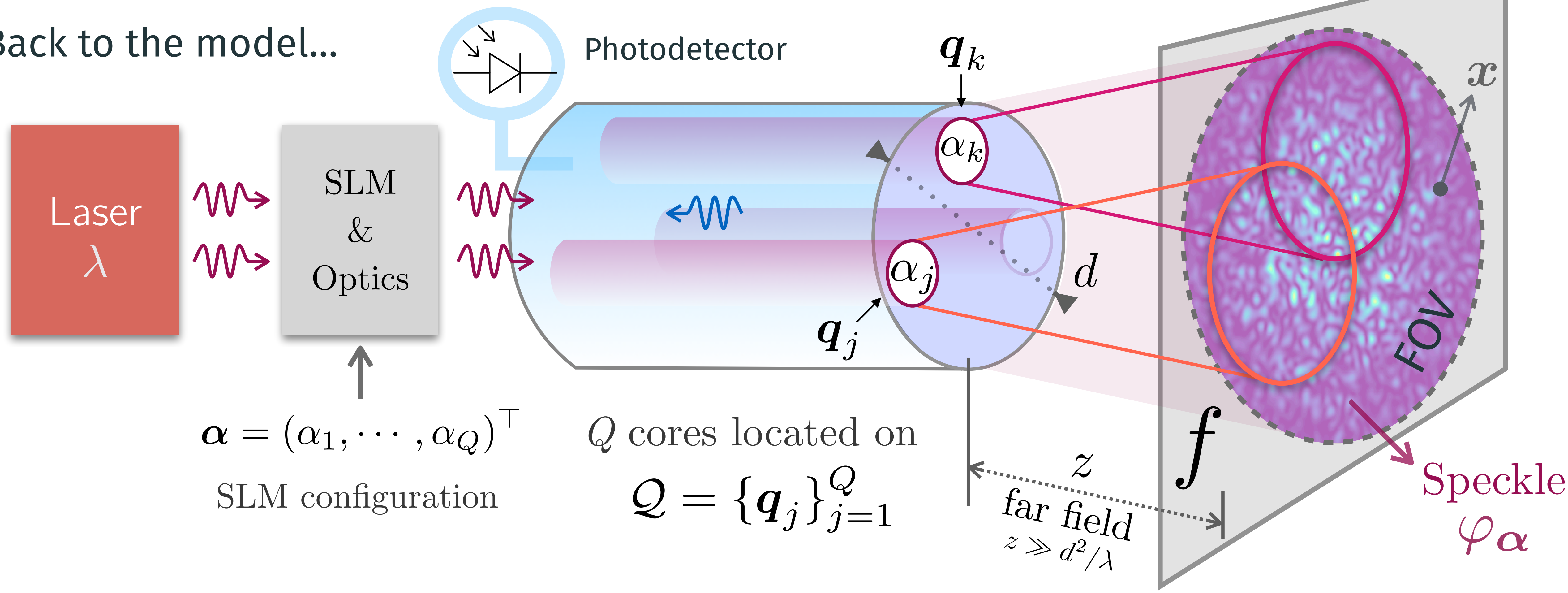


Speckle

$$y_{\boldsymbol{\alpha}} \propto \int_{\mathbb{R}^2} \varphi_{\boldsymbol{\alpha}}(\boldsymbol{x}) f(\boldsymbol{x}) \, \mathrm{d}^2\boldsymbol{x} = \langle \varphi_{\boldsymbol{\alpha}}, f \rangle$$

Measurement model

Back to the model...

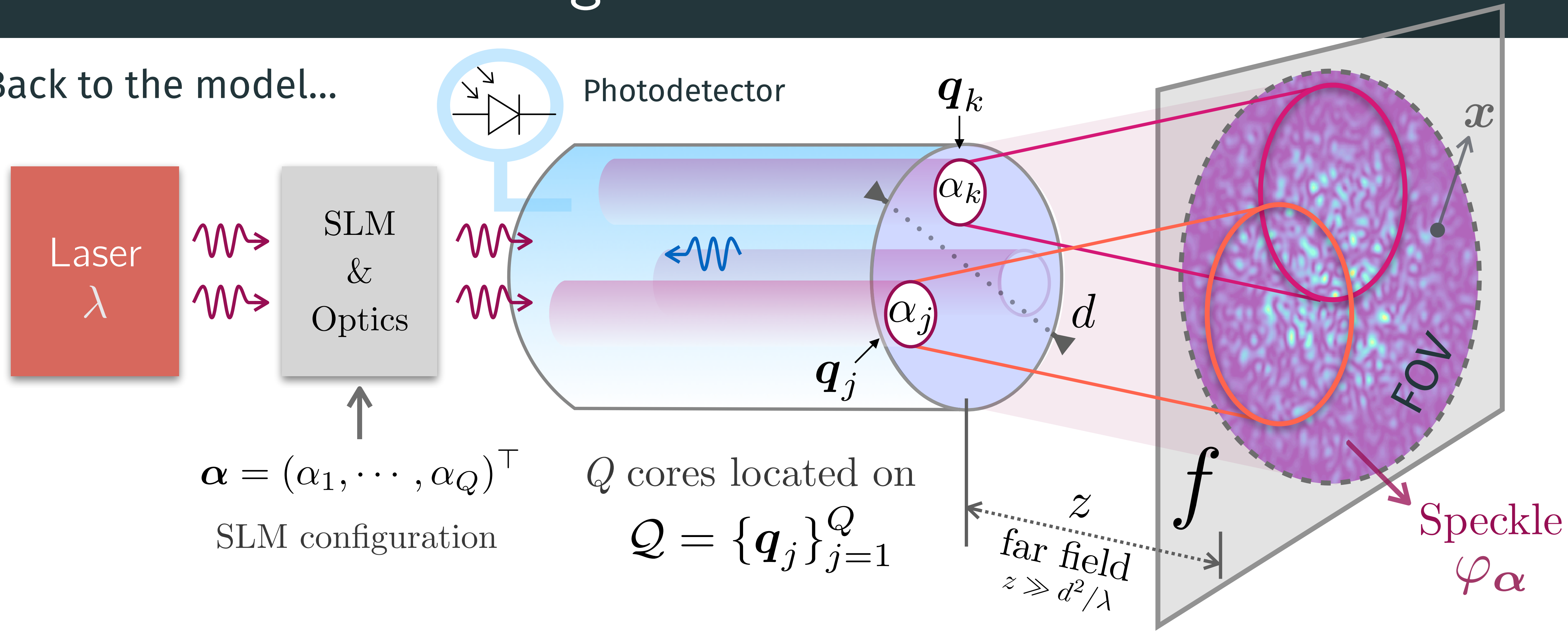


However, speckles are interferences: (Under far-field approximation)

$$\varphi_{\alpha}(x) \propto \underbrace{w(x)}_{\text{FOV window}} \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \underbrace{e^{\frac{2\pi i}{\lambda z}(q_j - q_k)^\top x}}_{\text{Core pair interference}}$$

Can we do compressive sensing?

Back to the model...



However, speckles are interferences: (Under far-field approximation)

$$\langle f(\boldsymbol x), \varphi_{\boldsymbol\alpha}(\boldsymbol x)\rangle \propto \langle w(\boldsymbol x) f(\boldsymbol x), \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* e^{\frac{2\pi i}{\lambda z}(\boldsymbol q_j - \boldsymbol q_k)^\top \boldsymbol x}\rangle$$

Can we do compressive sensing?

(noiseless) Interferometric sensing model

Therefore

$$\langle f, \varphi_{\boldsymbol\alpha} \rangle = \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol q_j - \boldsymbol q_k)^\top \boldsymbol x} w(\boldsymbol x) f(\boldsymbol x) \mathrm{d}\boldsymbol x \right]$$

$$\boldsymbol\alpha^* \boldsymbol{\mathcal{I}}[wf]\, \boldsymbol\alpha \quad \rightarrow \text{ROP!!}$$

with the (Hermitian) *interferometric matrix* $\boldsymbol{\mathcal{I}}[wf] \in \mathbb{C}^{Q \times Q}$ s.t.

$$(\boldsymbol{\mathcal{I}}[wf])_{j,k} := \int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol q_j - \boldsymbol q_k)^\top \boldsymbol x}\, w(\boldsymbol x) f(\boldsymbol x)\, \mathrm{d}\boldsymbol x$$

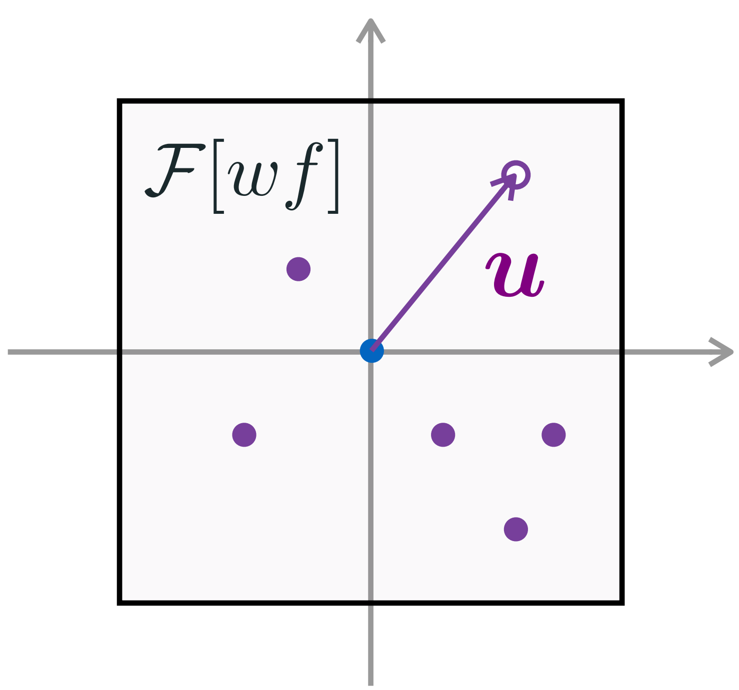
Therefore

$$\langle f, \varphi_{\boldsymbol\alpha}\rangle = \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol q_j - \boldsymbol q_k)^\top \boldsymbol x} w(\boldsymbol x) f(\boldsymbol x) \mathrm{d}\boldsymbol x \right]$$

$$\dashrightarrow \boldsymbol\alpha^* \, \boldsymbol{\mathcal{I}}[wf] \, \boldsymbol\alpha \quad \to \text{ROP!!}$$

with the (Hermitian) *interferometric matrix* $\boldsymbol{\mathcal{I}}[wf] \in \mathbb{C}^{Q\times Q}$ s.t.

$$(\boldsymbol{\mathcal{I}}[wf])_{j,k} := \int_{\mathbb{R}^2} e^{\underbrace{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol q_j - \boldsymbol q_k)^\top}_{\boldsymbol u} \boldsymbol x} w(\boldsymbol x) f(\boldsymbol x)\, \mathrm{d}\boldsymbol x = \mathcal{F}[wf](\mathcal{V})$$

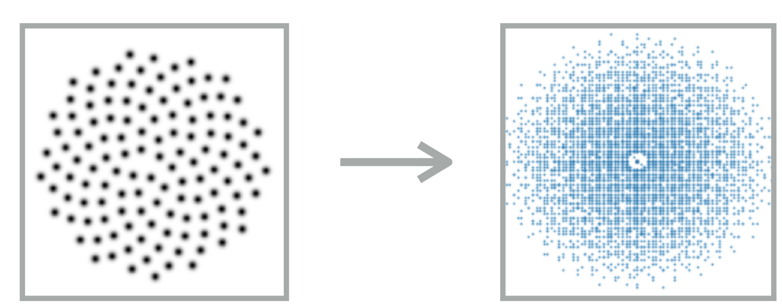


$$\boldsymbol u \in \mathcal{V} := \frac{1}{\lambda z}(\mathcal{Q} - \mathcal{Q})$$

Observation 1: denser Fourier sampling if

$$|\mathcal{V}| \simeq Q^2$$

- Lattices are bad core arrangements
- Fermat's spiral is not bad

$\mathcal{Q}$  $\mathcal{V}$

(noiseless) Interferometric sensing model

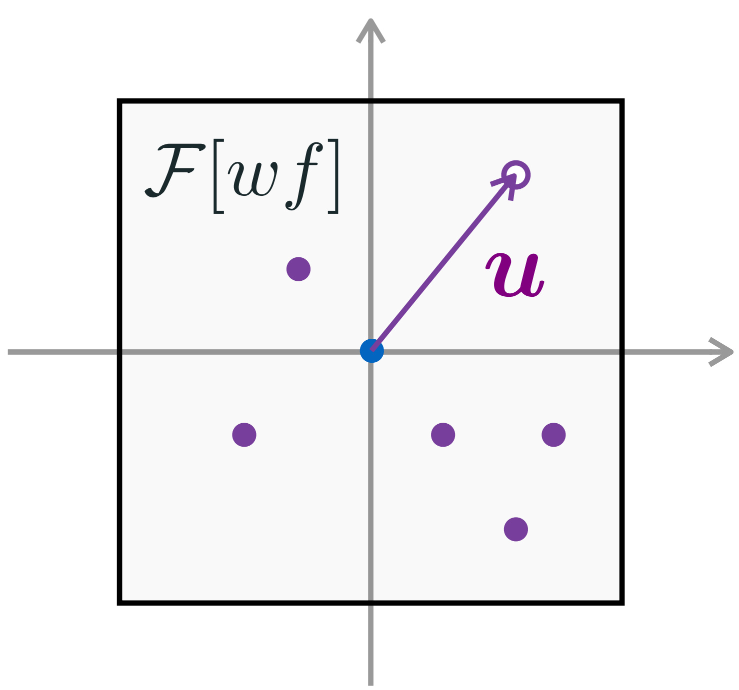
Therefore

$$\langle f, \varphi_{\boldsymbol\alpha}\rangle = \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol q_j - \boldsymbol q_k)^\top \boldsymbol x} w(\boldsymbol x) f(\boldsymbol x) \mathrm{d}\boldsymbol x \right]$$

$$\to \boldsymbol\alpha^* \boldsymbol{\mathcal{I}}[wf]\, \boldsymbol\alpha$$ → ROP!!

with the (Hermitian) *interferometric matrix* $\boldsymbol{\mathcal{I}}[wf] \in \mathbb{C}^{Q\times Q}$ s.t.

$$(\boldsymbol{\mathcal{I}}[wf])_{j,k} := \int_{\mathbb{R}^2} e^{\underbrace{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol q_j - \boldsymbol q_k)^\top}_{\boldsymbol u} \boldsymbol x}\, w(\boldsymbol x) f(\boldsymbol x)\, \mathrm{d}\boldsymbol x = \mathcal{F}[wf](\mathcal{V})$$



$$\boldsymbol u \in \mathcal{V} := \tfrac{1}{\lambda z}(\mathcal{Q} - \mathcal{Q})$$

Observation 2:

Low-complexity on f
→
Low-complexity on $\boldsymbol{\mathcal{I}}$.

e.g., sparsity → low-rank

Interferometric sensing model

Composition of two sensing methods

$Q \times Q$ (1)

$$\boldsymbol{y} = (y_{\boldsymbol{\alpha}_1}, \cdots, y_{\boldsymbol{\alpha}_m})^\top = \boldsymbol{\Phi}(\boldsymbol{\mathcal{I}}[wf]) + \text{noise},$$

(2) $m \times Q^2$

with $\boldsymbol{\Phi}(\boldsymbol{M}) := \{\langle \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j^*, \boldsymbol{M} \rangle_{\mathrm{F}}\}_{j=1}^m$.

Sample complexities of interest:

- (2) Does $\boldsymbol{\Phi}$ capture enough from $\boldsymbol{\mathcal{I}}$? $\leftrightarrow$ m big enough?
- (1) Does $\boldsymbol{\mathcal{I}}$ capture enough from f? $\leftrightarrow$ Q big enough?
  Core arrangement?

A few answers from a few simplifications ...

Theory + Simulations + Experimental results

Theoretical guarantees

Given

- a discretisation $\boldsymbol{f}$ of wf over N pixels
- a frequency coverage $\mathscr{V}$ respecting usual CS conditions (RIP)

(under specific simplifying assumptions)

If the $\{\boldsymbol{\alpha}_i\}$ are (sub)Gaussian, given a sparsity level K
and provided $M = O(K)$ and $Q^2 = O(K)$ (up to logs),
then, with high probability, given the observations $\boldsymbol{z} = \boldsymbol{\Phi}'[\boldsymbol{f}] + \underbrace{\text{noise}}_{\|\cdot\|_1 \leqslant \epsilon}$,
an ℓ_1-minimization program gives an estimate $\boldsymbol{f}'$ with

$$\|\boldsymbol{f} - \boldsymbol{f}'\|_2 \leqslant C \frac{\|\boldsymbol{f} - \boldsymbol{f}_K\|_1}{\sqrt{K}} + D \frac{\epsilon}{M}$$

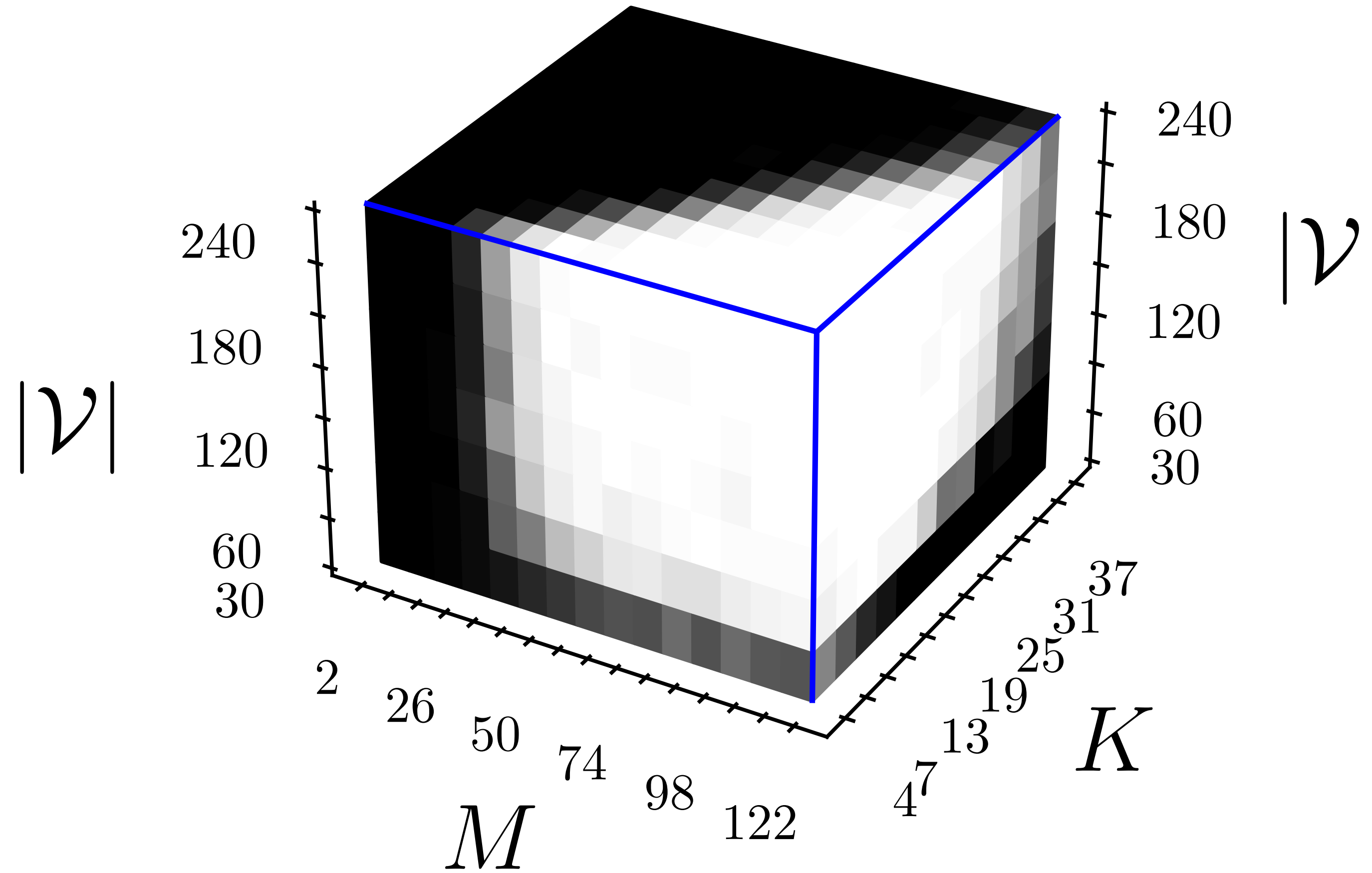
for some $C, D > 0$.

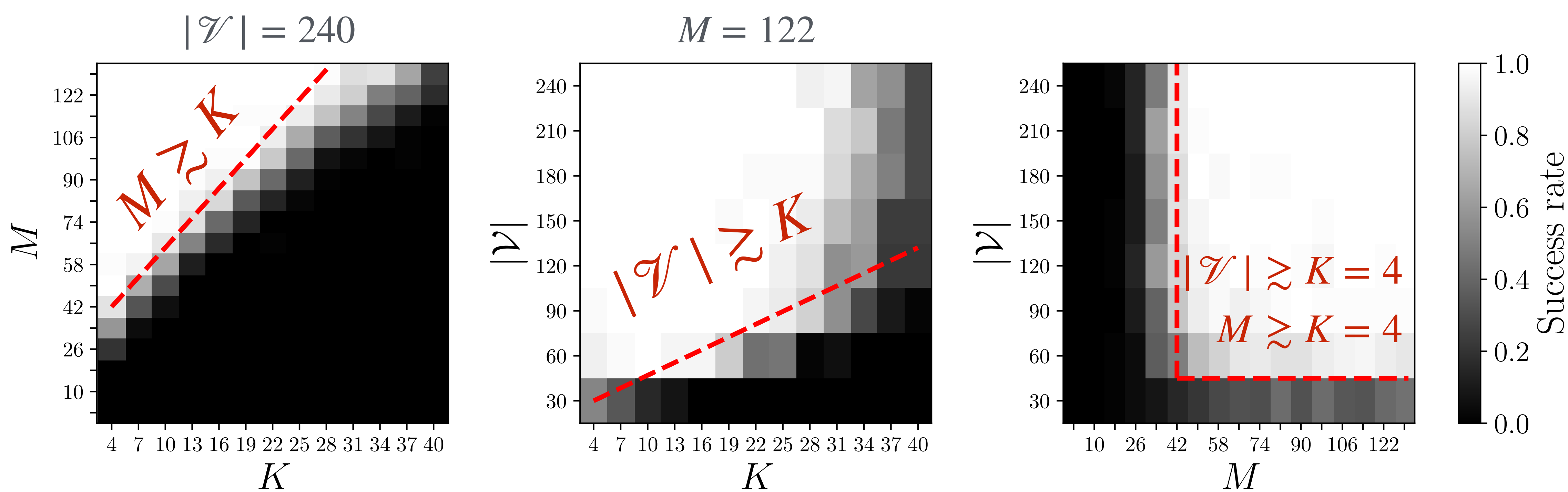
Proof idea: $\boldsymbol{\Phi}'$ = centering of $\boldsymbol{\Phi}$; show that $\boldsymbol{\Phi}'$ respects a variants of the restricted isometry property.

1-D simulations: phase transition diagrams

Simplified setting:

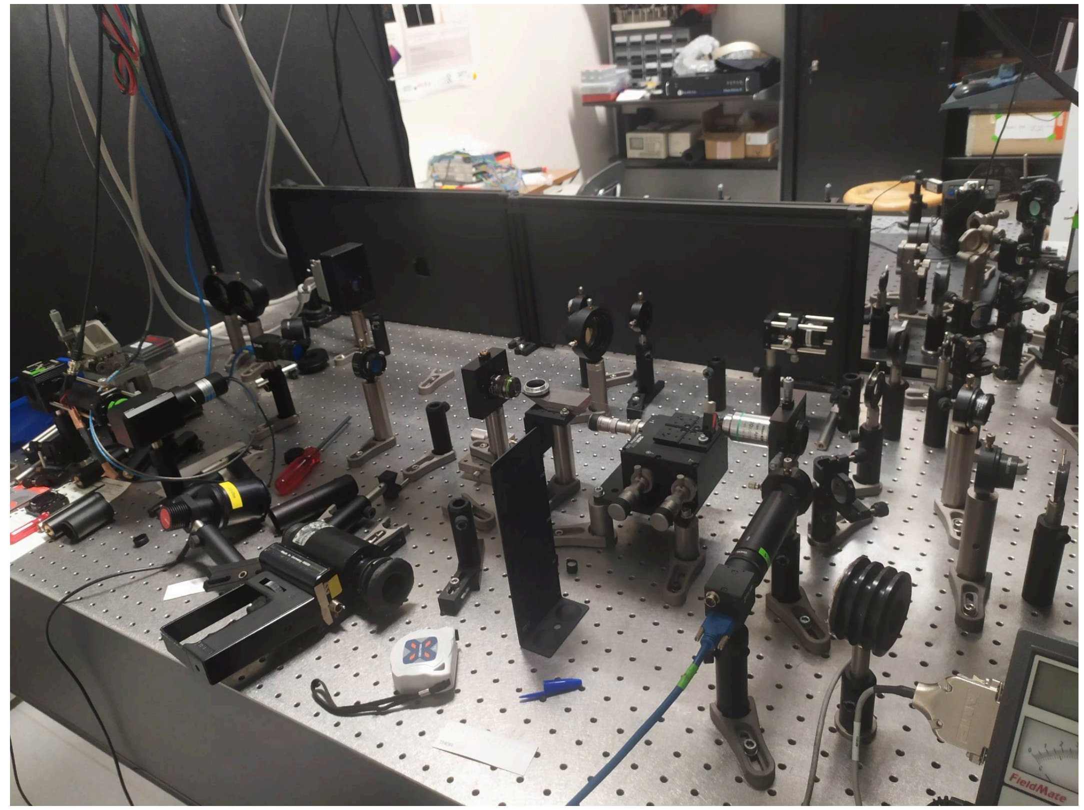
- 1-D core arrangement, $N = 256$
- K-sparse vectors
- Random $\{\boldsymbol{\alpha}_j\}_{j=1}^{M}$
- Q, M, K varying
- 80 trials, Success if $\geq$ 40 dB



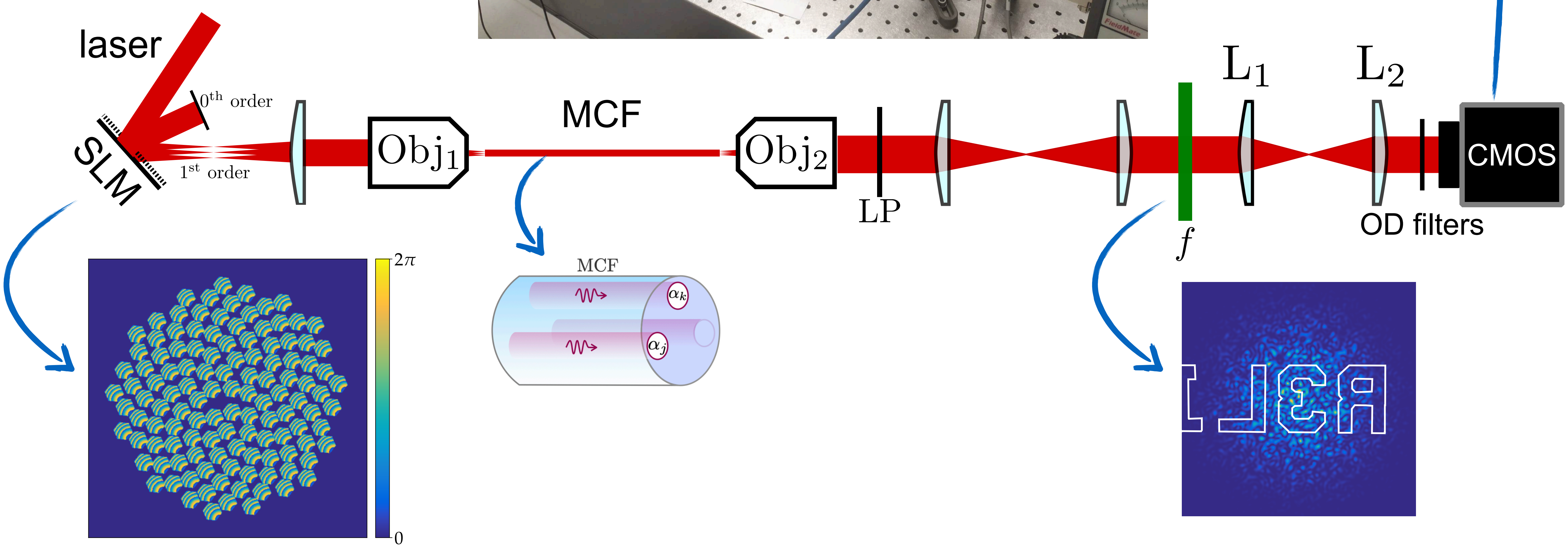


Experiments (in Institut Fresnel, France)

(Adapted from xkcd #1233)



Single pixel
=
Pixel summation



+ a lot of calibrations & validations

Experiments (in Institut Fresnel, France)

Average SNR (5 trials)

USAF target $Q = 55$ $Q = 110$