Time for dithering! Quantized random embeddings with RIP random matrices

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Outline

- (Brief introduction to CS)
- Quantizing Compressive Sensing theory
- Properties of dithered, quantized random projections
- Classification in a quantized world
- Quantized Random Sketching of Datasets

Brief introduction to CS

Compressed Sensing...

... in a nutshell:

> Generalize Dirac/Nyquist sampling:
>
> 1°) ask *few* (**linear**) *questions*
>
> about your ***informative*** signal
>
> 2°) and recover it *differently* (**non-linearly**)”

Compressed Sensing...

... in a nutshell:

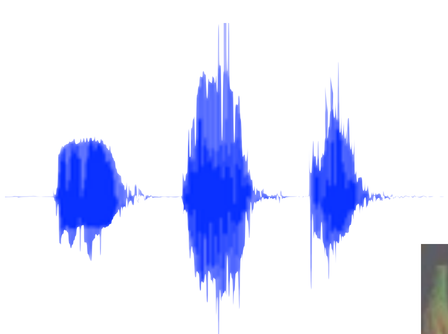
Generalize Dirac/Nyquist sampling:

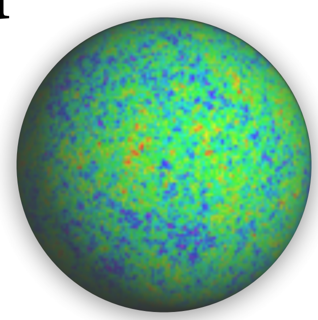
1°) ask *few* (**linear**) *questions*

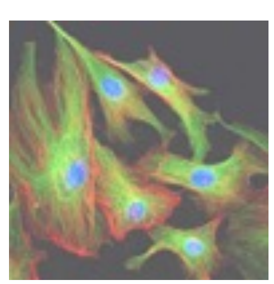
about your ***informative*** signal

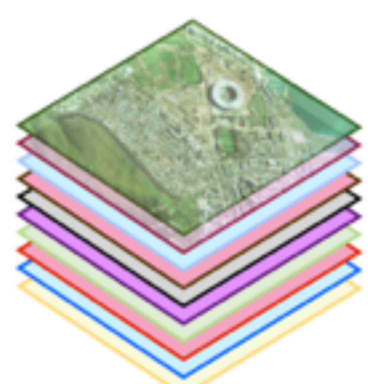
2°) and recover it *differently* (**non-linearly**)”

Signals composed of structures



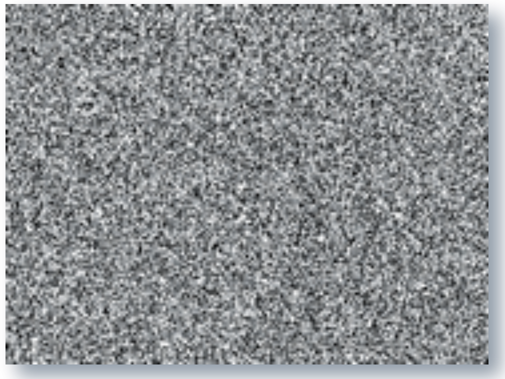






$\exists$ a prior!

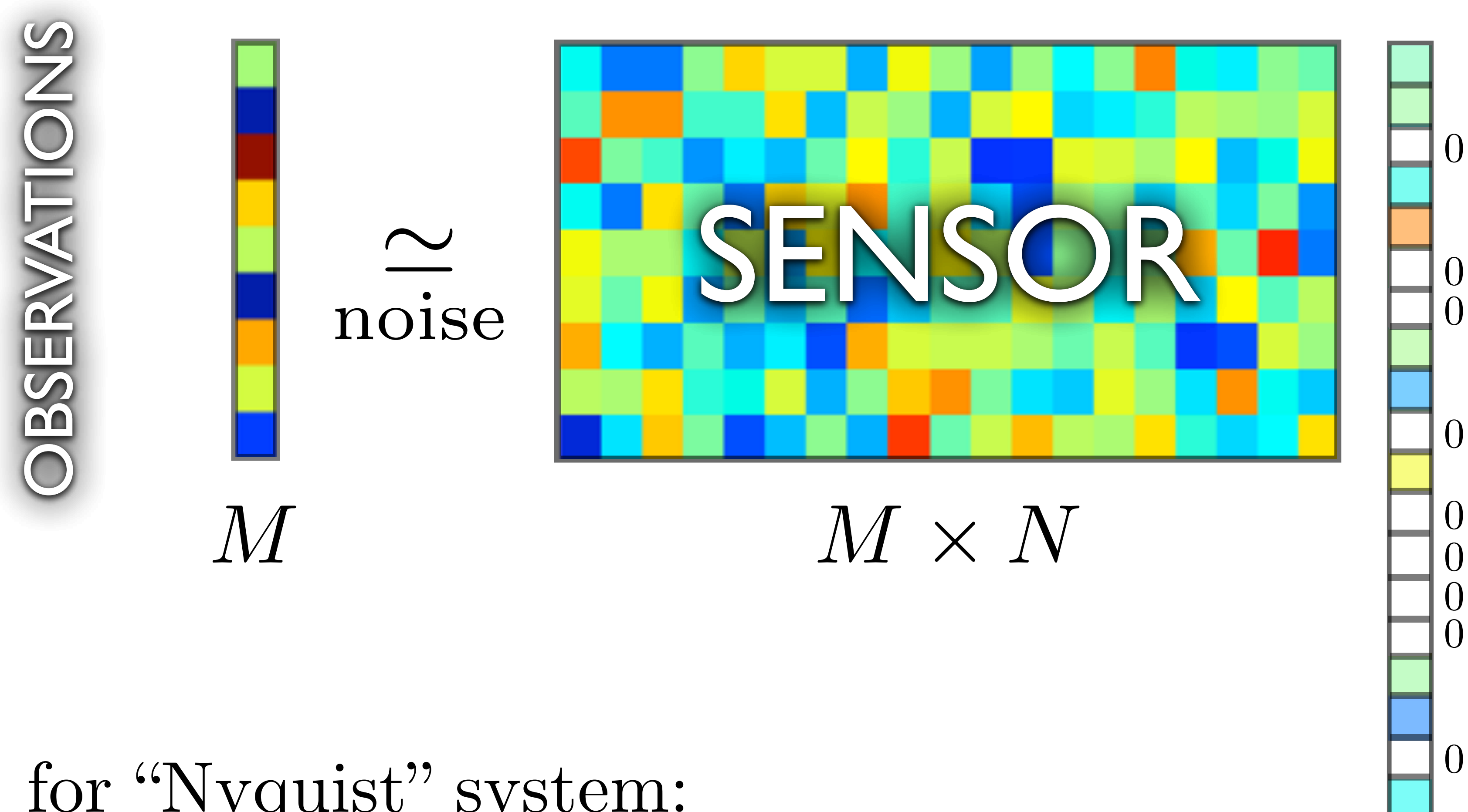
$\neq$



Compressed Sensing...

M questions — Sensing method — Signal

$$\boldsymbol{y} \qquad \boldsymbol{\Phi} \qquad \boldsymbol{x}$$



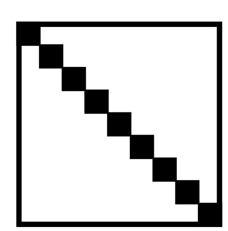
M $\qquad$ $M \times N$ $\qquad$ N

Sparsity
Prior
$(\Psi = \mathrm{Id})$

A signal in this discrete world

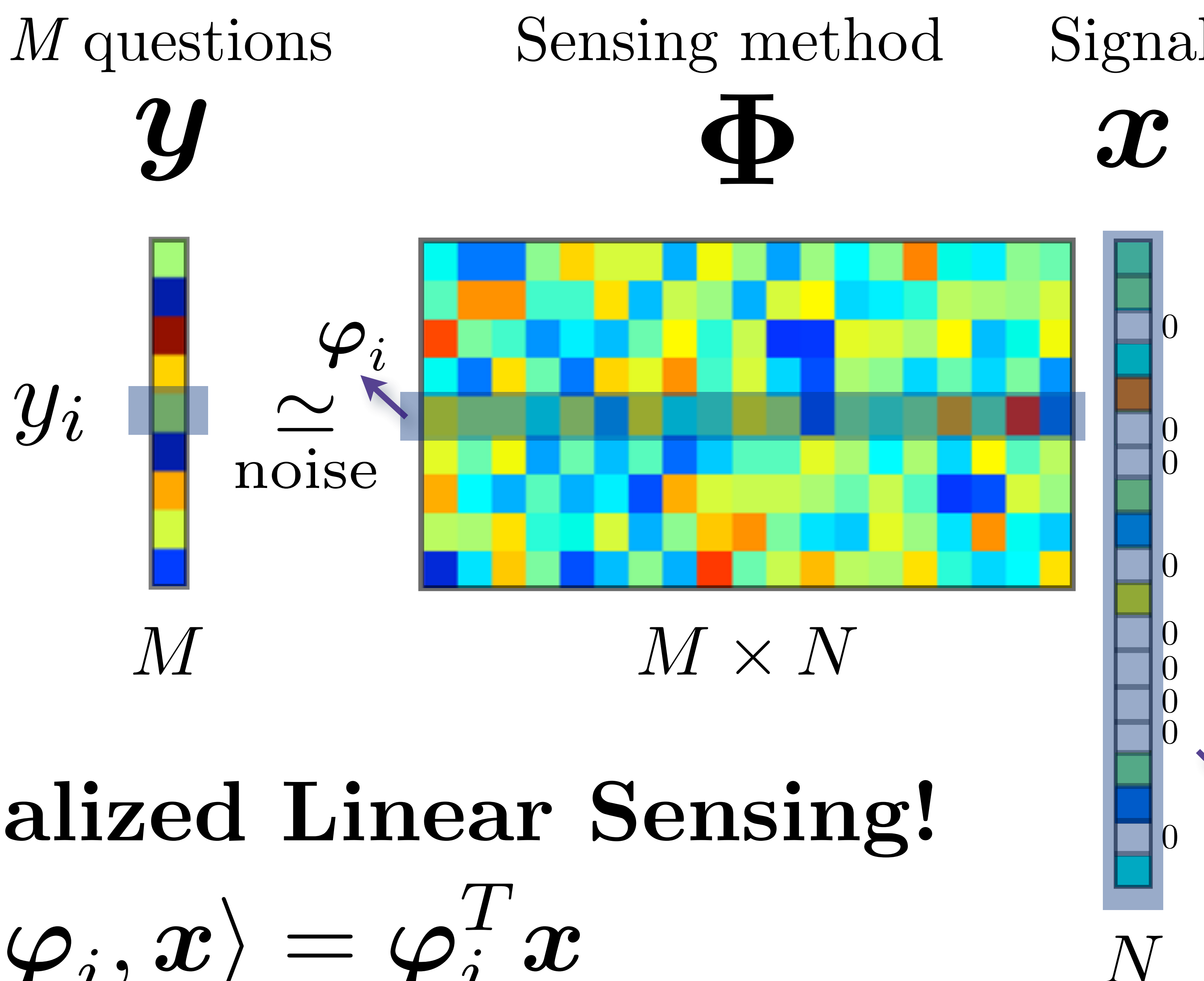
Reminder: for "Nyquist" system:

$\Leftrightarrow \boldsymbol{\Phi} \approx$



Identity

Compressed Sensing...



Sparsity
Prior
$(\Psi = \mathrm{Id})$

A signal
in this
discrete
world

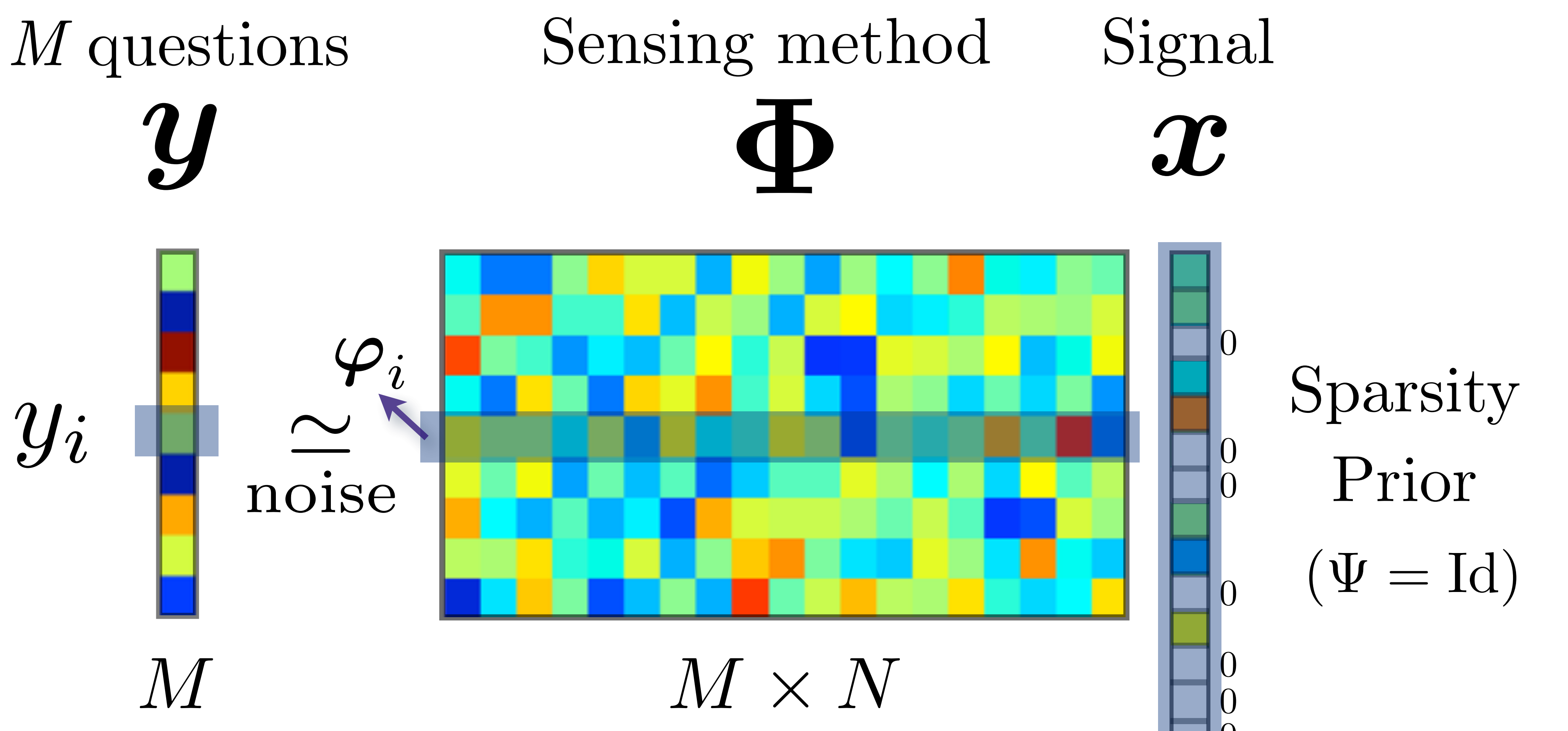
Generalized Linear Sensing!

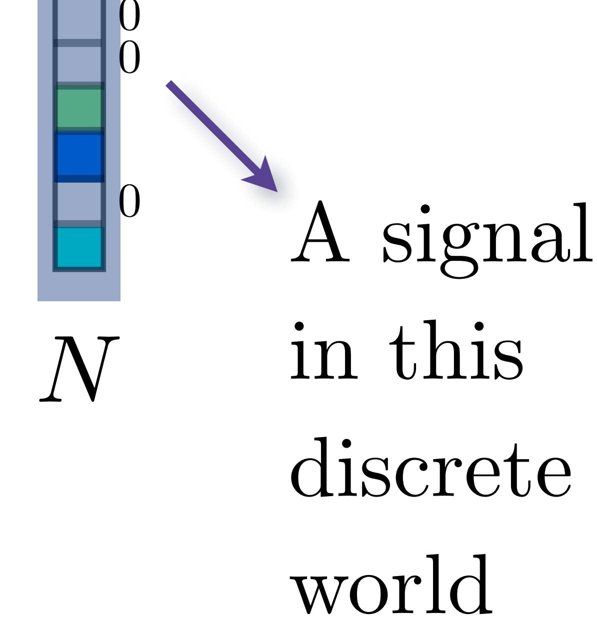
$$y_i \simeq \langle \boldsymbol{\varphi}_i, \boldsymbol{x} \rangle = \boldsymbol{\varphi}_i^T \boldsymbol{x}$$

$$1 \leq i \leq M$$

e.g., to be realized
optically/analogically

Compressed Sensing...



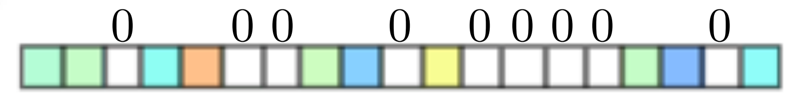


But why does it work?

Identifiability of $\boldsymbol{x}$ from $\boldsymbol{\Phi x}$?

$\Rightarrow$ use the prior!

Compressed Sensing...

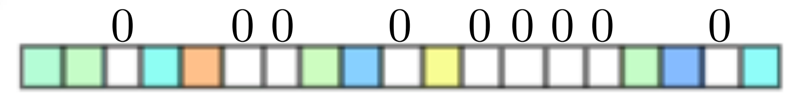


Two K-sparse signals $\boldsymbol{x}, \boldsymbol{x}' \in \Sigma_K := \{\boldsymbol{u} : \|\boldsymbol{u}\|_0 := |\operatorname{supp}\boldsymbol{u}| \leqslant K\}$

For many random constructions of $\boldsymbol{\Phi}$ (e.g., Gaussian, Bernoulli, structured) and "$M \gtrsim K \log(N/K)$", with high probability,

Geometry of $\boldsymbol{\Phi}(\Sigma_K)$
$\approx$ Geometry of Σ_K

Compressed Sensing...



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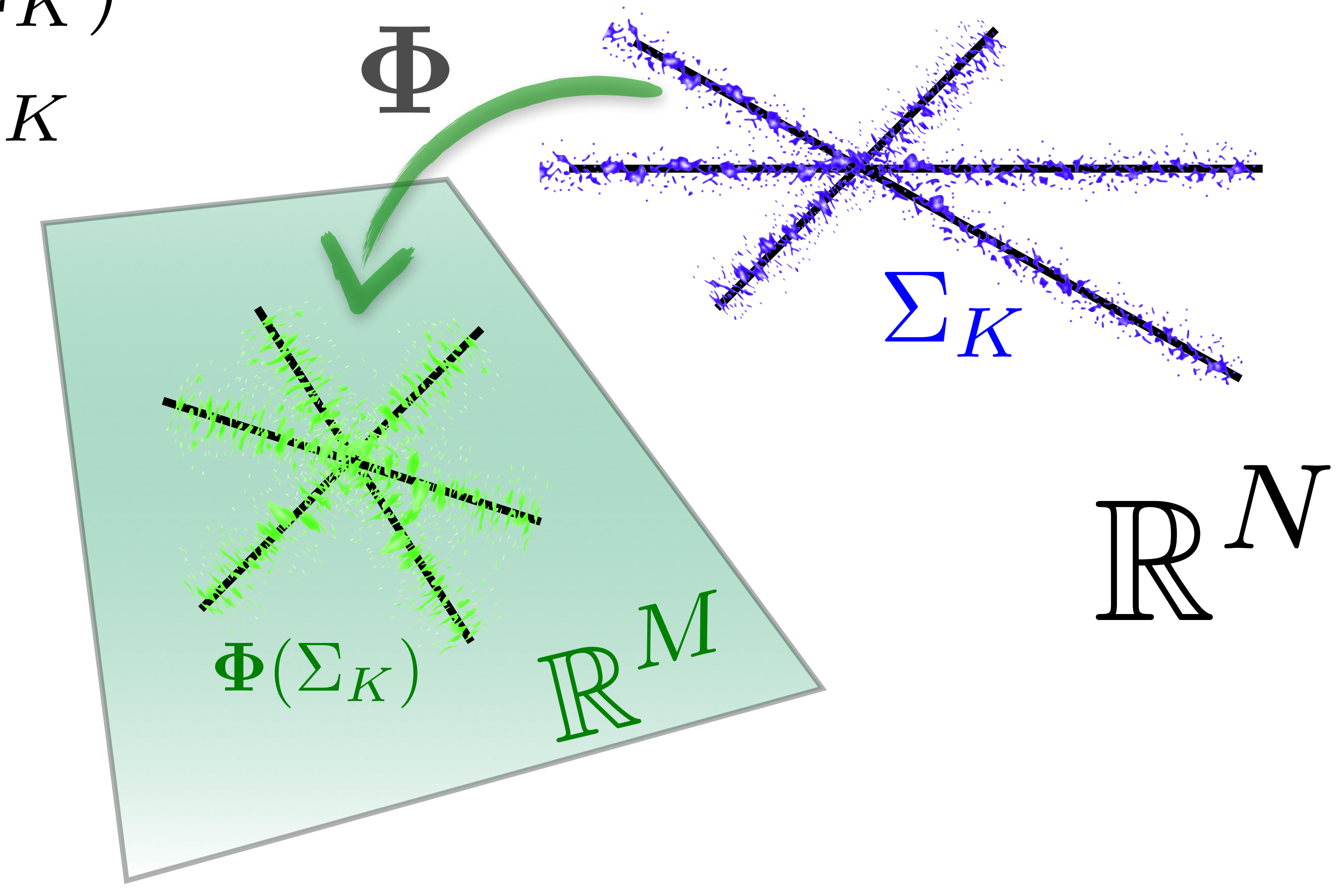


$\boldsymbol{\Phi x} \approx \boldsymbol{\Phi x}' \;\Leftrightarrow\; \boldsymbol{x} \approx \boldsymbol{x}'$

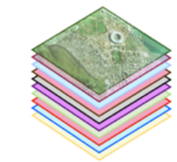
observations — true signals

For all $\boldsymbol{x}, \boldsymbol{x}' \in \Sigma_K$ and $0 < \rho < 1$,

Restricted **I**sometry **P**roperty

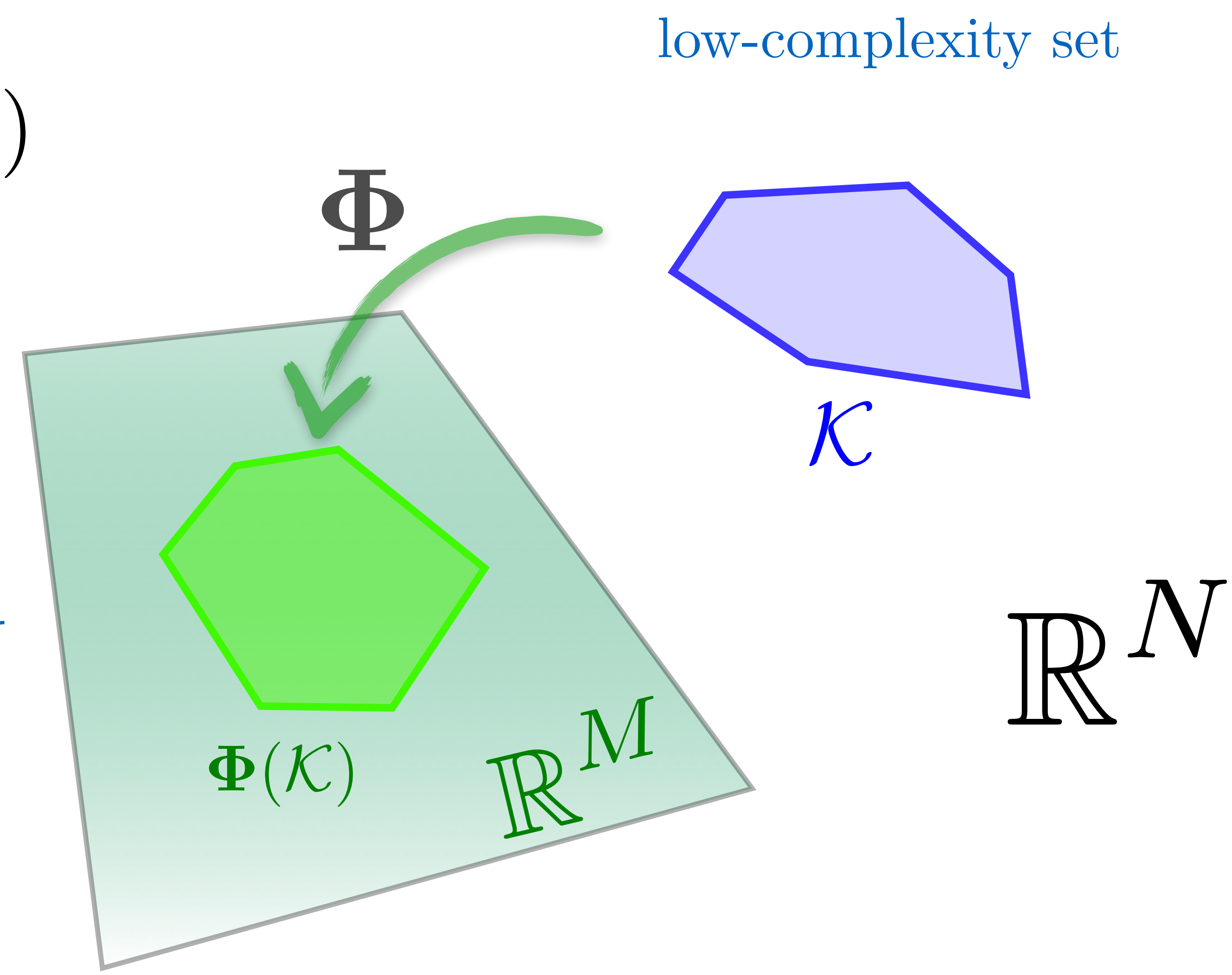
$$(1-\rho)\|\boldsymbol{x}-\boldsymbol{x}'\|^2 \leq \tfrac{1}{M}\|\boldsymbol{\Phi x}-\boldsymbol{\Phi x}'\|^2 \leq (1+\rho)\|\boldsymbol{x}-\boldsymbol{x}'\|^2$$

Compressed Sensing...

Two low-complexity signals $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{K}$ (e.g., low-rank data )

For many random constructions of $\boldsymbol{\Phi}$ (e.g., Gaussian, Bernoulli, structured) and "$M \gtrsim C_{\mathcal{K}}$", with high probability,

Geometry of $\boldsymbol{\Phi}(\mathcal{K})$
$\approx$ Geometry of $\mathcal{K}$



$$\boldsymbol{\Phi x} \approx \boldsymbol{\Phi x}' \;\Leftrightarrow\; \boldsymbol{x} \approx \boldsymbol{x}'$$

And generalization to many other low-complexity sets!

For all $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{K}$ and $0 < \rho < 1$,

Restricted **I**sometry **P**roperty

$$(1-\rho)\|\boldsymbol{x}-\boldsymbol{x}'\|^2 \leq \tfrac{1}{M}\|\boldsymbol{\Phi x}-\boldsymbol{\Phi x}'\|^2 \leq (1+\rho)\|\boldsymbol{x}-\boldsymbol{x}'\|^2$$

Compressed Sensing...

Use non-linear reconstruction methods: e.g.,

Basis Pursuit DeNoise **[Chen, Donoho, Saunders, 98]**

$$\hat{\boldsymbol{x}} \in \arg\min_{\boldsymbol{u} \in \mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{y} - \boldsymbol{\Phi u}\| \leq \epsilon$$

If $\frac{1}{\sqrt{M}}\boldsymbol{\Phi}$ respects the Restricted Isometry Property (RIP)

Then, if $\rho < \sqrt{2} - 1$ **[Candès, 09]**

(with $f \lesssim g \equiv \exists c > 0 : f \leqslant c\,g$)

Robustness: *vs* sparse deviation + noise.

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \lesssim \underbrace{\frac{1}{\sqrt{K}}\|\boldsymbol{x} - \boldsymbol{x}_K\|_1}_{e_0(K)} + \underbrace{\frac{\epsilon}{\sqrt{M}}}_{\text{noise}}$$

hidden constant

$e_0(K)$: error of the model

noise

The Power of Random Projections

At the heart of CS: *random projections*!

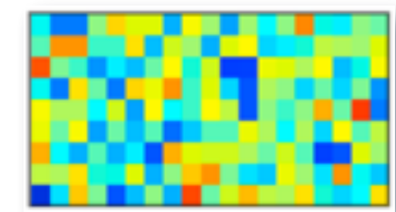
as realized by random sensing matrices

For instance:

- random sub-Gaussian ensembles (e.g., Gaussian, Bernoulli)

  e.g., Gaussian: $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$, with $\Phi_{ij} \sim_{\rm iid} \mathcal{N}(0,1)$

  or $\Phi_{ij} \sim_{\rm iid} \pm 1$ (eq. prob), $\cdots$

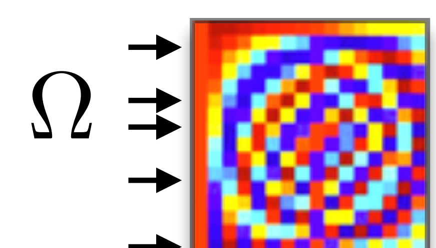


or *structured* sensing matrices (less memory, fast computations):

- random Fourier/Hadamard ensembles (e.g., for CT, MRI);

  e.g., $\boldsymbol{\Phi} = \boldsymbol{F}_\Omega$, with $\boldsymbol{F} \in \mathbb{C}^{n \times n}$

  and random $\Omega \subset \{1, \cdots, n\}$, $|\Omega| = m$

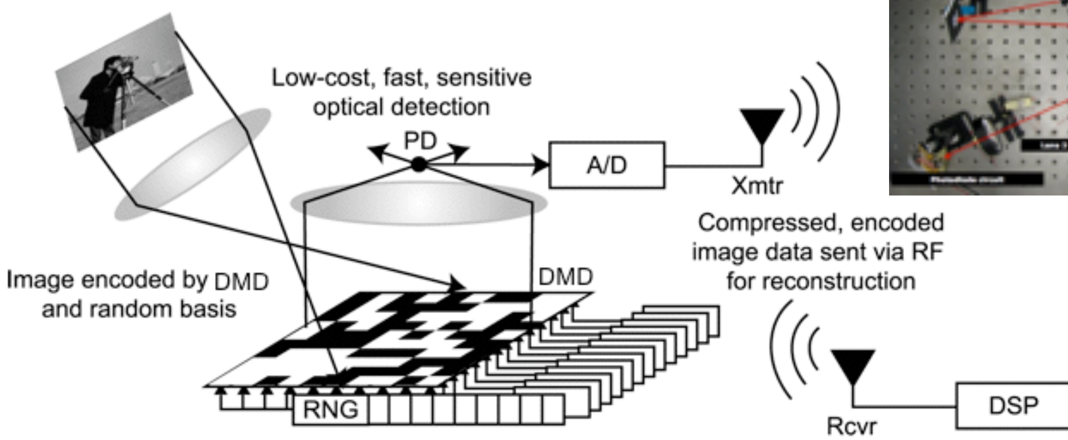


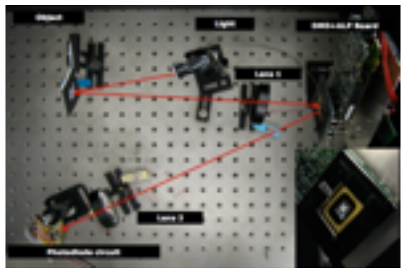
- random convolutions, spread-spectrum (e.g., for imaging), ...

(see, *e.g.*, **[Foucart, Rauhut, 2013]**)

CS Applications

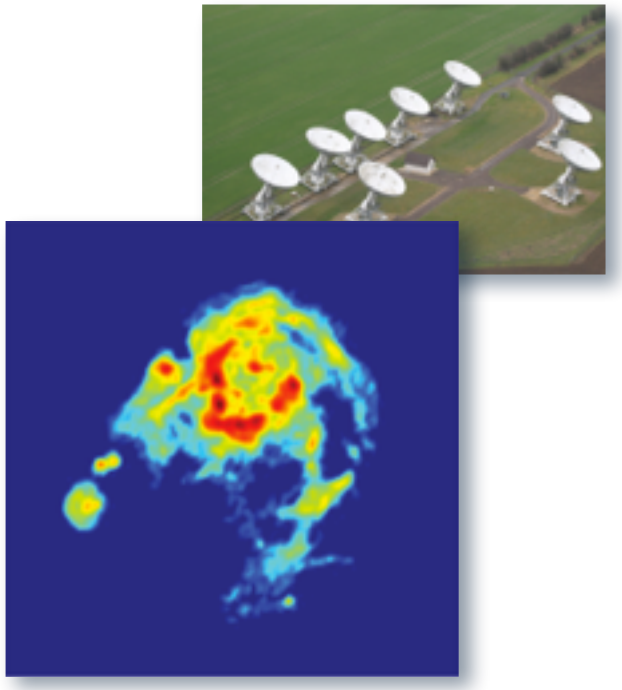
Rice Single Pixel Camera





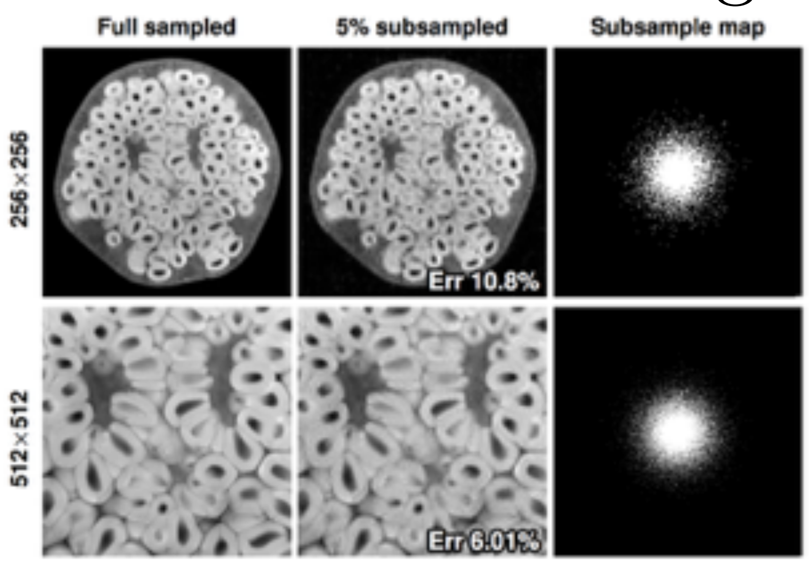
[Duarte, Davenport, Takbar, Laska, Sun, Kelly, Baraniuk, 08]

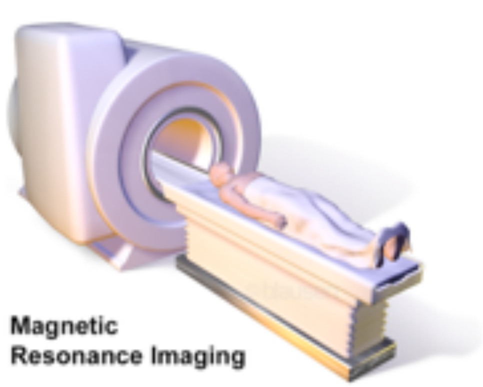
Radio-Astronomy



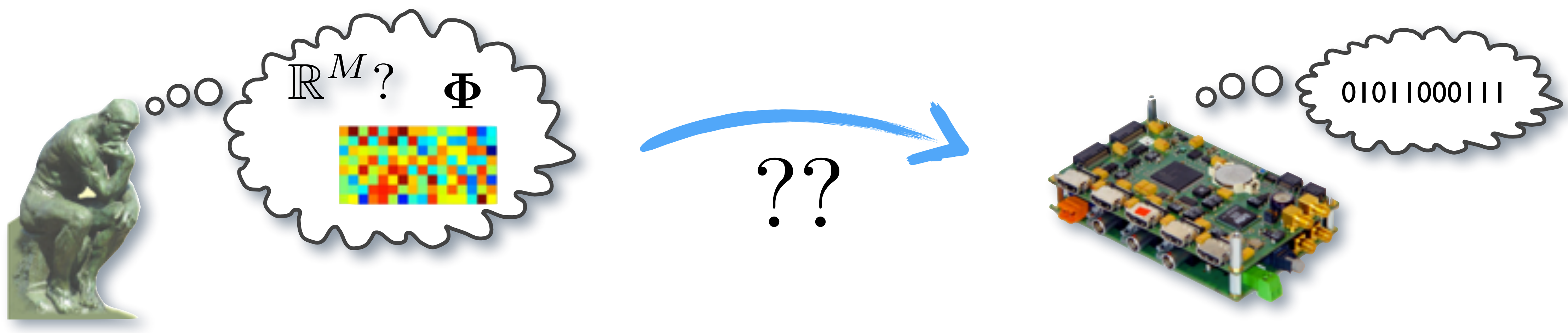
[Carrillo, McEwen, Wiaux, 12]

MRI & Fourier Imaging





[Roman, Adcock, Hansen, 14 + Siemens]

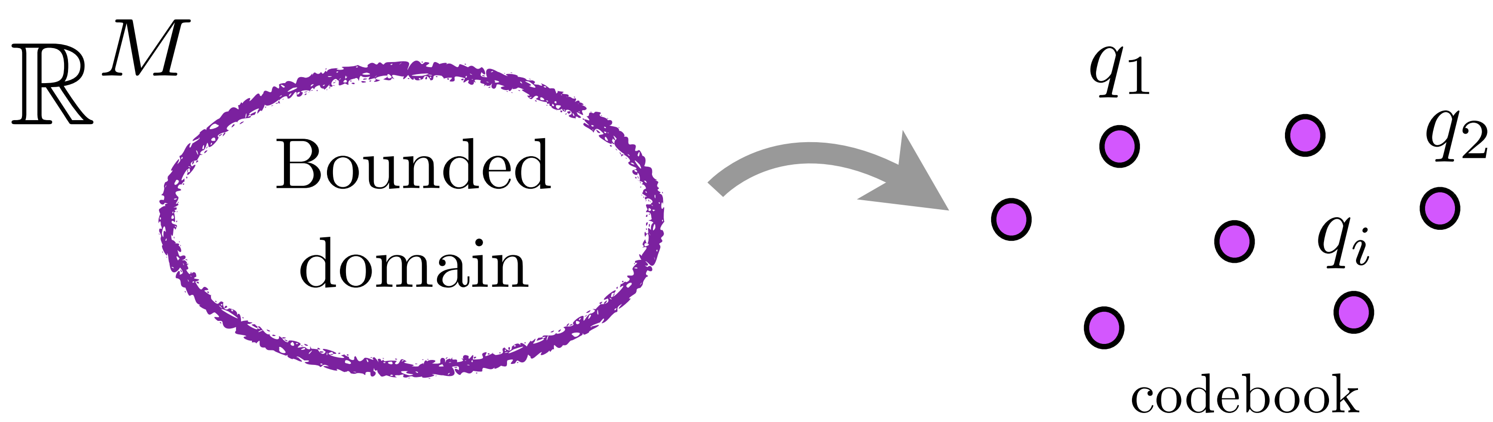


Quantizing Compressive Sensing theory

What is quantization?

- <u>Generality</u>:

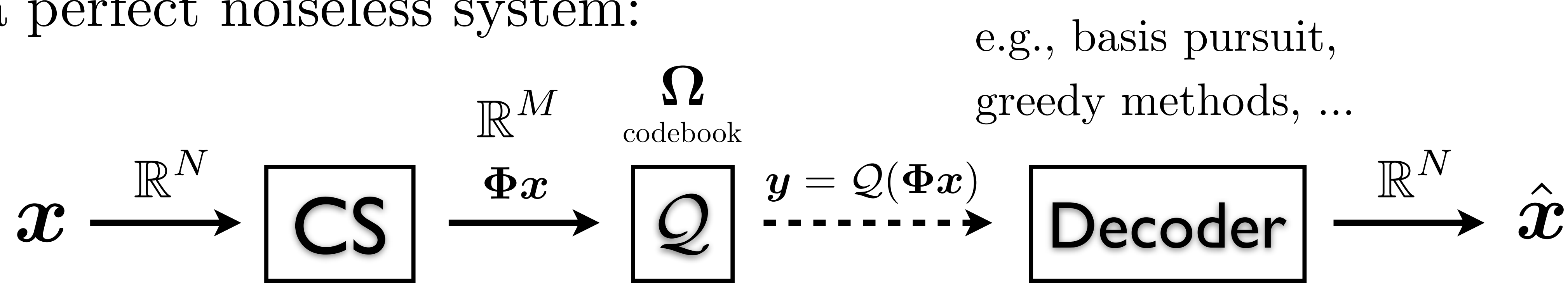
  Intuitively: "*Quantization maps a bounded continuous domain to a set of finite elements (or codebook)*"



$$\mathcal{Q}[\boldsymbol{u}] \in \{\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots\}$$

Quantizing Compressed Sensing?

In a perfect noiseless system:



Finite codebook $\Rightarrow \hat{\boldsymbol{x}} \neq \boldsymbol{x}$

i.e., impossibility to encode continuous domain
in a finite number of elements.

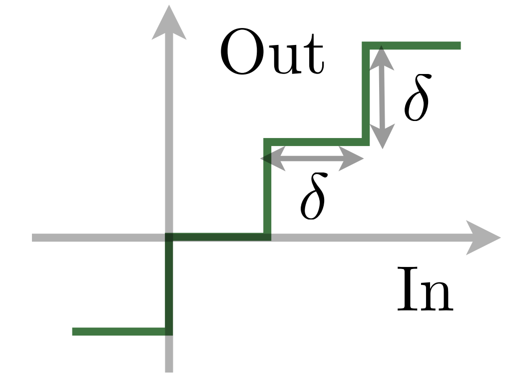
Objective: Minimize $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|$

given a certain number of:
bits, measurements, or bits/meas.

Best tradeoff? Large M and low bits/meas. *or* low M and high resolution?

Examples of quantization

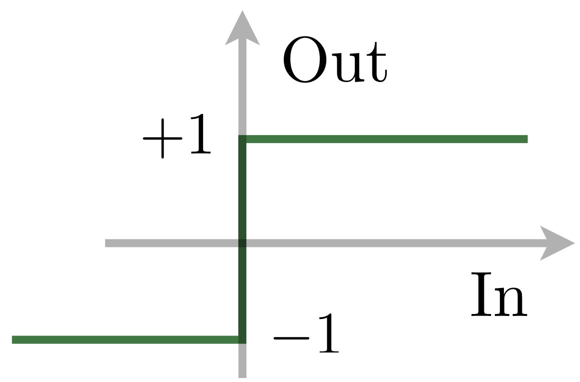
- Simple example: rounding/flooring*



$$\mathcal{Q}[\lambda] = \delta \lfloor \tfrac{\lambda}{\delta} \rfloor \in \delta\mathbb{Z}$$

for some resolution $\delta > 0$ and $\mathcal{Q}(\boldsymbol{u}) = (\mathcal{Q}(u_1), \mathcal{Q}(u_2), \cdots)$.

- Even simpler: 1-bit quantizer



$$\mathcal{Q}[\lambda] = \operatorname{sign} \lambda \in \pm 1$$

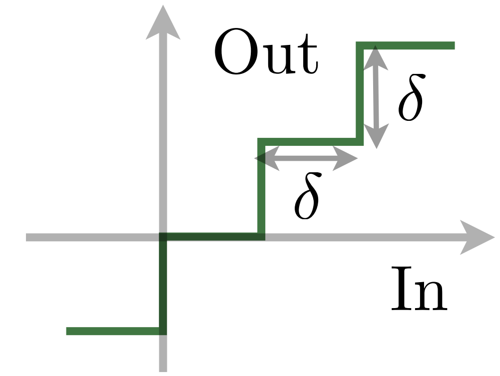
(with lost of the global measurement amplitude)

*: Also known as a special case of Pulse Code Modulation - PCM, or Memoryless Scalar Quantization - MSQ

Examples of quantization

- Simple example: rounding/flooring

$$\mathcal{Q}[\lambda] = \delta\lfloor \tfrac{\lambda}{\delta} \rfloor \in \delta\mathbb{Z}$$



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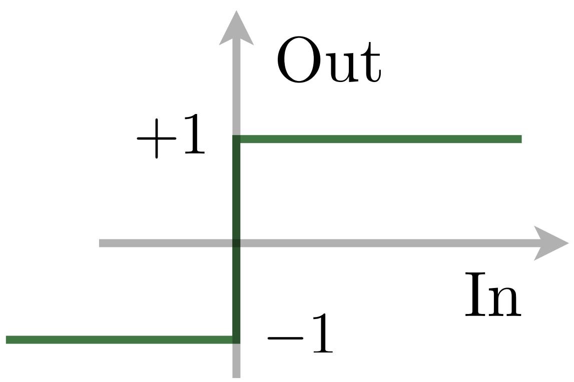
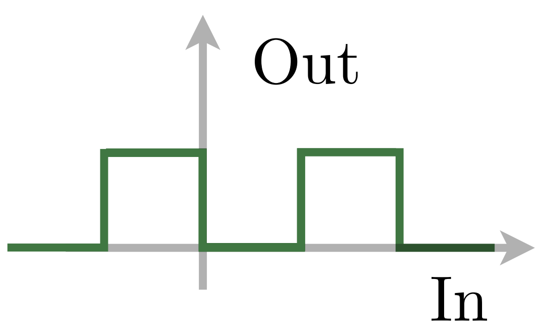


- Non-regular: square wave (or LSB)

$$\mathcal{Q}[\lambda] := \delta(\lfloor \tfrac{\lambda}{\delta} \rfloor \bmod 2)$$



- Other examples (not covered here): Non-uniform scalar quantizer, vector quantizer, ΣΔ quantizer/noise shaping, ...

(see the works of, e.g., **[Gunturk, Lammers, Powell, Saab, Yilmaz, Goyal]**)

First attempts [Candès, Tao, 04]

- Quantization is like a noise! (e.g., for $\mathcal{Q}[\lambda] = \delta\lfloor \frac{\lambda}{\delta} \rfloor \in \delta\mathbb{Z}$)

$$\boldsymbol{y} = \mathcal{Q}(\boldsymbol{\Phi x}) = \boldsymbol{\Phi x} + \boldsymbol{n}, \quad \text{with } \boldsymbol{n} = \mathcal{Q}(\boldsymbol{\Phi x}) - \boldsymbol{\Phi x}.$$

$$\text{and } \|\boldsymbol{n}\|^2 = O(m\,\delta^2)$$

- **Problem**:

e.g., for BPDN, with

$$\hat{\boldsymbol{x}} \in \operatorname*{arg\,min}_{\boldsymbol{u} \in \mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{y} - \boldsymbol{\Phi u}\| \leq \epsilon$$

$\Rightarrow$ $\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \lesssim \frac{\epsilon}{\sqrt{m}} = O(\delta)$ does not decay if m increases!

counterintuitive?

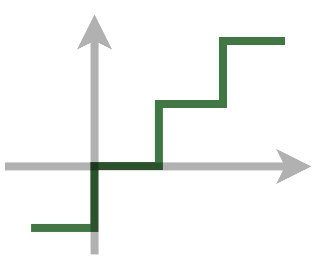
First attempts [Candès, Tao, 04]

- Causes of the problem:

1 Solution is **not** **consistent** (*i.e.*, we lost something):

$$\|\boldsymbol{\Phi}\hat{\boldsymbol{x}} - \boldsymbol{y}\| \text{ small} \quad \not\Rightarrow \quad \mathcal{Q}(\boldsymbol{\Phi}\hat{\boldsymbol{x}}) = \boldsymbol{y}$$

2 Quantization is discontinuous (it does not "**dither**")



$$\exists\, \boldsymbol{x} : \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| > C \quad \text{(e.g., if } \Phi_{ij} \in \{\pm 1\}\text{)!}$$

[Plan, Vershynin, 13] [LJ, 17]

1 & **2** = two main ingredients in this talk

The power of consistency

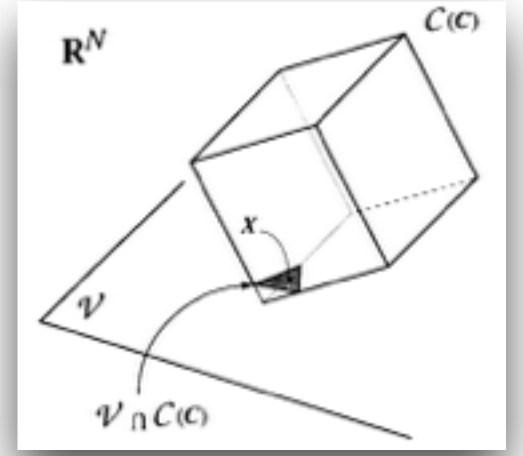
1

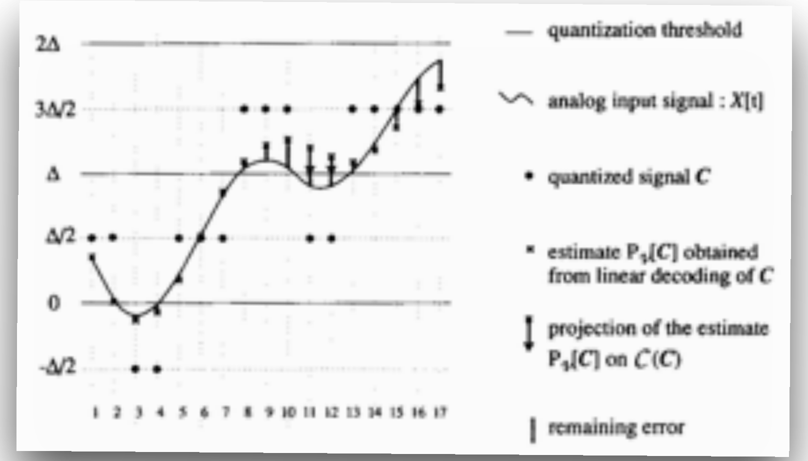
- Former insight: [Thao, Vetterli, 94]

  Quantization of oversampled, band-limited signals

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(\delta/R)$$

with R the oversampling rate (i.e., M in CS),
and $\hat{\boldsymbol{x}}$ an estimate that is both band-limited (same prior)
and consistent (non-linear reconstruction needed)





N. Thao and M. Vetterli, "Deterministic analysis of oversampled A/D conversion and decoding improvement based on consistent estimates," Signal Processing, IEEE Transactions on Signal Processing, vol. 42, no. 3, pp. 519–531, 1994.

The power of dithering

2

- Inject a pre-quantization, uniform "noise":

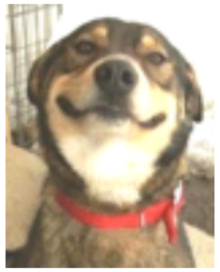
  Given a resolution $\delta > 0$,

  a scalar quantizer $\mathcal{Q}(\cdot) := \delta\lfloor \cdot/\delta \rfloor$,

  a "well-behaved" $\boldsymbol{\Phi}$ (see later),

  and a *dithering* $\boldsymbol{\xi} \in \mathbb{R}^m$ with $\xi_j \sim_{\text{iid}} \mathcal{U}([0,\delta])$

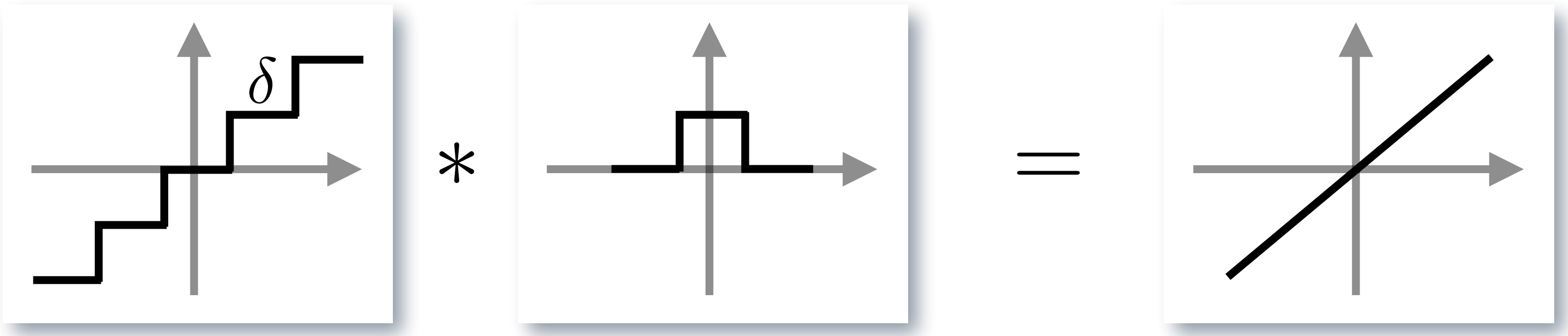
The good boy!



$$\boxed{\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})}$$

- Motivation? $\mathbb{E}_{\boldsymbol{\xi}} \mathcal{Q}(\boldsymbol{u} + \boldsymbol{\xi}) = \boldsymbol{u}$

  $\Rightarrow \mathsf{A}(\boldsymbol{x}) \approx \boldsymbol{\Phi x}$ if M large



The power of dithering

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$$\boxed{\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi})}$$

- Motivation? $\mathbb{E}_{\boldsymbol{\xi}}\mathcal{Q}(\boldsymbol{u} + \boldsymbol{\xi}) = \boldsymbol{u}$

  $\Rightarrow \mathsf{A}(\boldsymbol{x}) \approx \boldsymbol{\Phi}\boldsymbol{x}$ if M large

- Interesting quantized (discrete) random embedding of low-complexity signals!

Properties of dithered, quantized random projections

Parenthesis:

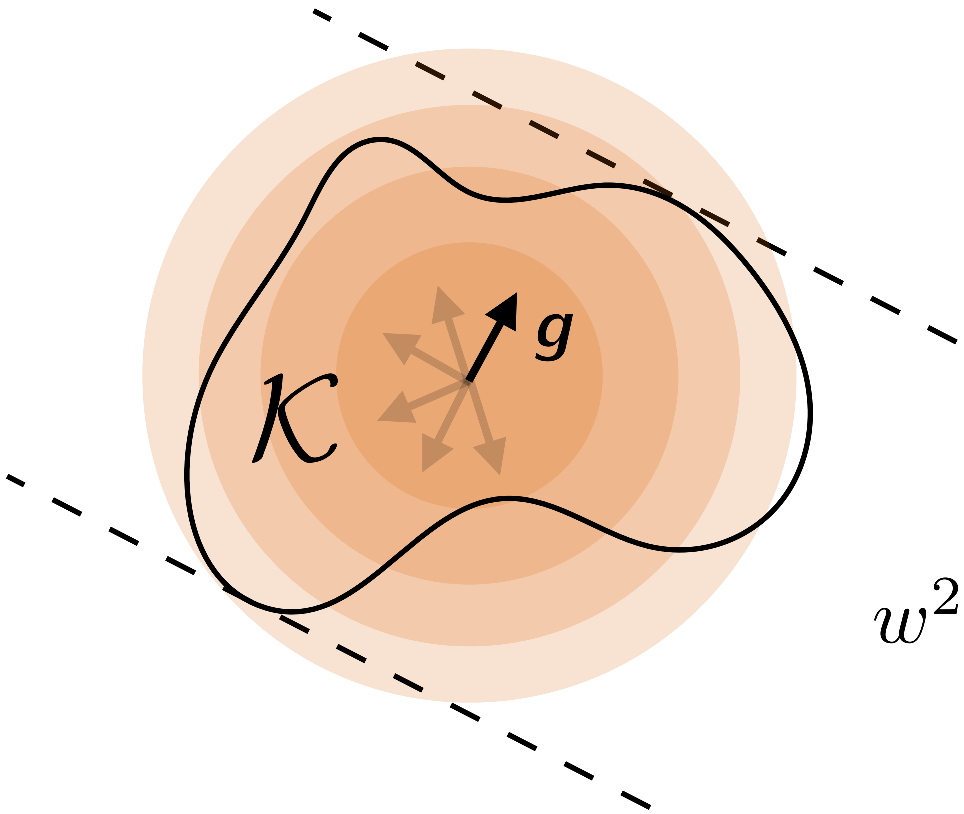
How to measure the dimension
of a low-complexity set?

Parenthesis:

Gaussian (Mean) Width (GW):

Let $\mathcal{K} \subset \mathbb{R}^n$, $\boldsymbol{g} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_n)$,

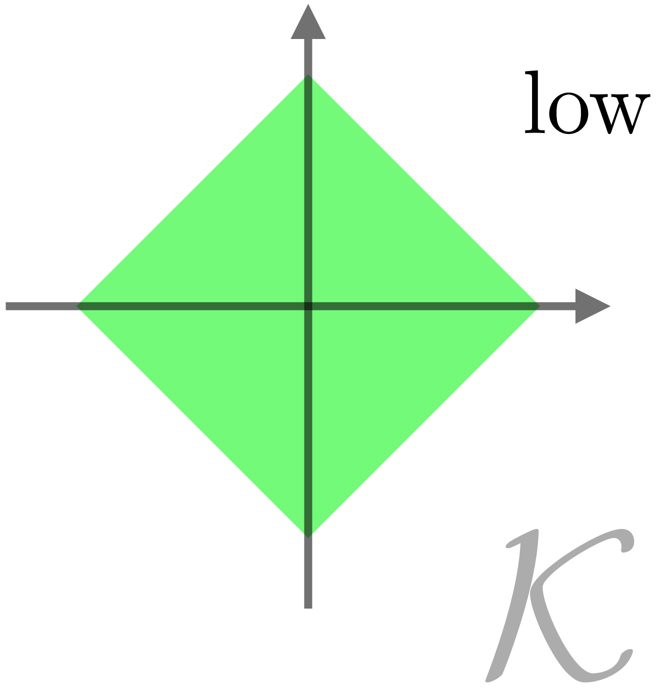
$$w(\mathcal{K}) = \mathbb{E}_{\boldsymbol{g}} \sup_{\boldsymbol{x} \in \mathcal{K}} |\langle \boldsymbol{x}, \boldsymbol{g} \rangle|$$



Examples:

$$w^2(\mathcal{K}) \lesssim \log|\mathcal{K}|$$
$$w^2(\mathbb{B}^n) \lesssim n$$
$$w^2(\Sigma_k^n \cap \mathbb{B}^n) \lesssim k \log(n/k)$$
$$w^2(\mathcal{M}_r \cap \mathbb{B}_F^{n\times n}) \lesssim rn$$
$$w^2(\cup_{i=1}^T \mathcal{K}_i) \lesssim \log T + \max_i w^2(\mathcal{K}_i)$$
$$\vdots$$

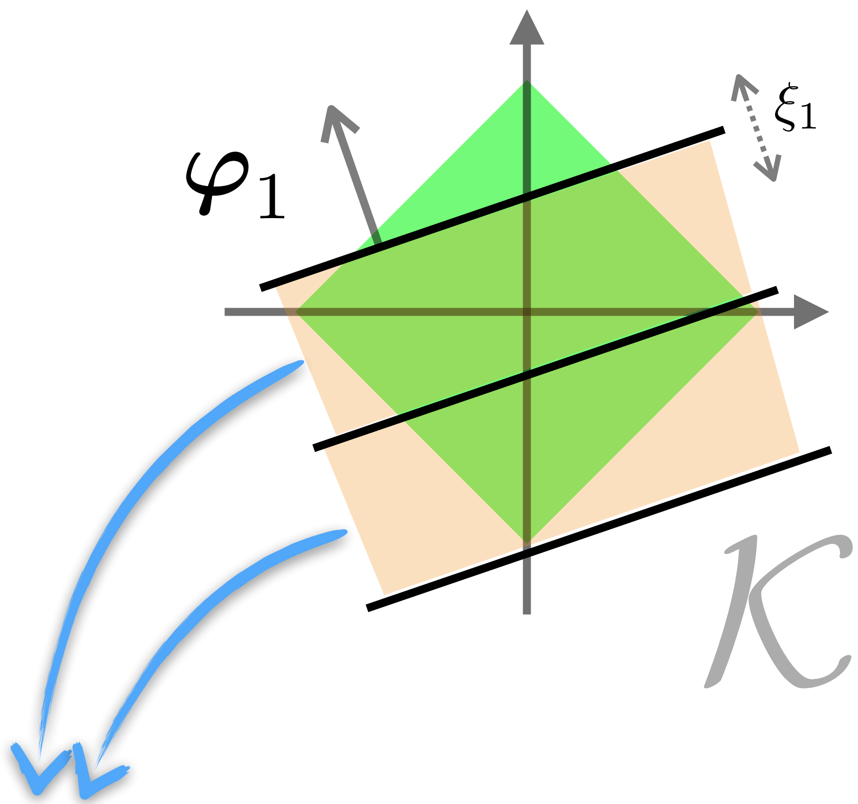
(A) Control of the "consistency width"



low complexity set $\mathcal{K}$

(e.g., sparse signals,
low-rank matrix,
compressible signals, ...)

Ⓐ Control of the "consistency width"

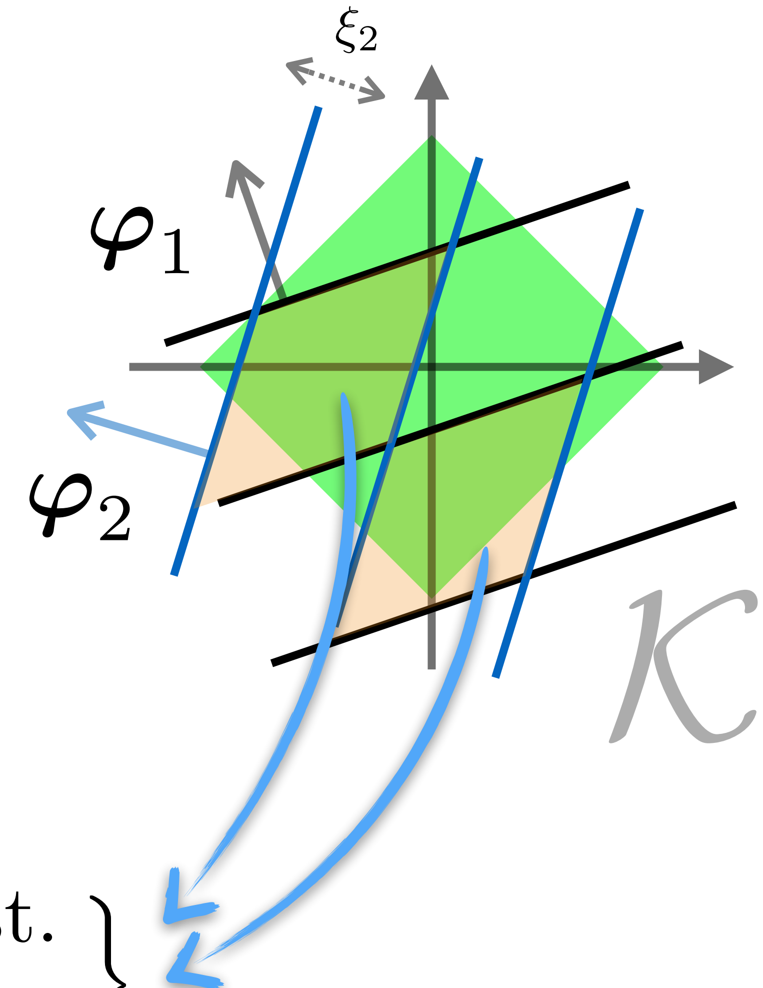


$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\underbrace{\mathcal{Q}(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1)}_{\delta\lfloor(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1)/\delta\rfloor} = \text{cst.}$$

Ⓐ Control of the "consistency width"

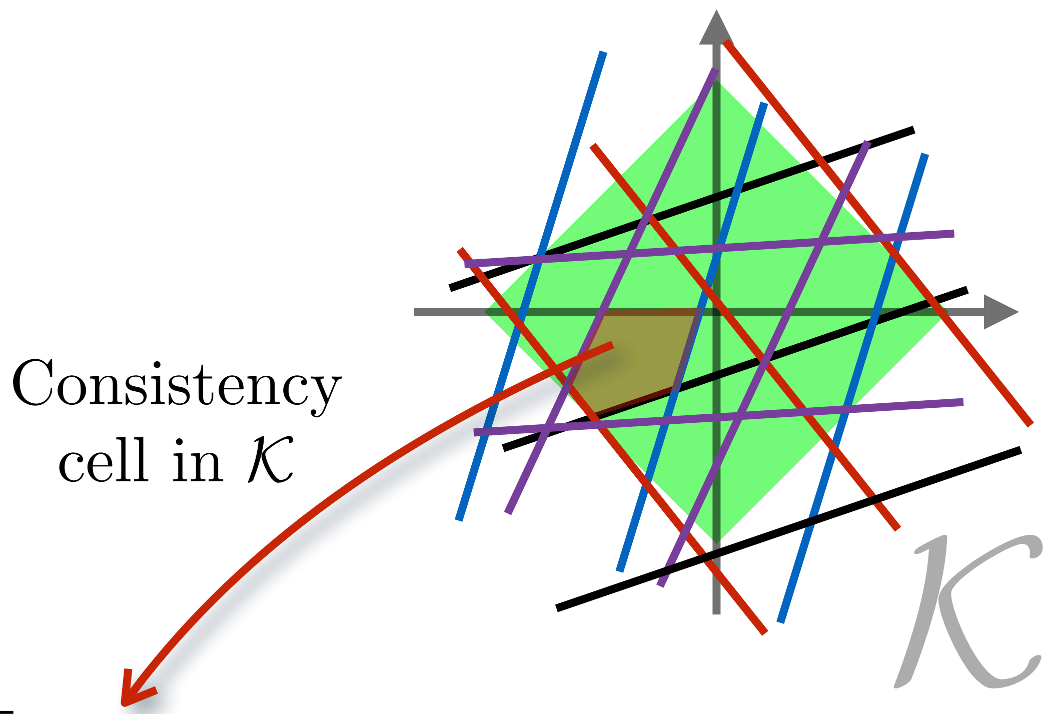


$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\left.\begin{aligned} \mathcal{Q}(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1) &= \text{cst.} \\ \mathcal{Q}(\boldsymbol{\varphi}_2^\top \boldsymbol{u} + \xi_2) &= \text{cst.} \end{aligned}\right\}$$

Control of the "consistency width"



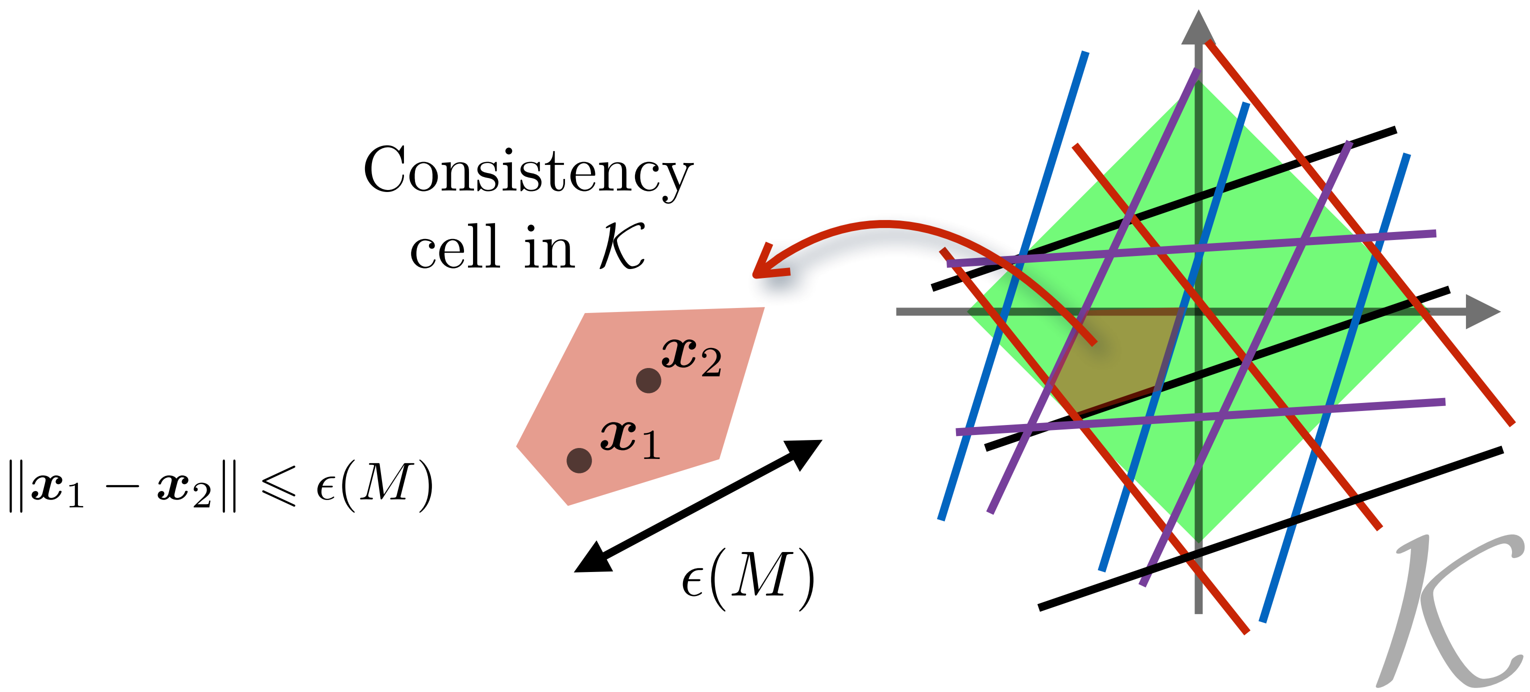
$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\mathsf{A}(\boldsymbol{u}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{u} + \boldsymbol{\xi}) = \boldsymbol{y}$$

for some $\boldsymbol{y} \in \delta\mathbb{Z}^M$

A Control of the "consistency width"



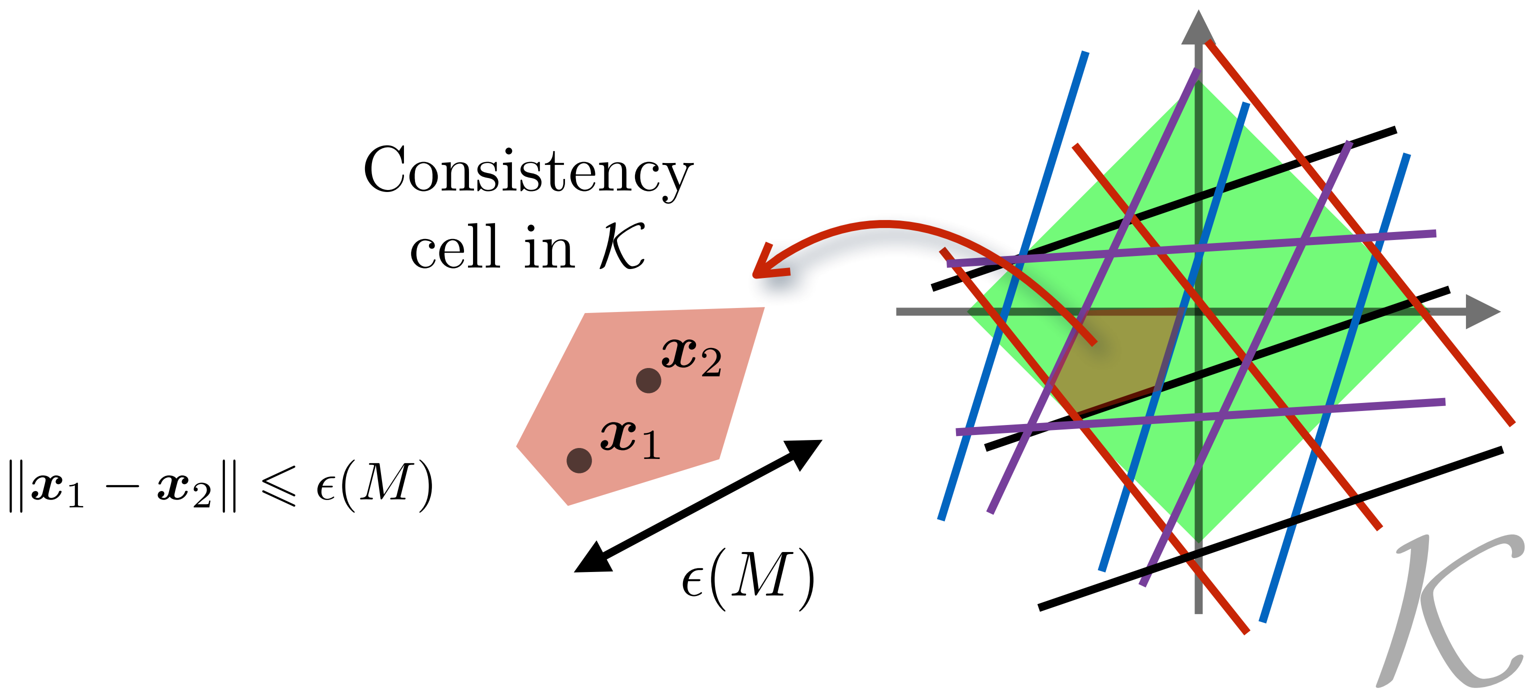
$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Consistency width:

$$\epsilon(M;\mathcal{K},\delta) = \max_{\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{K}} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \text{ s.t. } \mathsf{A}(\boldsymbol{x}_1) = \mathsf{A}(\boldsymbol{x}_2)$$

Size $\epsilon(M)$ should decay for large M!

Ⓐ Control of the "consistency width"



$$\mathbf{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

For $\mathbf{\Phi}$ a random Gaussian matrix, with high probability,

[LJ, 16], [LJ, 17]

$$\epsilon(M) \leqslant C_{\mathcal{K},\delta}\, M^{-1/q}.$$

For $\mathcal{K}$ a structured set (*e.g.*, sparse signals, low-rank matrices)

$$q = 1, \quad C_{\mathcal{K},\delta} = (1+\delta) w(\mathcal{K} \cap \mathbb{B}^n)^2 \times \text{log factors},$$

otherwise, for a bounded set,

$$q = 4, \quad C_{\mathcal{K},\delta} = \left(\tfrac{1+\delta}{\delta^{1/2}}\right) w(\mathcal{K})^{1/2}.$$

Open problem:
Extension to RIP matrices?

Ⓐ CW for consistent reconstruction

Let us take $\mathcal{K} = \mathcal{C}_s = \{\boldsymbol{u} \in \mathbb{R}^N : \|\boldsymbol{u}\|_1 \leqslant \sqrt{s}, \|\boldsymbol{u}\| \leqslant 1\}$ & $\boldsymbol{x}_0 \in \Sigma_s \cap \mathbb{B}^N$

$$\boldsymbol{x}^* \in \operatorname*{argmin}_{\boldsymbol{u} \in \mathbb{R}^N} \|\boldsymbol{u}\|_1 \ \text{ s.t. } \ \cancel{\|\boldsymbol{\Phi u} - \boldsymbol{\Phi x}_0\| \leqslant \epsilon,} \qquad \text{(BPDN)}$$

$$\underbrace{\boldsymbol{A}(\boldsymbol{u}) = \boldsymbol{A}(\boldsymbol{x}_0),}_{\text{convex}} \ \boldsymbol{u} \in \mathbb{B}^N. \quad \text{(CoBP)}$$

A CW for consistent reconstruction

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$$\boldsymbol{x}^* \in \operatorname*{argmin}_{\boldsymbol{u} \in \mathbb{R}^N} \|\boldsymbol{u}\|_1 \text{ s.t. } \boldsymbol{A}(\boldsymbol{u}) = \boldsymbol{A}(\boldsymbol{x}_0),\ \boldsymbol{u} \in \mathbb{B}^N. \text{ (CoBP)}$$

$\|\boldsymbol{x}^*\| \leqslant 1$ and $\|\boldsymbol{x}^*\|_1 \leqslant \|\boldsymbol{x}_0\|_1 \leqslant \sqrt{s}$

$\Rightarrow \boldsymbol{x}^* \in \mathcal{C}_s$

Moreover $A(\boldsymbol{x}^*) = A(\boldsymbol{x}_0)$ (consistency)

$\Rightarrow \|\boldsymbol{x}^* - \boldsymbol{x}_0\| \leqslant \epsilon(M; \mathcal{C}_S, \delta)$

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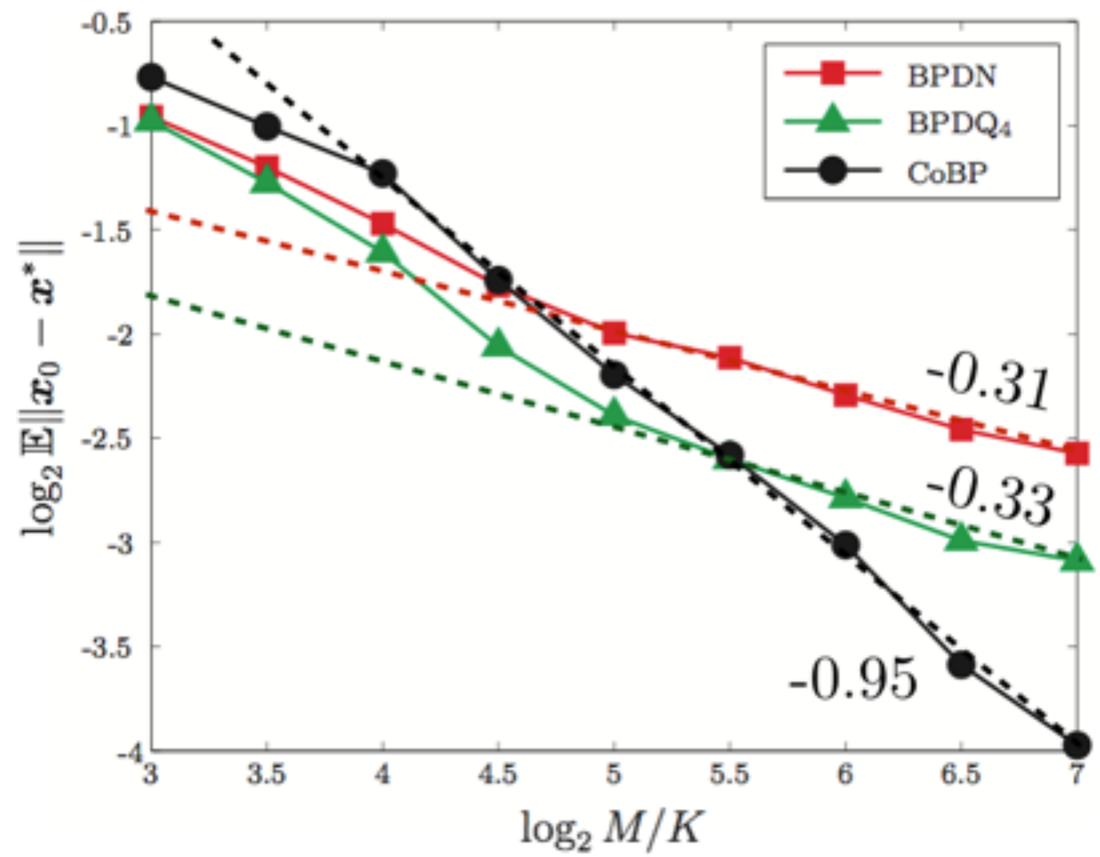
K-sparse signals with random Gaussian sensing

$N = 2048$, $K = 16$,
$B = 3$, $M/K \in [8, 128]$
20 trials per points



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* ↔ $\|\boldsymbol{u}\|_\sharp$

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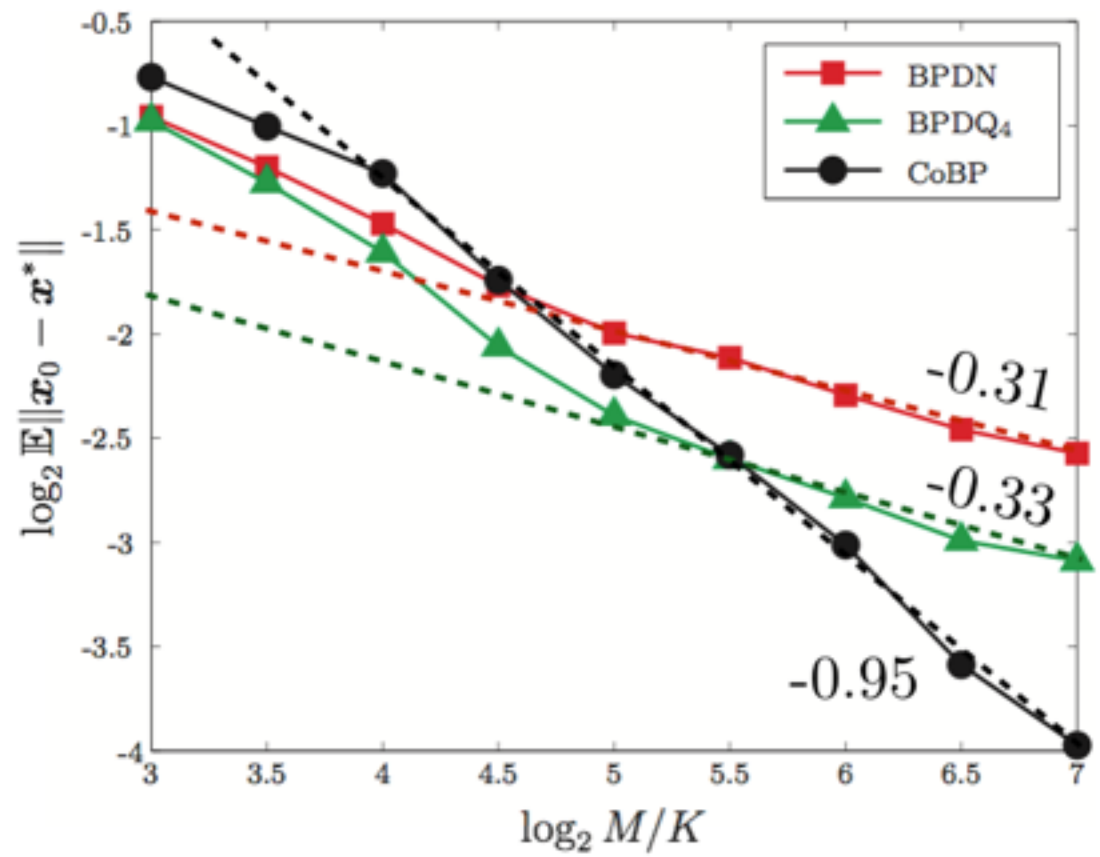
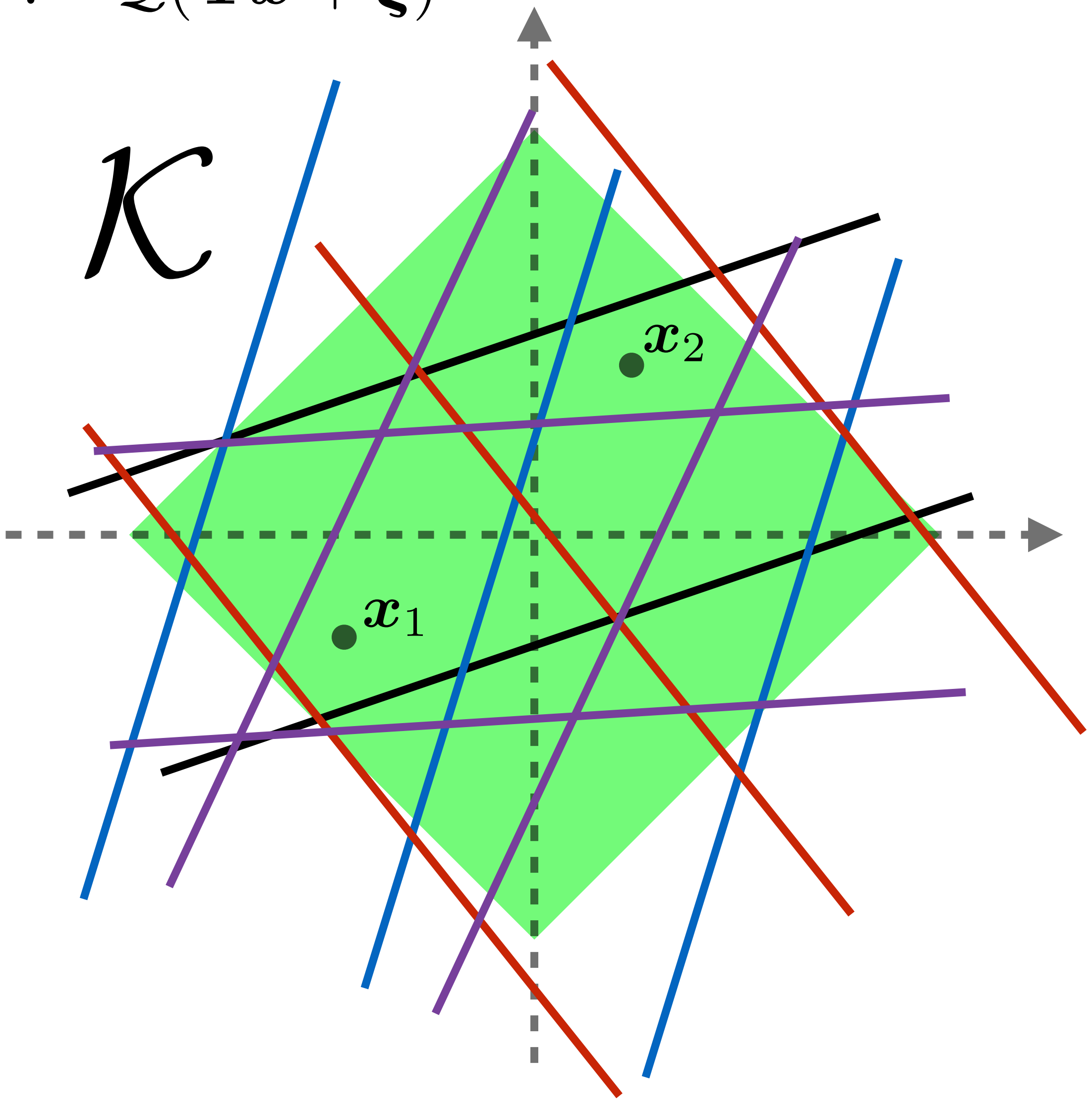


*Can be extended to other gauge/ atomic norms (e.g., nuclear norm)

[Moshtaghpour, LJ, Cambareri, Degraux, De Vleeschouwer, 17]

B Quantizing the RIP (approximate consistency)

$\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$



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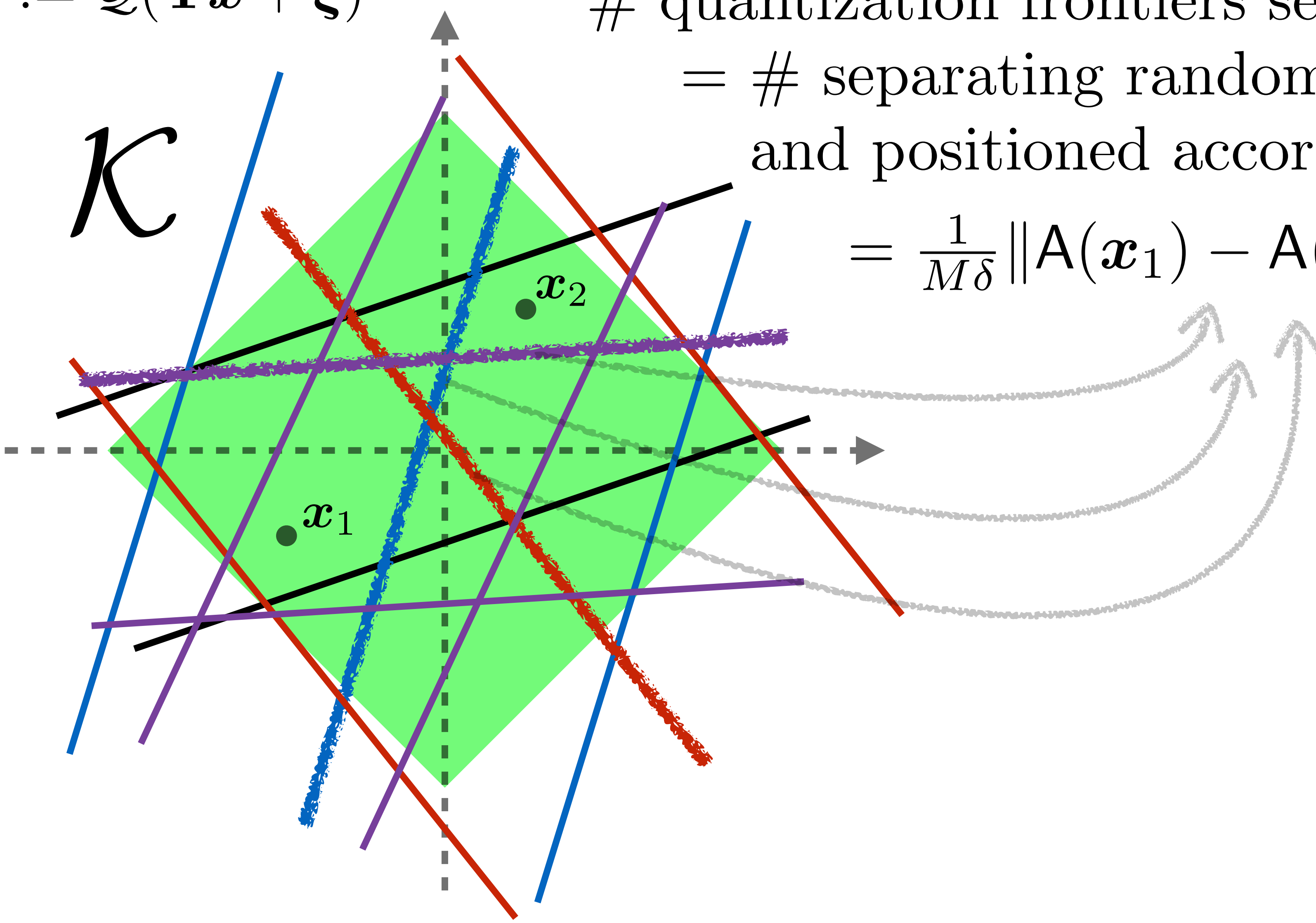

quantization frontiers separating $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$
= # separating random hyperplanes oriented and positioned according to $(\boldsymbol{\Phi}, \boldsymbol{\xi})$

$$= \tfrac{1}{M\delta}\|\mathsf{A}(\boldsymbol{x}_1) - \mathsf{A}(\boldsymbol{x}_2)\|_1 \approx \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \quad ??$$

Hope: dithering sufficiently smoothen discontinuities to allow for RIP matrices.

B Quantizing the RIP (approximate consistency)

Let $\mathcal{K} \subset \mathbb{R}^N$ be a structured set (*e.g.*, sparse signals, low-rank matrices).

Let $\boldsymbol{\Phi}$ be a (ℓ_1, ℓ_2)-RIP$(\epsilon, \mathcal{K} - \mathcal{K})$ matrix, *i.e.*,

$$(1-\epsilon)\|\boldsymbol{x}\|^2 \leqslant \tfrac{c_{\boldsymbol{\Phi}}}{m}\|\boldsymbol{\Phi x}\|_1^2 \leqslant (1+\epsilon)\|\boldsymbol{x}\|^2, \forall \boldsymbol{x} \in \mathcal{K} - \mathcal{K},$$

(*e.g.*, Gaussian random matrix, circulant Gaussian random matrix for $\mathcal{K} = \Sigma_k$)

[Dirksen, Jung, Rauhut, 17]

Provided that $M \gtrsim \epsilon^{-2} C_{\mathcal{K}} \log(1 + \frac{1}{\delta\epsilon})$, (with $C_{\mathcal{K}} > 0$ an upper bound on $w(\mathcal{K})^2$)
with probability exceeding $1 - C\exp(-\epsilon^2 m)$,

$$(1-\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| - c'\epsilon\delta \leqslant \tfrac{1}{m}\|\mathsf{A}(\boldsymbol{x}_1) - \mathsf{A}(\boldsymbol{x}_2)\|_1 \leqslant (1+\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + c'\epsilon\delta,$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{K} \cap \mathbb{B}^N$.

($\exists$ other variants with ℓ_2/ℓ_2 and standard RIP)

[LJ, Cambareri, 17]

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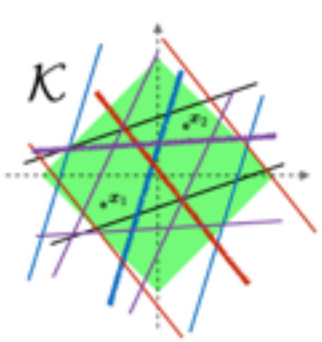


In other words, we can potentially classify signals from their quantized observations!

[LJ, Cambareri, 17]

Classification in a quantized world

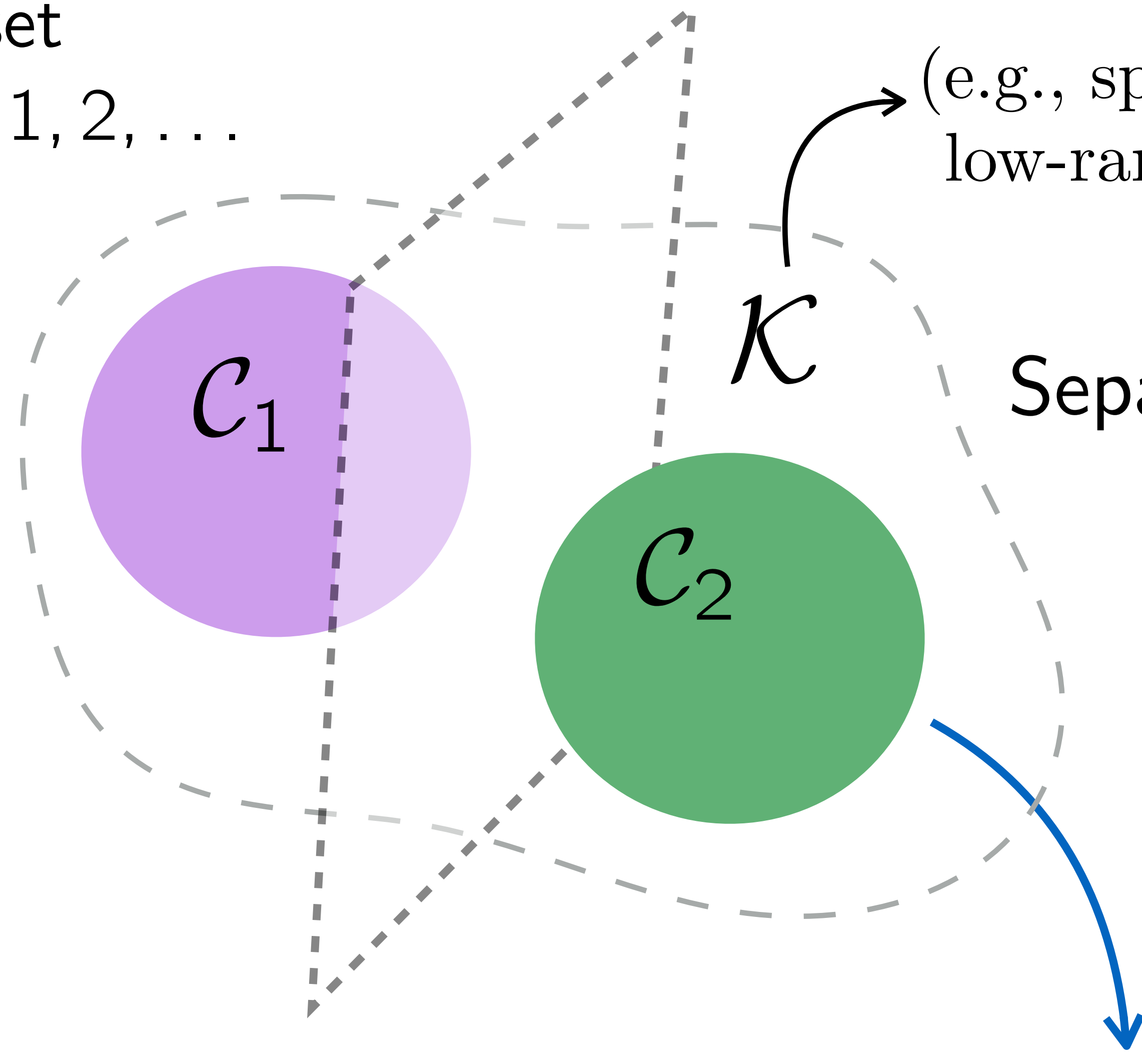
“The Rare Eclipse Problem on Tiles”

The Big Picture (an easy classification problem)

$\mathcal{K} \subset \mathbb{R}^n$ dataset

$\mathcal{C}_i \subset \mathcal{K}$ classes, $i = 1, 2, \ldots$

(e.g., sparse signals, low-rank matrices, $\mathbb{R}^n$)



Separable (*e.g.*, linearly)

$\text{Classify}[\mathcal{C}_1 \cup \mathcal{C}_2]$

(e.g., LDA, SVM, PCA, K-Means, K-NN, ...)

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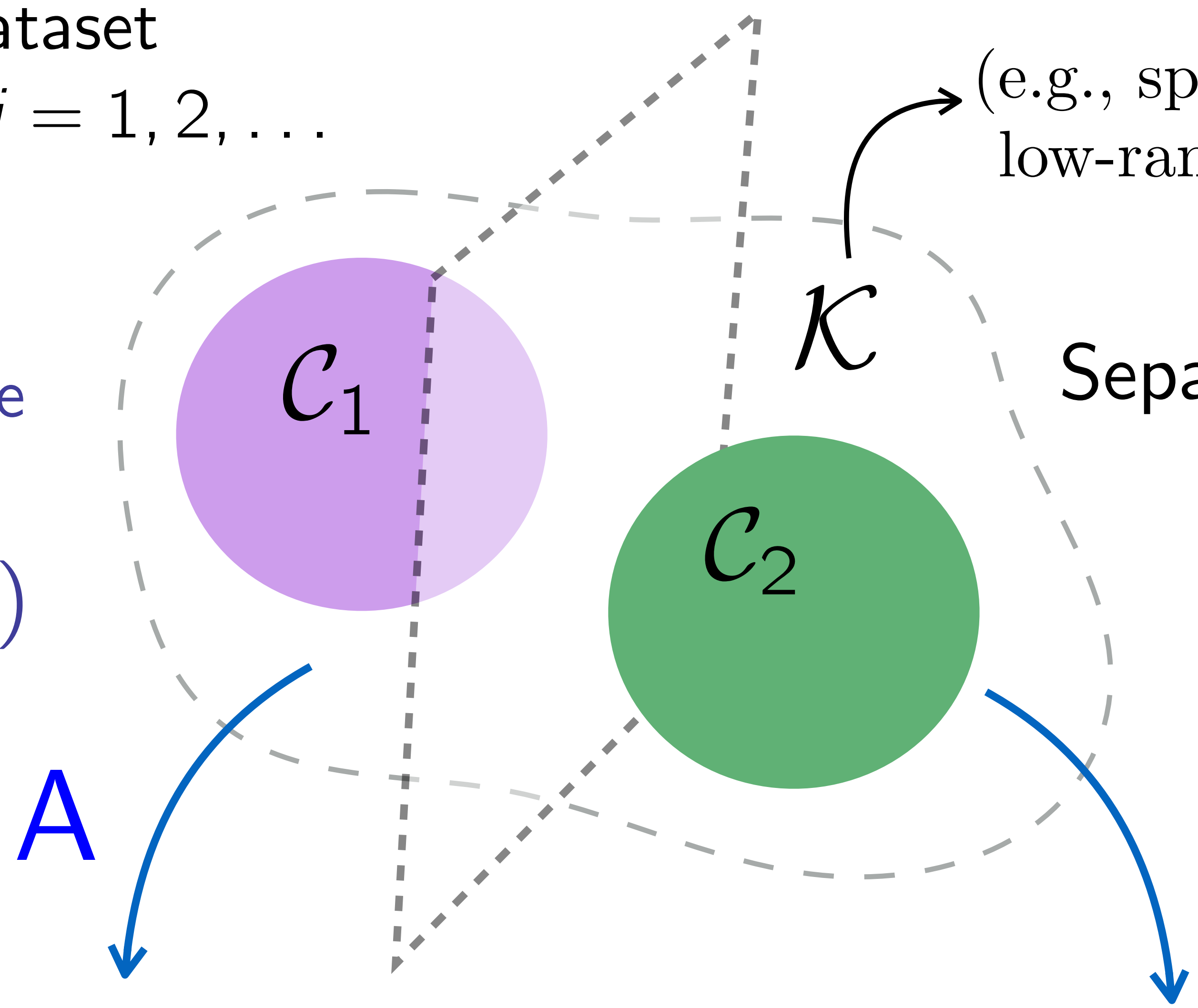
Non-Adaptive Mapping

$y = \mathsf{A}(x)$



Separable (*e.g.*, linearly)

$$\text{Classify}[\mathsf{A}(\mathcal{C}_1) \cup \mathsf{A}(\mathcal{C}_2)] \approx ??? \ \text{Classify}[\mathcal{C}_1 \cup \mathcal{C}_2]$$

(e.g., LDA, SVM, PCA, K-Means, K-NN, ...)

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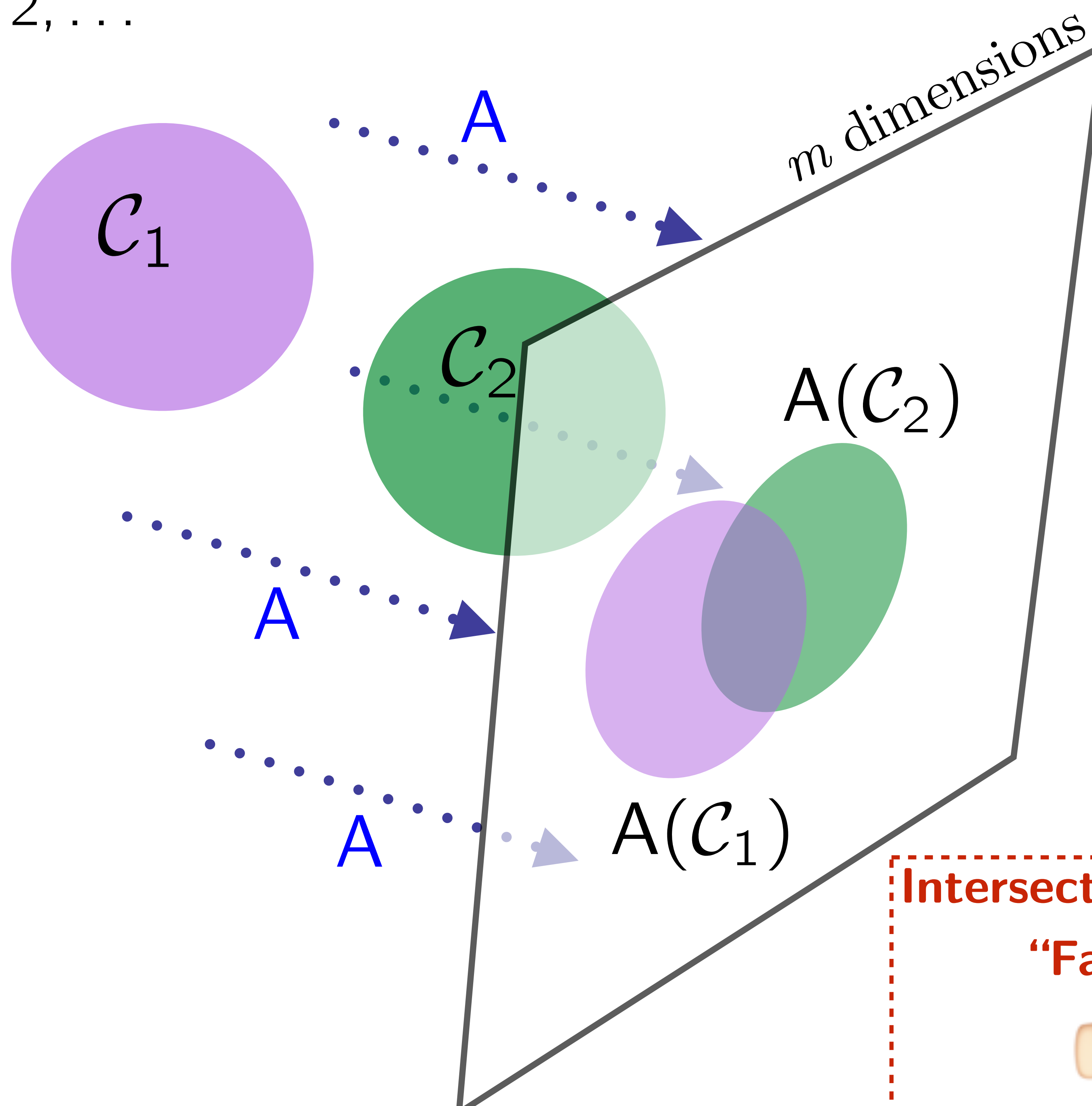

Intersecting/Eclipse

"Failure"

The Big Picture

$\mathcal{K} \subset \mathbb{R}^n$ dataset
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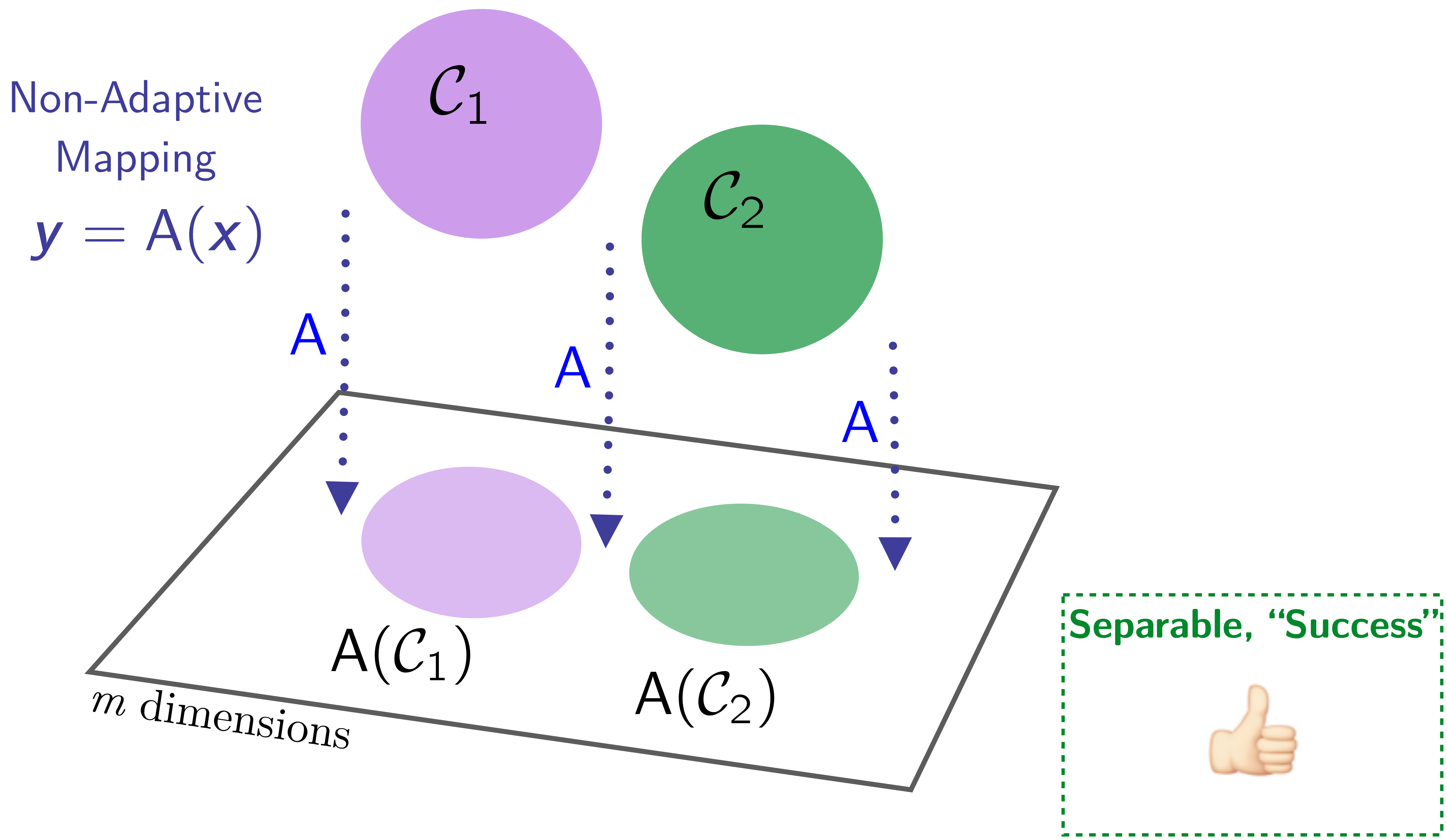


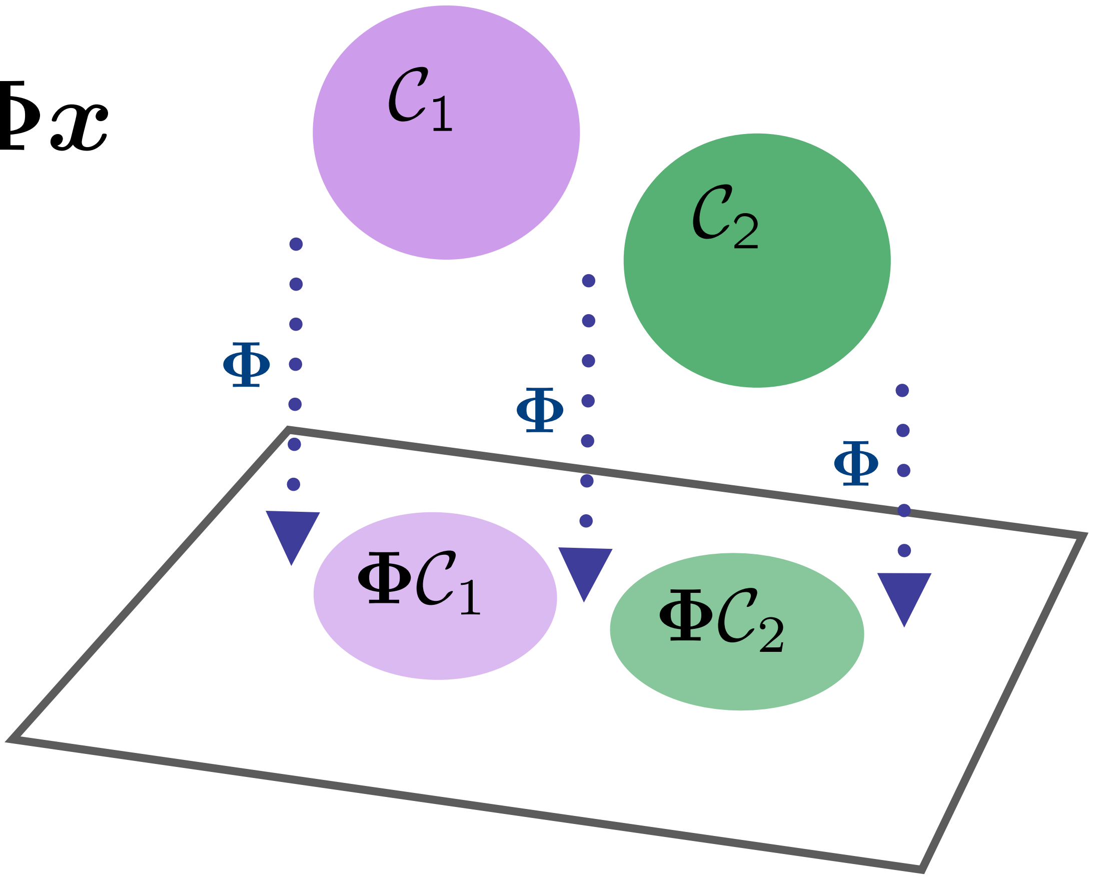
The Rare Eclipse Problem (Linear case)

Problem (Rare Eclipse Problem (Bandeira *et al.* '14)).

Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n : \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ be closed convex sets, $\mathbf{\Phi} \sim \mathcal{N}^{m \times n}(0,1)$. Given $\eta \in (0,1)$, find the smallest m so that

$$p_0 := \mathbb{P}_{\mathbf{\Phi}}[\mathbf{\Phi}\mathcal{C}_1 \cap \mathbf{\Phi}\mathcal{C}_2 = \emptyset] \geq 1 - \eta.$$

$$\mathrm{A}(\boldsymbol{x}) = \mathbf{\Phi}\boldsymbol{x}$$



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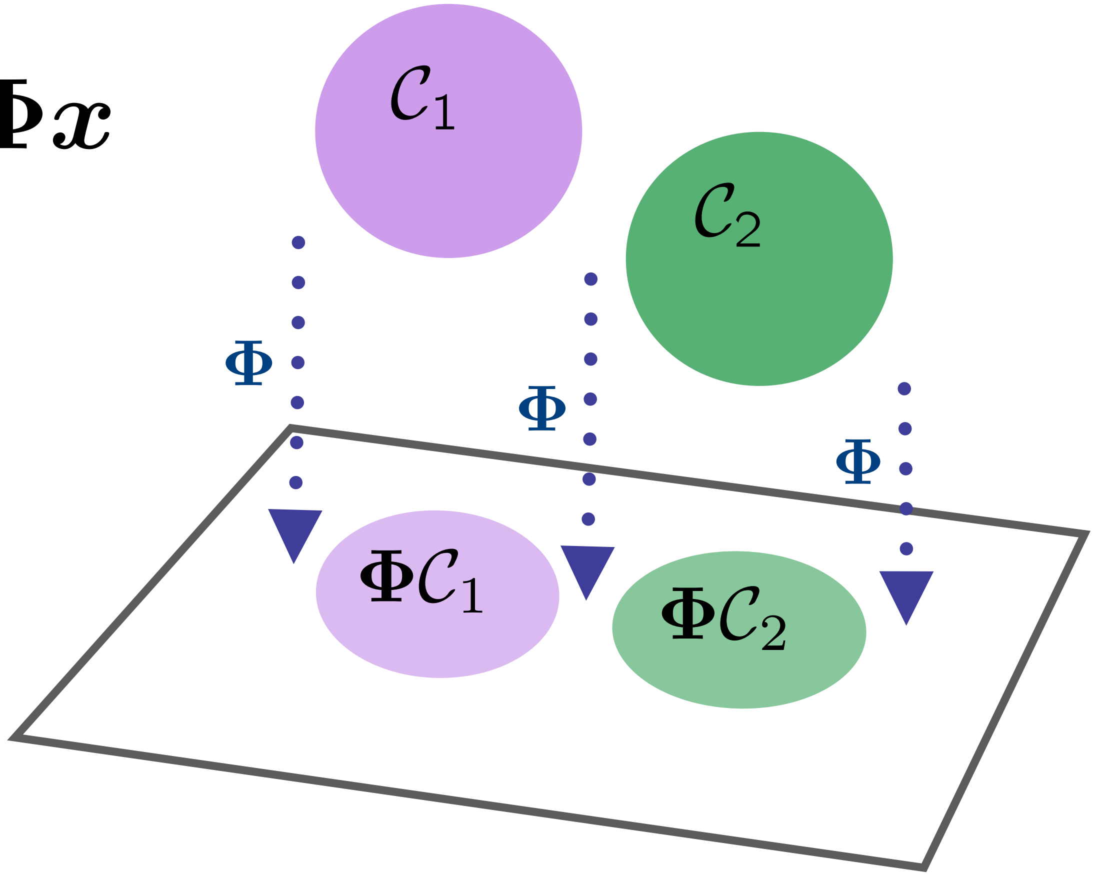
$$p_0 = \mathbb{P}_{\boldsymbol{\Phi}}[\forall \boldsymbol{x}_1 \in \mathcal{C}_1, \forall \boldsymbol{x}_2 \in \mathcal{C}_2, \boldsymbol{\Phi}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \neq 0] \geq 1 - \eta.$$

$$\mathsf{A}(\boldsymbol{x}) = \boldsymbol{\Phi}\boldsymbol{x}$$



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$$p_0 = \mathbb{P}_{\boldsymbol{\Phi}}[\mathcal{C}^{\ominus} \cap \ker \boldsymbol{\Phi} = \emptyset] = \mathbb{P}_{\boldsymbol{\Phi}}[\,\underline{\mathcal{S}}\, \cap \ker \boldsymbol{\Phi} = \emptyset] \geq 1 - \eta.$$

$$\mathcal{S} = (\mathbb{R}_+ \mathcal{C}^{\ominus}) \cap \mathbb{S}_2^{n-1}$$

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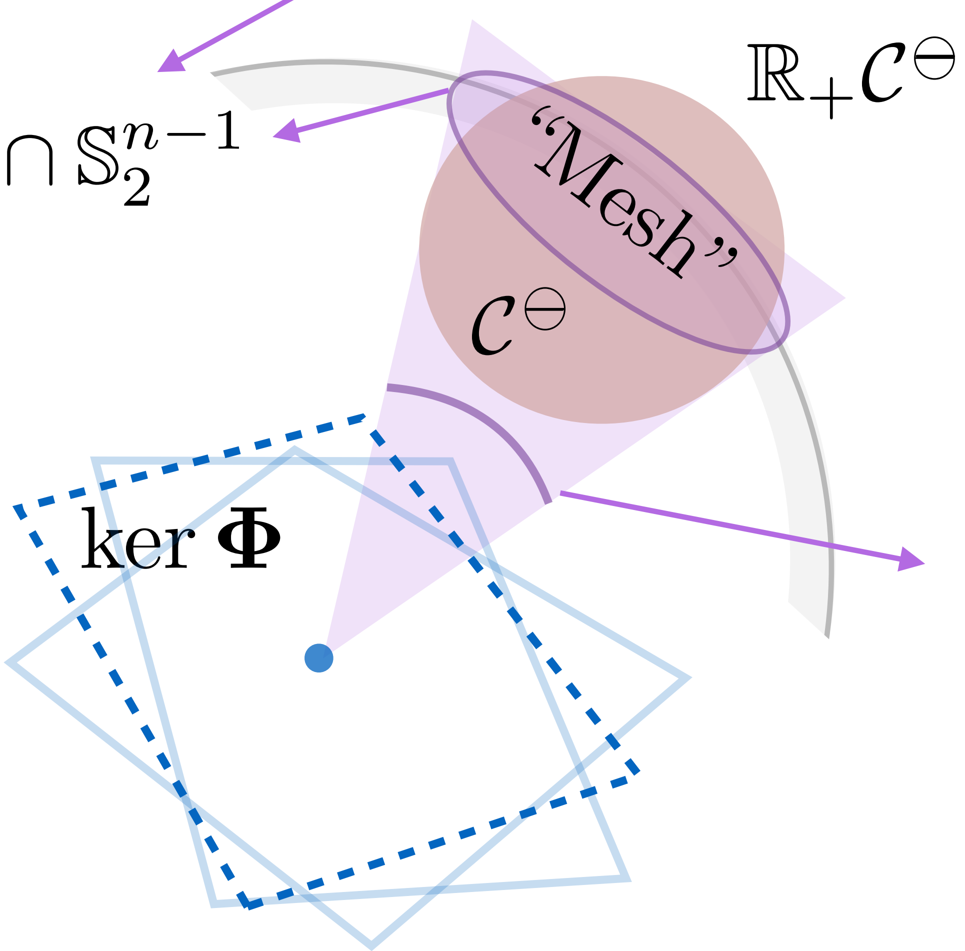


Small GW:
$w_\ominus = w(\mathcal{S})$

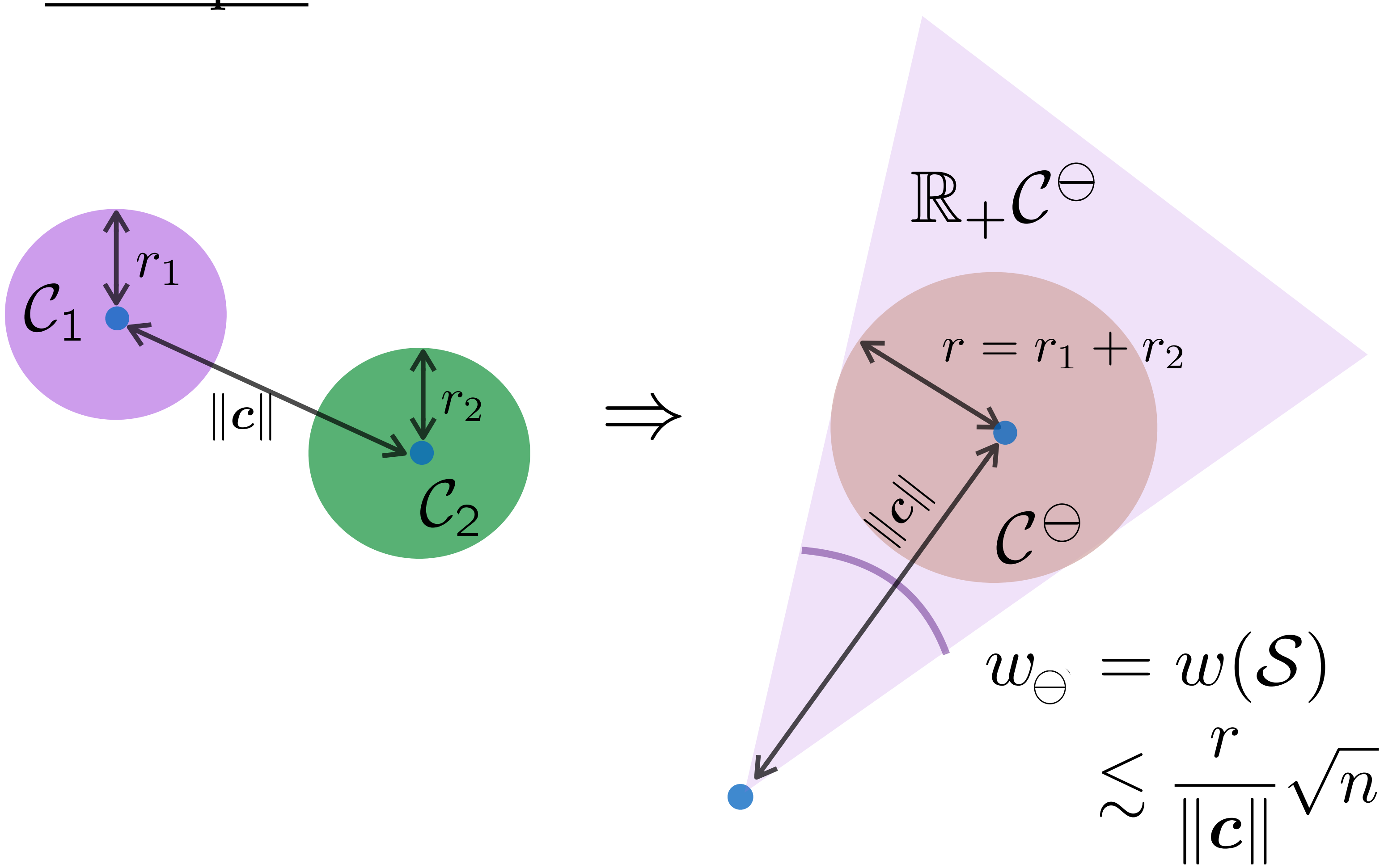
The Rare Eclipse Problem (Linear case)

BMR '14: "Gordon's escape through a mesh" theorem

Proposition (Corollary 3.1 in BMR '14). (& *really* tight [Amelunxen et al, 13])

Given $\eta \in (0,1)$, if $m > \left(w_{\ominus} + \sqrt{2\log\frac{1}{\eta}}\right)^2 + 1$ then $p_0 \geq 1 - \eta$.

<u>Example:</u>



$\Rightarrow$

$$m \gtrsim \frac{r^2}{\|\boldsymbol{c}\|^2} n$$

The Rare Eclipse Problem (alternate proof)

Restricted Isometry Property: (ℓ_1, ℓ_2)-RIP$(\mathcal{K}, \epsilon)$

$$\forall \boldsymbol{x} \in \mathcal{K},\ (1-\epsilon)\,\|\boldsymbol{x}\| \leqslant \|\boldsymbol{\Phi x}\|_1 \leqslant (1+\epsilon)\,\|\boldsymbol{x}\|$$

[Schechtman, 06] [Plan, Vershynin, 14]

If $\mathcal{S} \subset \mathbb{S}^{n-1}$ and $m \gtrsim \epsilon^{-2} w^2(\mathcal{S})$, then, w.h.p.*,

$$(1-\epsilon) \leqslant \sqrt{\tfrac{\pi}{2}}\tfrac{1}{m}\|\boldsymbol{\Phi u}\|_1 \leqslant (1+\epsilon)$$

for $\boldsymbol{\Phi} \in \mathbb{R}^{m\times n}$ and $\Phi_{ij} \sim_{\rm iid} \mathcal{N}(0,1)$.

(note: extendable to subsets of $\mathbb{B}^n$, *e.g.*, compressible signals, [Xu, LJ, 18])

*: *i.e.*, $\mathbb{P} \geqslant 1 - C\exp(-c\epsilon^2 m)$.

The Rare Eclipse Problem (alternate proof)

Restricted Isometry Property: (ℓ_1, ℓ_2)-RIP$(\mathcal{K}, \epsilon)$

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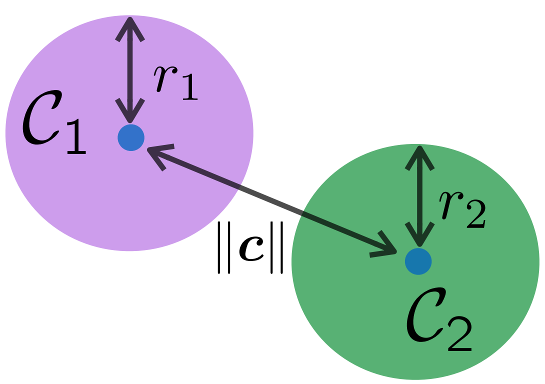
for $\boldsymbol{\Phi} \in \mathbb{R}^{m\times n}$ and $\Phi_{ij} \sim_{\text{iid}} \mathcal{N}(0,1)$.

Therefore:

For $\mathcal{S} = (\mathbb{R}^+\mathcal{C}^\ominus) \cap \mathbb{S}^{n-1}$, if $m \gtrsim \epsilon^{-2} w^2_\ominus$, w.h.p*, (RIP$_\ominus$)

$$(1-\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leqslant \sqrt{\tfrac{\pi}{2}}\,\tfrac{1}{m}\|\boldsymbol{\Phi x}_1 - \boldsymbol{\Phi x}_2\|_1 \leqslant (1+\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|,$$

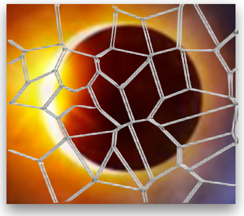
for all $\boldsymbol{x}_1 \in \mathcal{C}_1$ and all $\boldsymbol{x}_2 \in \mathcal{C}_2$.



This result also explains REP,
e.g., with $\epsilon = 1/2$, (but less sharply)

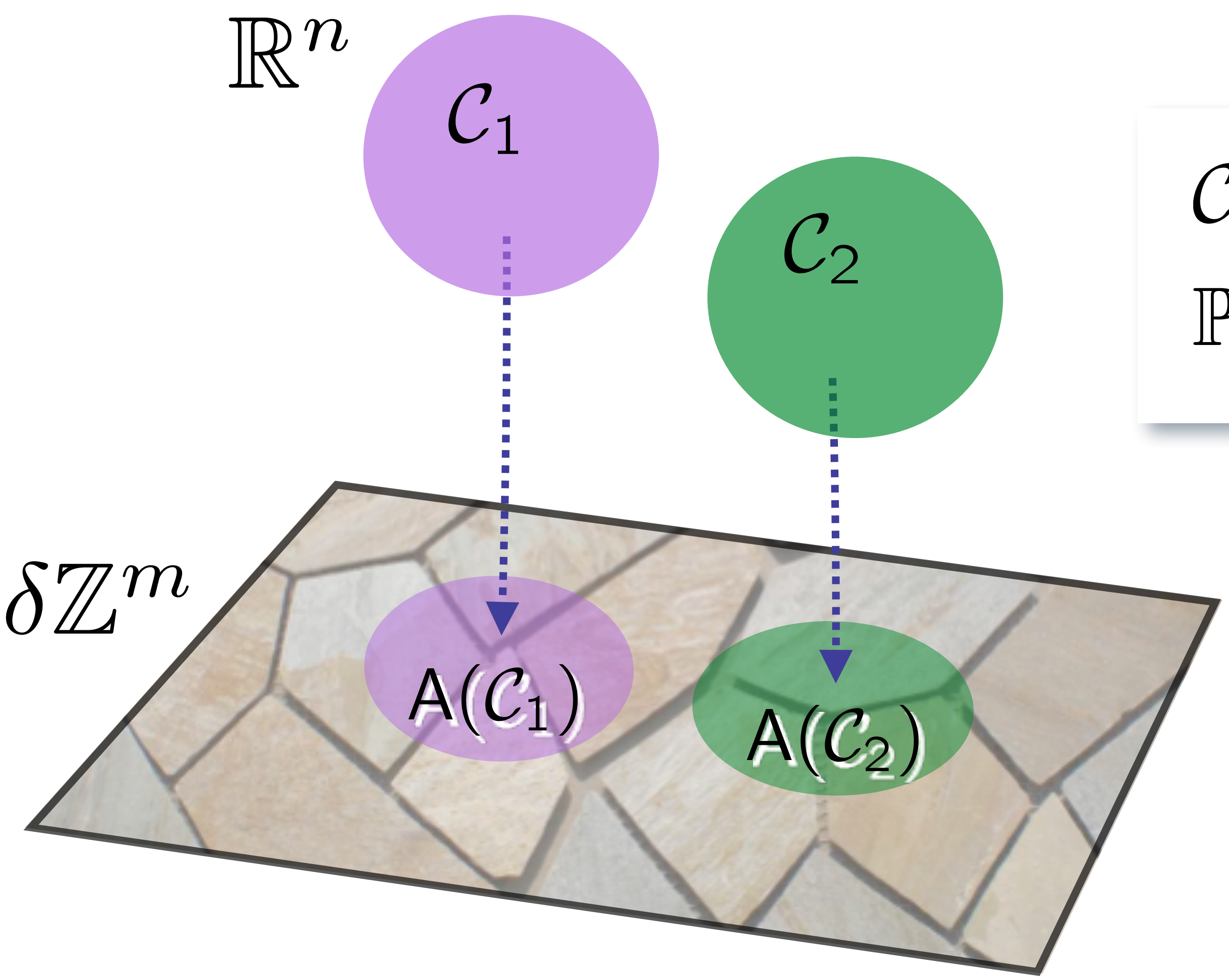
$*$: *i.e.*, $\mathbb{P} \geqslant 1 - C\exp(-c\epsilon^2 m)$.

The Rare Eclipse Problem "on Tiles"



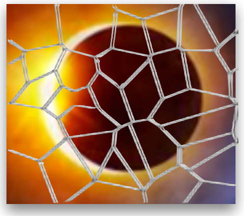
$$\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi})$$

with $\boldsymbol{\Phi}$ Gaussian random matrix,
$\mathcal{Q}(\lambda) = \delta\lfloor \frac{\lambda}{\delta} \rfloor$, $\xi_i \sim \mathcal{U}([0,\delta])$.



$\mathcal{C}_1$, $\mathcal{C}_2$, m and δ such that
$\mathbb{P}[\mathsf{A}(\mathcal{C}_1) \cap \mathsf{A}(\mathcal{C}_2) = \emptyset] \geqslant 1 - \eta$?

The Rare Eclipse Problem "on Tiles"



Given $\sigma := \min_{\boldsymbol{z}\in\mathcal{C}^{\ominus}} \|\boldsymbol{z}\|$ and $w_{\cap} = w((\mathbb{R}_+\mathcal{C}^{\ominus}) \cap \mathbb{S}^{n-1})$.

Provided

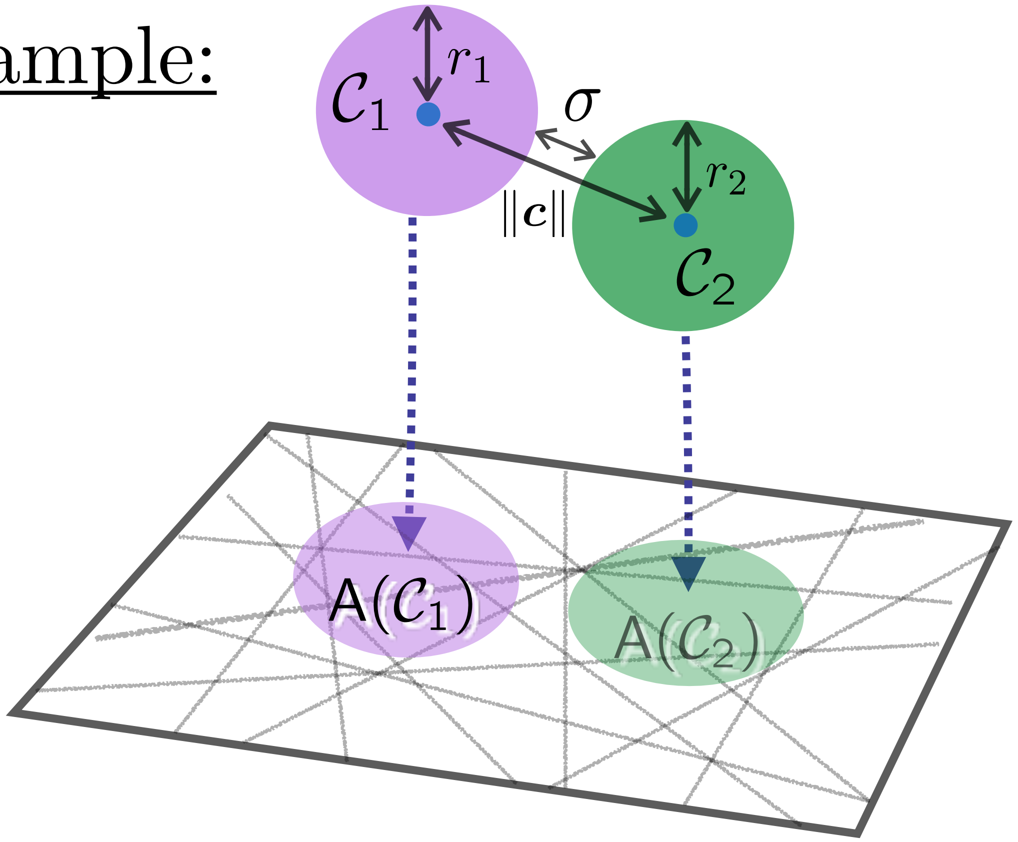
$$m \gtrsim \Big(w_{\ominus}^2 + n\tfrac{\delta^2}{\sigma^2}\Big)\ \Big(1 + \log\big(1 + \tfrac{rm}{\delta n}\big) + w_{\ominus}^{-2}\log\tfrac{1}{\eta}\Big),$$

linear, quantiz., proof artifact?, linear

we have

$$\mathbb{P}[\mathsf{A}(\mathcal{C}_1) \cap \mathsf{A}(\mathcal{C}_2) = \emptyset] \geqslant 1 - \eta.$$

Example:



$$\Rightarrow \quad m \gtrsim \Big(\frac{r^2}{\|\boldsymbol{c}\|^2} + \frac{\delta^2}{(\|\boldsymbol{c}\|-r)^2}\Big)\, n$$

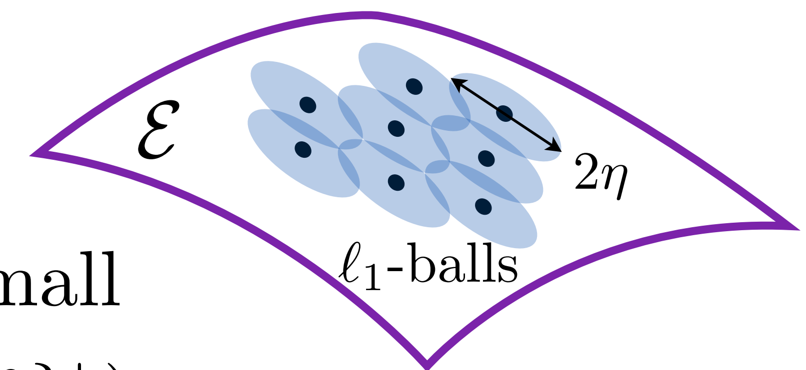
Note: $\delta > \sigma$ is allowed (dithering effect!)

Note bis: $m > n$ not specially bad ($\delta\mathbb{Z}^m$).

Proof sketch:

A. Embedding $\mathcal{E} \subset \mathbb{R}^m$ into $\delta\mathbb{Z}^m$ with

$$\mathsf{A}'(\boldsymbol{y}) := \underset{\uparrow\atop\delta}{\mathcal{Q}}(\boldsymbol{y} + \underset{\uparrow\atop\delta}{\boldsymbol{\xi}})$$



If $\mathcal{E}$ has small ℓ_1-Kolmogorov entropy, *i.e.*, small

$$\mathcal{H}_1(\mathcal{E}, \eta) = \log(\,\min_{\mathcal{G}} |\{\mathcal{G} : \mathcal{G} \subset \mathcal{E} \subset \mathcal{S} + \eta\mathbb{B}_1^n\}|\,)$$

Given $\epsilon > 0$, if $m \gtrsim \epsilon^{-2}\mathcal{H}_1(\mathcal{E}, \frac{m\delta\epsilon^2}{1+\epsilon})$, then, w.h.p*, for all $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{E}$ and some $c > 0$,

$$\tfrac{1}{m}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_1 - c\delta\epsilon \;\leqslant\; \tfrac{1}{m}\|\mathsf{A}'(\boldsymbol{y}_1) - \mathsf{A}'(\boldsymbol{y}_2)\|_1 \;\leqslant\; \tfrac{1}{m}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_1 + c\delta\epsilon. \tag{P1}$$

B. For $\mathcal{E} = \boldsymbol{\Phi}\mathcal{K}$ and $\boldsymbol{\Phi}$ an (ℓ_1, ℓ_2)-RIP$(\mathcal{K} - \mathcal{K}, \epsilon' < 1)$,

$$\mathcal{H}_1(\mathcal{E}, 2m\eta) \leqslant \mathcal{H}_2(\mathcal{K}, \eta) \tag{P2}$$

with $\mathcal{H}_2$ is bounded for sets (cones, convex spaces, ...)

*: [LJ, Cambareri, '16] [Cambareri, Xu, LJ, '17]

Proof sketch:

C. We apply A and B for $\mathcal{E} = \boldsymbol{\Phi}\mathcal{C}_\cup$ so that (P1) holds if

$$m \gtrsim \epsilon^{-2} n \log(1 + \tfrac{2r(1+\epsilon)^2}{\delta\epsilon^2})$$

D. We rely on $(\mathrm{RIP}_\ominus)$, as in the linear case, i.e.,

For $\mathcal{S} = (\mathbb{R}^+\mathcal{C}^\ominus) \cap \mathbb{S}^{n-1}$, if $m \gtrsim \epsilon_0^{-2} w_\ominus^2$, w.h.p*,

$$(1-\epsilon_0)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leqslant \sqrt{\tfrac{\pi}{2}}\,\tfrac{1}{m}\|\boldsymbol{\Phi}\boldsymbol{x}_1 - \boldsymbol{\Phi}\boldsymbol{x}_2\|_1 \leqslant (1+\epsilon_0)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|,$$

for all $\boldsymbol{x}_1 \in \mathcal{C}_1$ and all $\boldsymbol{x}_2 \in \mathcal{C}_2$.

E. For $\boldsymbol{x}_1 \in \mathcal{C}_1$, $\boldsymbol{x}_2 \in \mathcal{C}_2$,

$$\begin{aligned}\tfrac{1}{m}\|\mathsf{A}'(\boldsymbol{\Phi}\boldsymbol{x}_1) - \mathsf{A}'(\boldsymbol{\Phi}\boldsymbol{x}_2)\|_1 &\geqslant \tfrac{1}{m}\|\boldsymbol{\Phi}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\|_1 - c\delta\epsilon\\ &\geqslant (\tfrac{2}{\pi})^{1/2}(1-\epsilon_0)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| - c\delta\epsilon\\ &\geqslant (\tfrac{2}{\pi})^{1/2}(1-\epsilon_0)\sigma - c\delta\epsilon.\end{aligned}$$

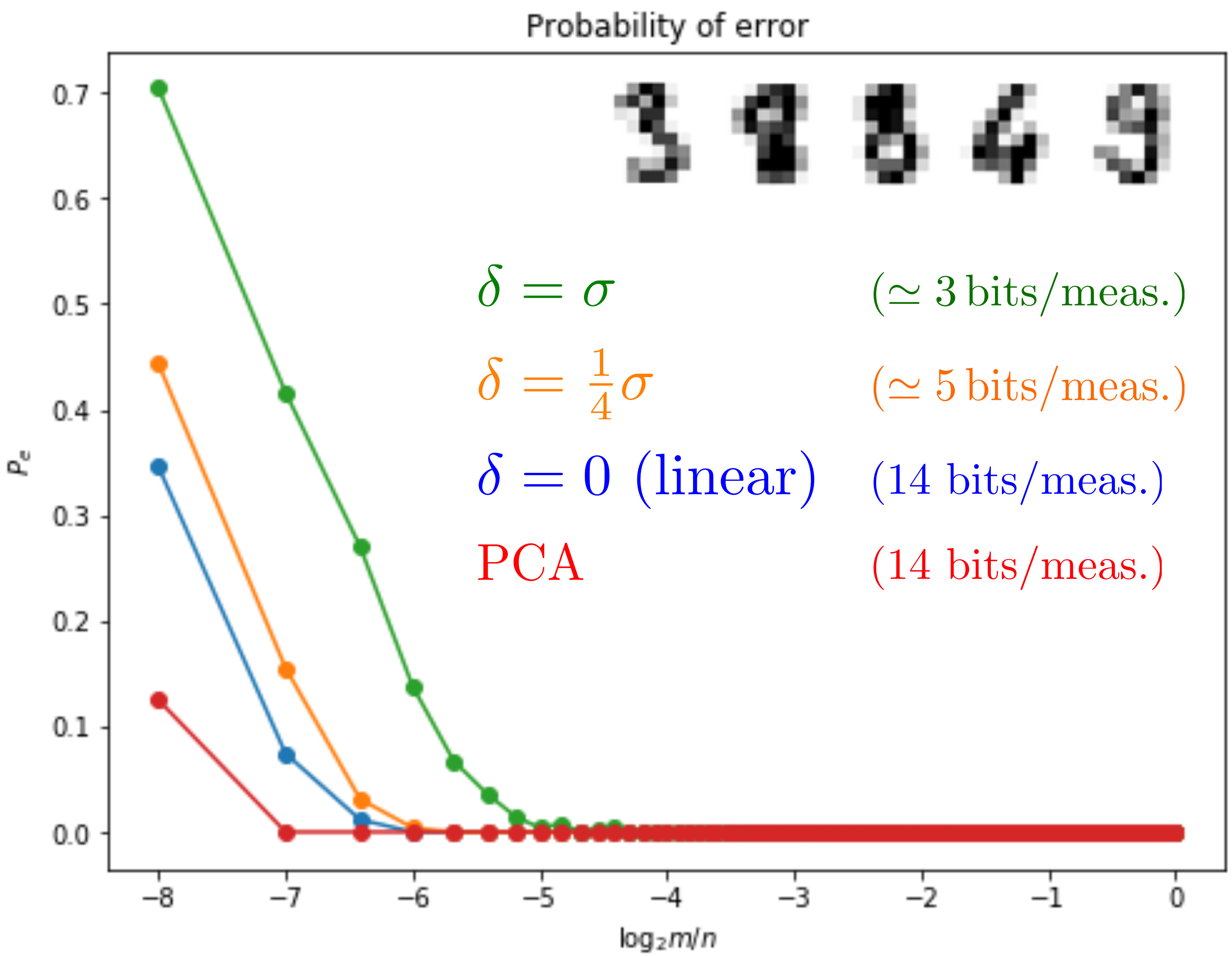
Fixing $\epsilon = \frac{\sqrt{n}}{w_\cap}\epsilon_0$ (to get similar conditions), and forcing $(\frac{2}{\pi})^{1/2}(1-\epsilon_0)\sigma - c\delta\epsilon > c\delta\epsilon > 0$ (for separability) gives the result (after solving for ϵ_0). $\square$

Simulations: Digit dataset (from scikit learn)

10 handwritten digits, 8x8 pixels (n=64), samples/class ≃ 12.

Training/Test sets = 50%/50%. $\sigma = \min_{i,j:i\neq j} \min_{\boldsymbol{u}\in\mathcal{C}_i,\boldsymbol{v}\in\mathcal{C}_j} \|\boldsymbol{u}-\boldsymbol{v}\|$

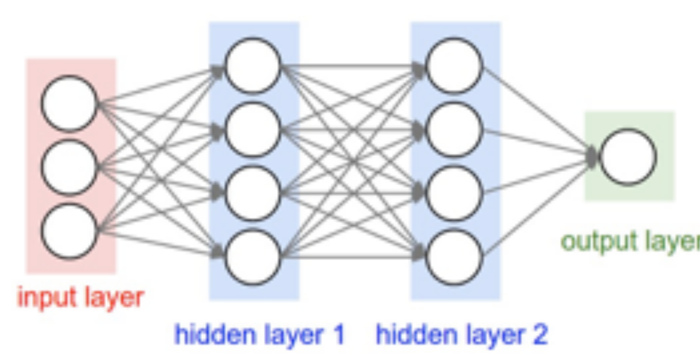
Classification: 5-NN Classifier.

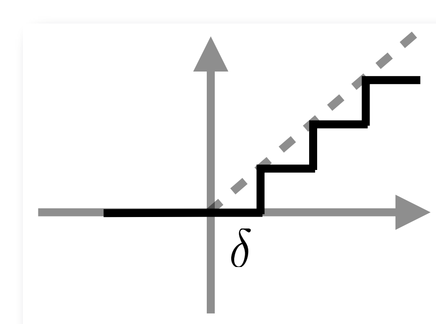


Try some code out here: **github.com/VC86/MLSPbox**

Take-away messages

- From CS to QCS (for scalar quantizers)
  - Importance of consistency and dithering
- Reconstruction still possible in QCS
  with decaying error as m increases
- Learning/Classification possible in QCS domain
- Open problems
  - CW for (other/all?) RIP matrices?
  - Quantizing non-linear embedding (clipping, ReLU)?
  - ...





Thank you for your attention!

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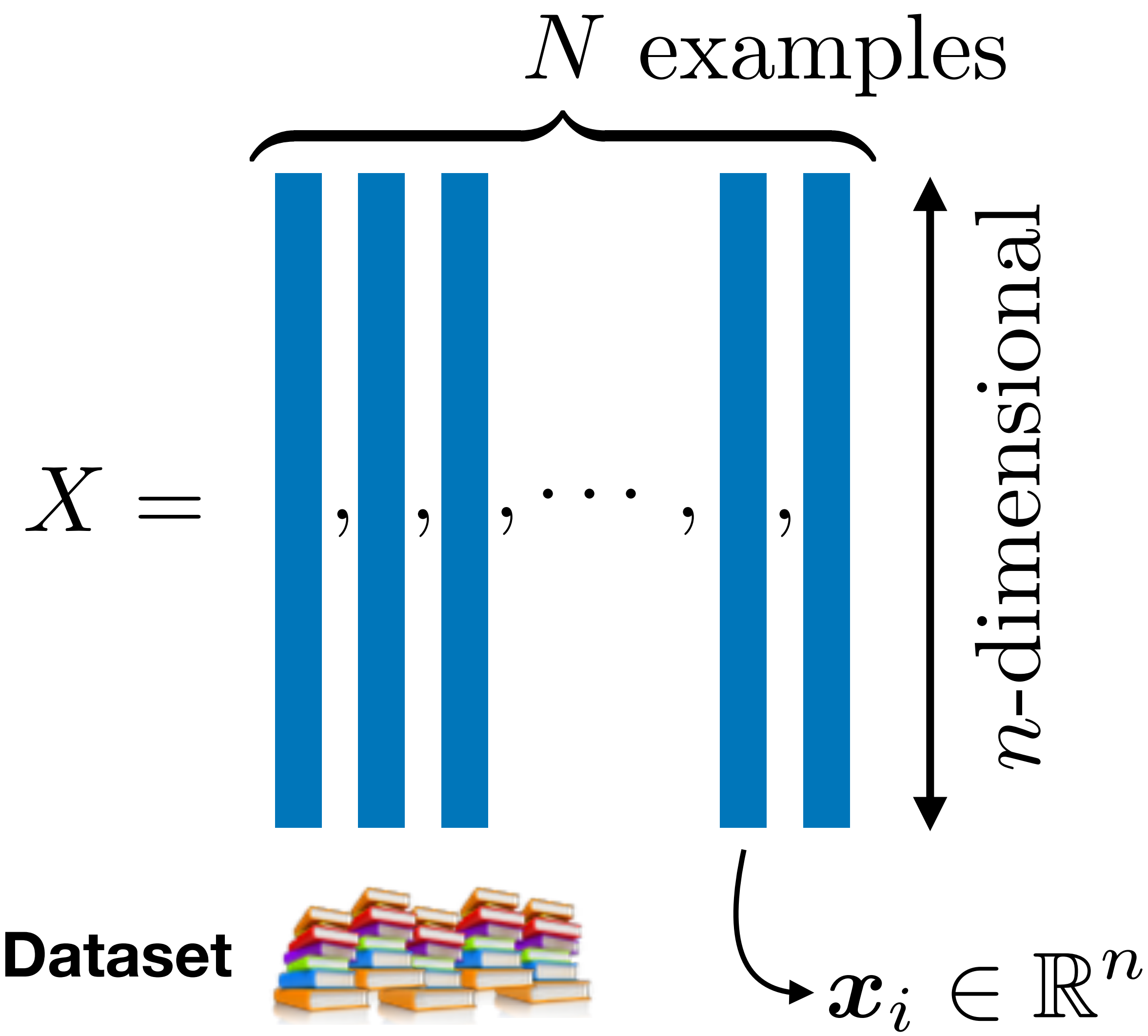
Backup slides

Quantized Random Sketching of Datasets

Compressing a dataset?



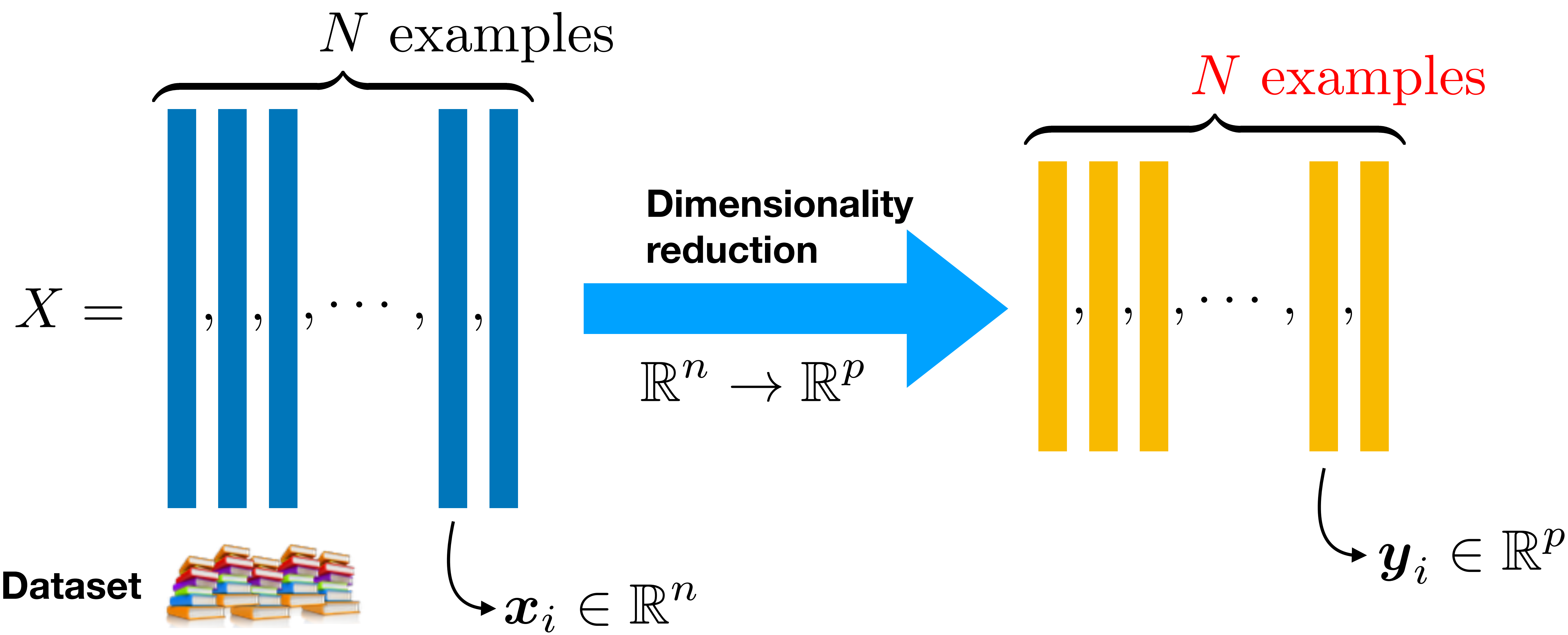
Large datasets means:

- Large memory required
- Slow learning algorithm
  (e.g., K-Means, (K)SVM, ...)

Complexity:
$O(nNK)$ per iteration

BUT extracted "knowledge" is "simple"
Do we really need all this data?

Compressing a dataset?



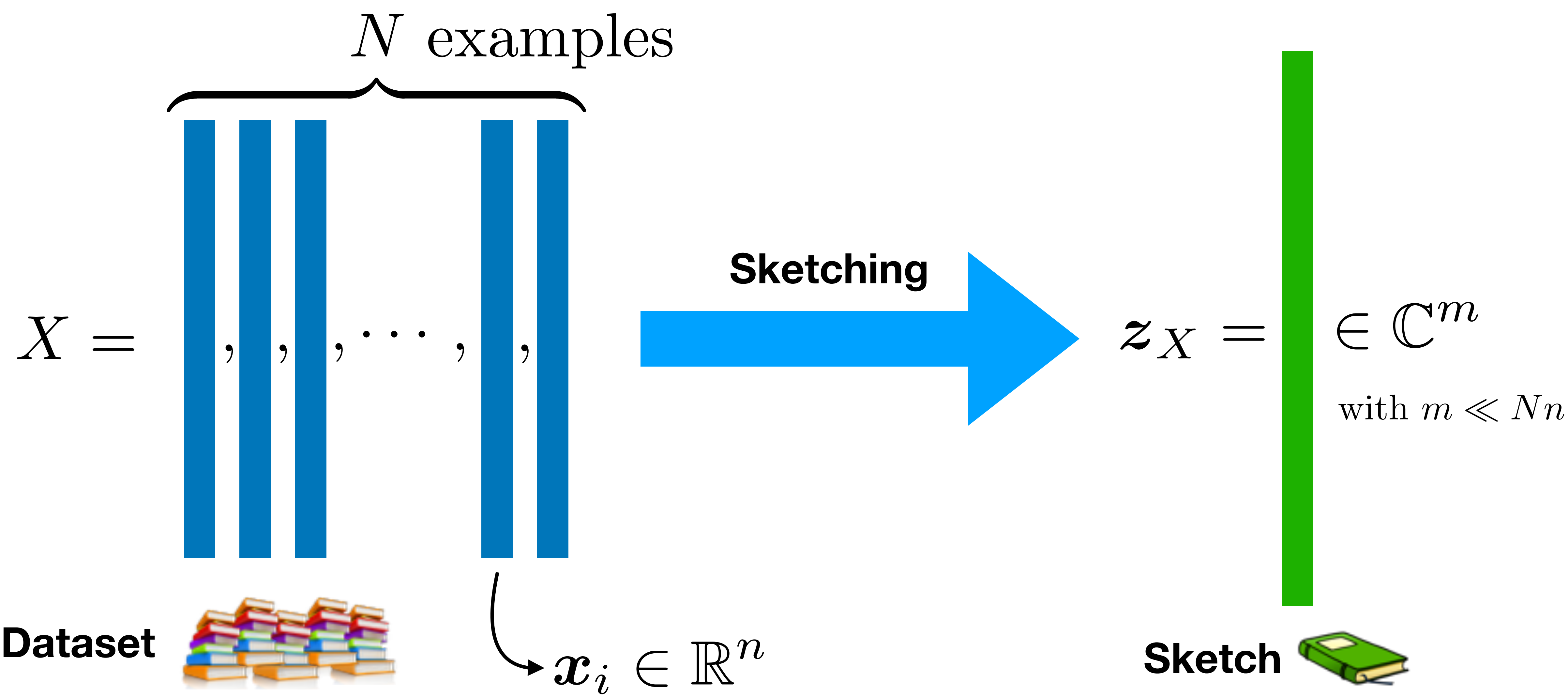
Classical CS methods

[Davenport et al. '07-'10], [Haupt et al. '06]
[Reboredo et al. '13-'16],
[Bandeira, Mixon, Recht '14]

- Compressed representation ✓
- Preserves relevant information ✓
- Constant number of examples ✗

N can be VERY large ("big data")!

Compressing a dataset?



- Compressed representation ✓
- Preserves relevant information ✓
- Dataset summary = single vector ✓

[Keriven, Bourrier, Gribonval, Pérez, 16]
[Gribonval, Blanchard, Keriven, Traonmilin, 17]

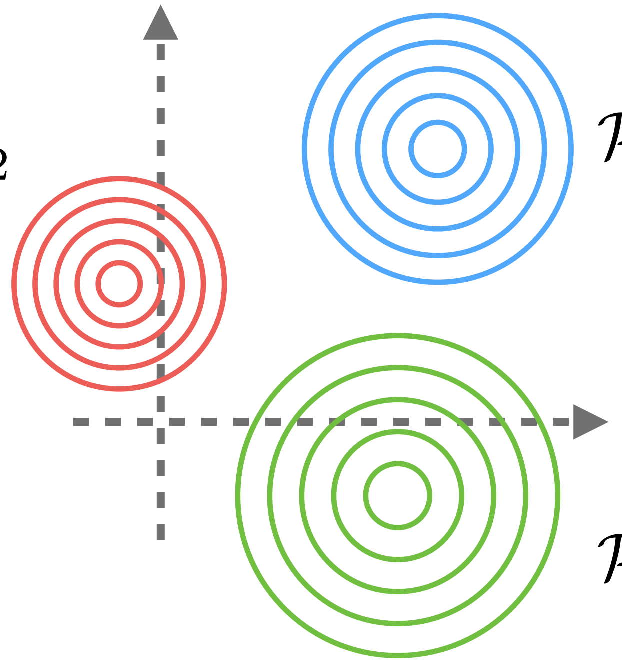
Compressing a dataset?

Dataset 

$$\boldsymbol{X} = \{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\} \subset \mathbb{R}^N$$

Prior information:

$$\boldsymbol{x}_i \sim_{\text{iid}} \mathcal{P} = \sum_{k=1}^{K} \alpha_k \mathcal{P}_k$$



"Sparse" Distribution

(e.g., Gaussian mixture)

[Keriven, Bourrier, Gribonval, Pérez, 16]
[Gribonval, Blanchard, Keriven, Traonmilin, 17]

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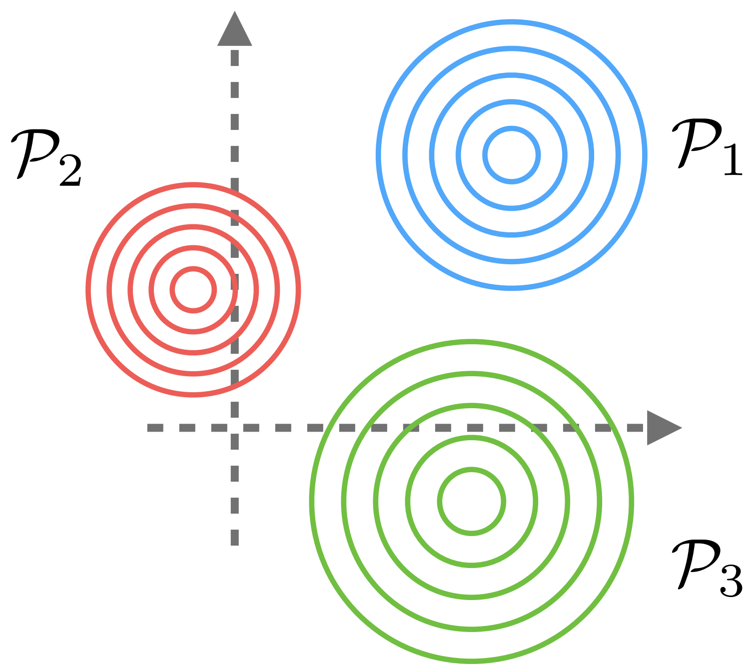


"Sparse" Distribution

(e.g., Gaussian mixture)

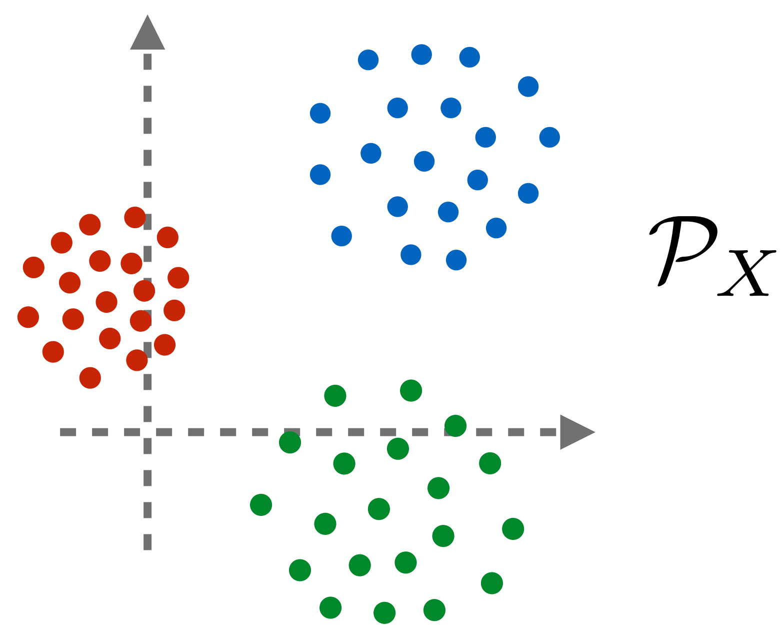
[Keriven, Bourrier, Gribonval, Pérez, 16]
[Gribonval, Blanchard, Keriven, Traonmilin, 17]

Sketch 

$z_X \leftarrow$ information about $\mathcal{P}$

Observable? Not $\mathcal{P}$ but

$$\mathcal{P}_X := \frac{1}{N} \sum_{j=1}^{N} \delta_{\boldsymbol{x}_j}$$



Objective: $z_X = \mathcal{A}(\mathcal{P}_X) \approx \mathcal{A}(\mathcal{P})$

We need to sketch distributions!

Sketch of a distribution

Linear sketch on a distribution $\mathcal{P}$:

Given m "frequencies" $\{\boldsymbol{\omega}_1, \cdots, \boldsymbol{\omega}_m\}$,

$$\mathcal{A}(\mathcal{P}) := \mathbb{E}_{\boldsymbol{x}\sim\mathcal{P}} \left[e^{-\mathrm{i}\boldsymbol{\omega}_j^T \boldsymbol{x}} \right]_{j=1}^{m} \in \mathbb{C}^m,$$

with $\mathbb{E}_{\boldsymbol{x}\sim\mathcal{P}}(e^{-\mathrm{i}\boldsymbol{\omega}^\top \boldsymbol{x}}) = \int_{\mathbb{R}^n} e^{-\mathrm{i}\boldsymbol{\omega}^\top \boldsymbol{x}} \mathcal{P}(\boldsymbol{x})\, \mathrm{d}^n\boldsymbol{x}$ for $\boldsymbol{\omega} \in \mathbb{R}^n$.

$\Rightarrow \mathcal{A} =$ sampling the *characteristic function* of $\mathcal{P}$ (*i.e.*, its Fourier transform) over m frequencies

$\Rightarrow \mathcal{A} =$ generalized moments of $\mathcal{P}$

Reminiscent of CS
with partial Fourier sensing!

Related to
Random Fourier Features
[Rahimi, Recht, 07]

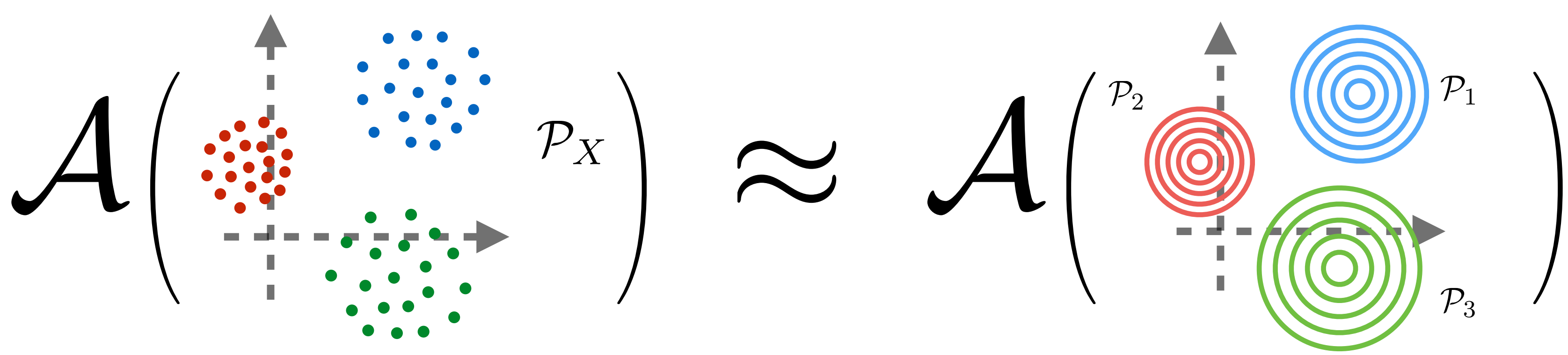
Sketch of a distribution

Sketch of $\boldsymbol{X}$? (the only observable!)

$$\mathcal{A}(\mathcal{P}_X) := \frac{1}{N}\sum_{i=1}^{N}\left[e^{-\mathrm{i}\boldsymbol{\omega}_j^T\boldsymbol{x}_i}\right]_{j=1}^{m} \in \mathbb{C}^m,$$

<u>Properties:</u>

- *oblivious* of each involved signal (as expected)
- easily updatable (from the sum, or *pooling*)
- we "see" $\mathcal{P}_X$ only over certain frequencies!



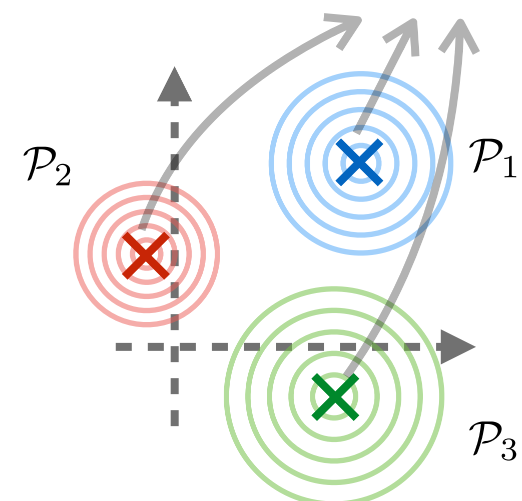
Provided we take low-frequencies!

Select $\{\boldsymbol{\omega}_j\}_{j=1}^{m}$ conveniently!

Compressive K-Means

$\mathcal{P}_X$

$\boldsymbol{z}_X = \; \in \mathbb{C}^m$ **?** $\boldsymbol{C} = \{\boldsymbol{c}_k\}_{k=1}^K$



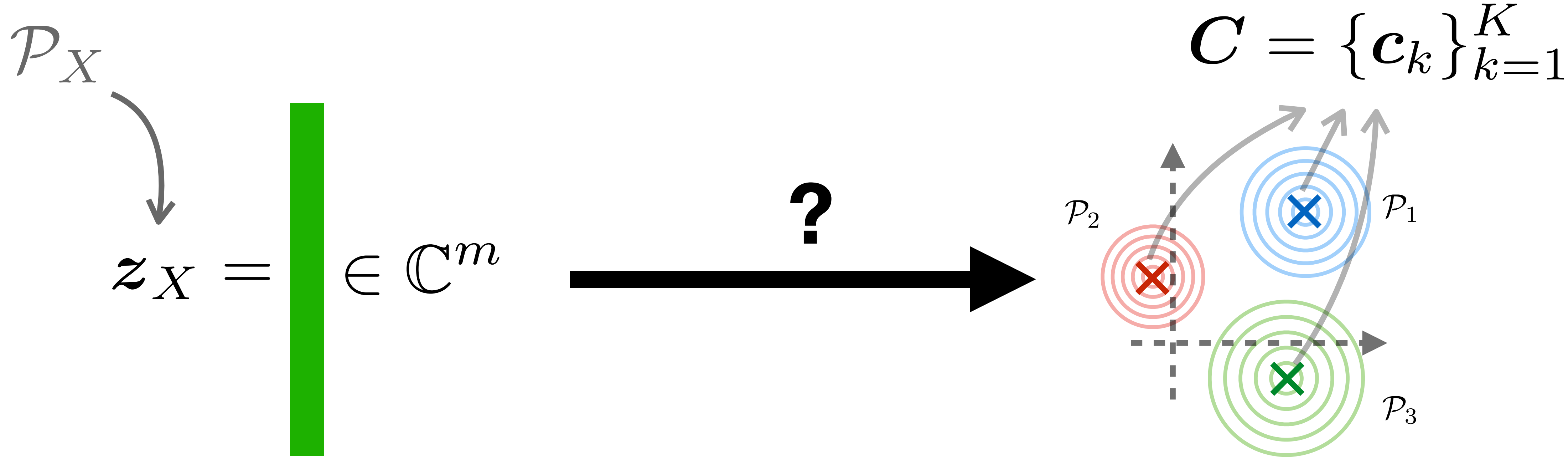
<u>Sketch matching</u>: finding centroids & weights from

$$\min_{\boldsymbol{C},\boldsymbol{\alpha}} \|\boldsymbol{z}_X - \mathcal{A}\big(\textstyle\sum_{k=1}^K \alpha_k \delta_{\boldsymbol{c}_k}\big)\|^2$$

Sparse mixture
of K Diracs

with $\omega_j \sim_{\rm iid} \Lambda$, $1 \leqslant j \leqslant m$
and Λ a pdf promoting low-frequencies

Compressive K-Means

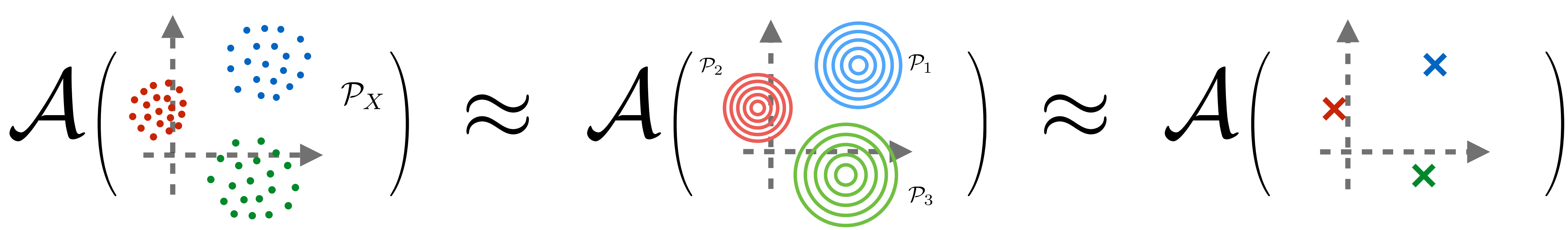


<u>Sketch matching</u>: finding centroids & weights from

$$\min_{\boldsymbol{C},\boldsymbol{\alpha}} \|\boldsymbol{z}_X - \mathcal{A}\big(\textstyle\sum_{k=1}^K \alpha_k \delta_{\boldsymbol{c}_k}\big)\|^2$$

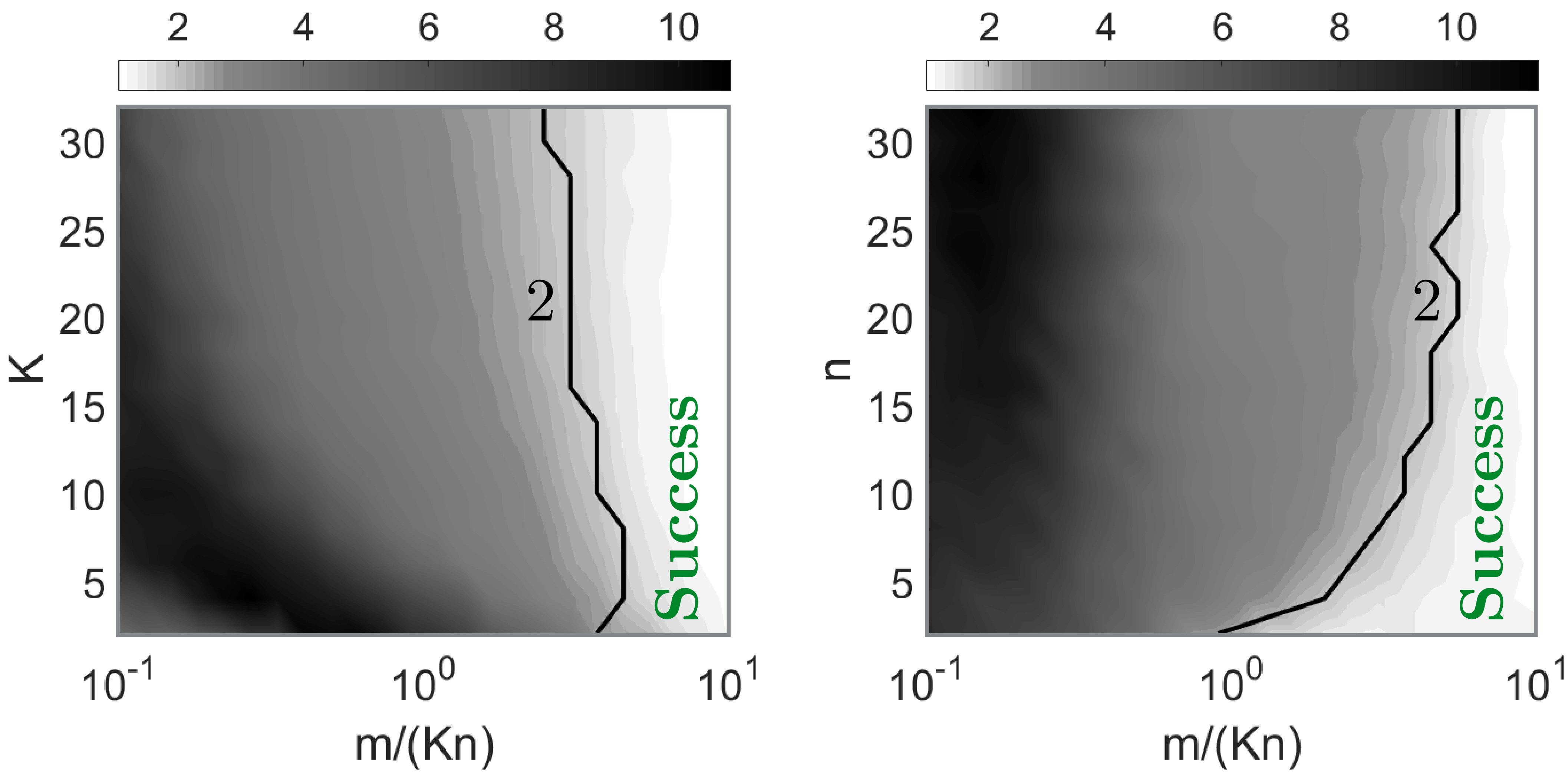
non-convex, infinite dimensional optimization.

At low frequencies,



Compressive K-Means

$$\mathrm{SSE}(\boldsymbol{X}, \boldsymbol{C}_{\mathrm{KM}})/\mathrm{SSE}(\boldsymbol{X}, \boldsymbol{C}_{\mathrm{CKM}})$$

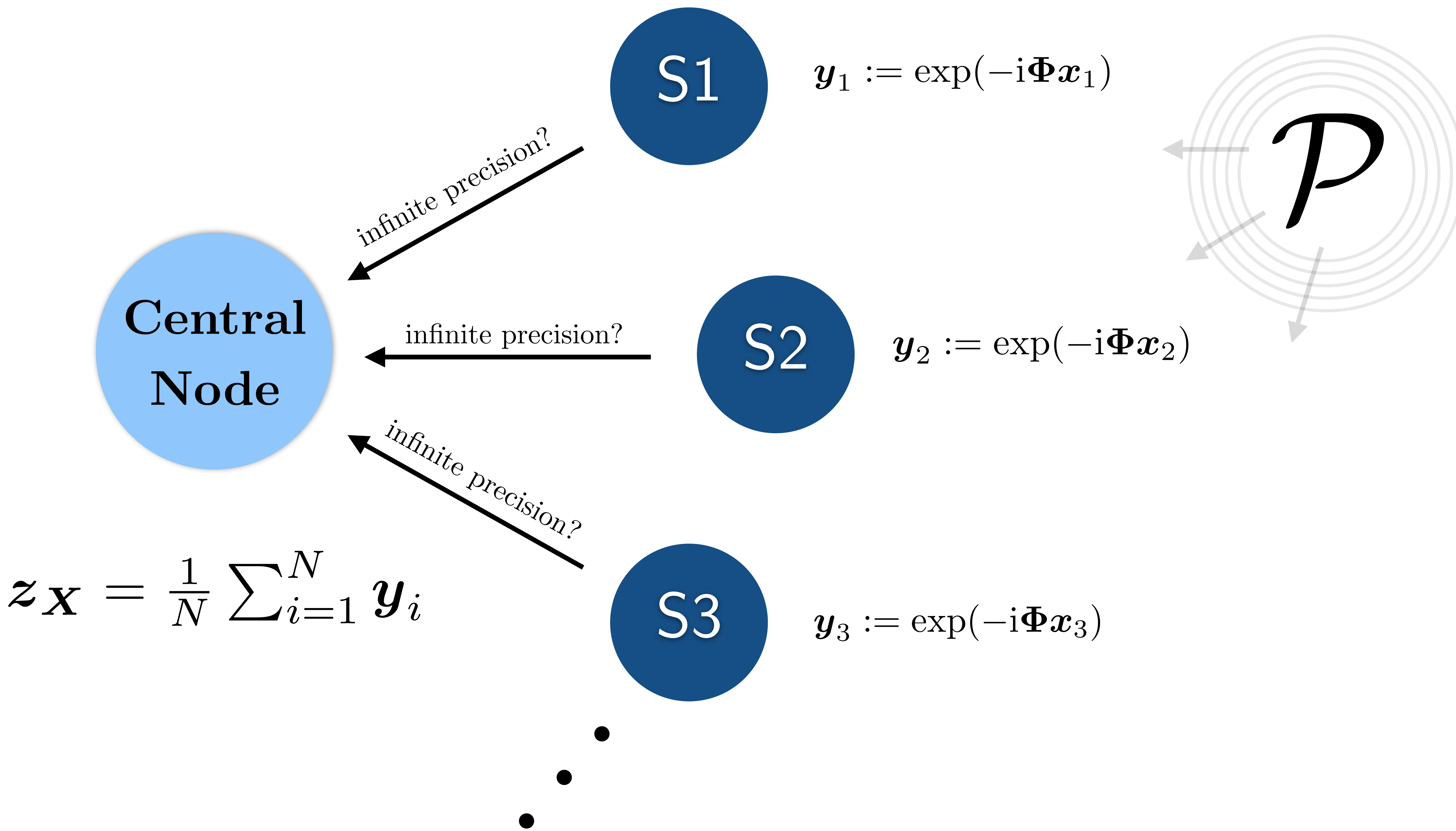


$$SSE(\boldsymbol{X}, \boldsymbol{C}) := \sum_{i=1}^{N} \min_k \|\boldsymbol{x}_i - \boldsymbol{c}_k\|^2$$

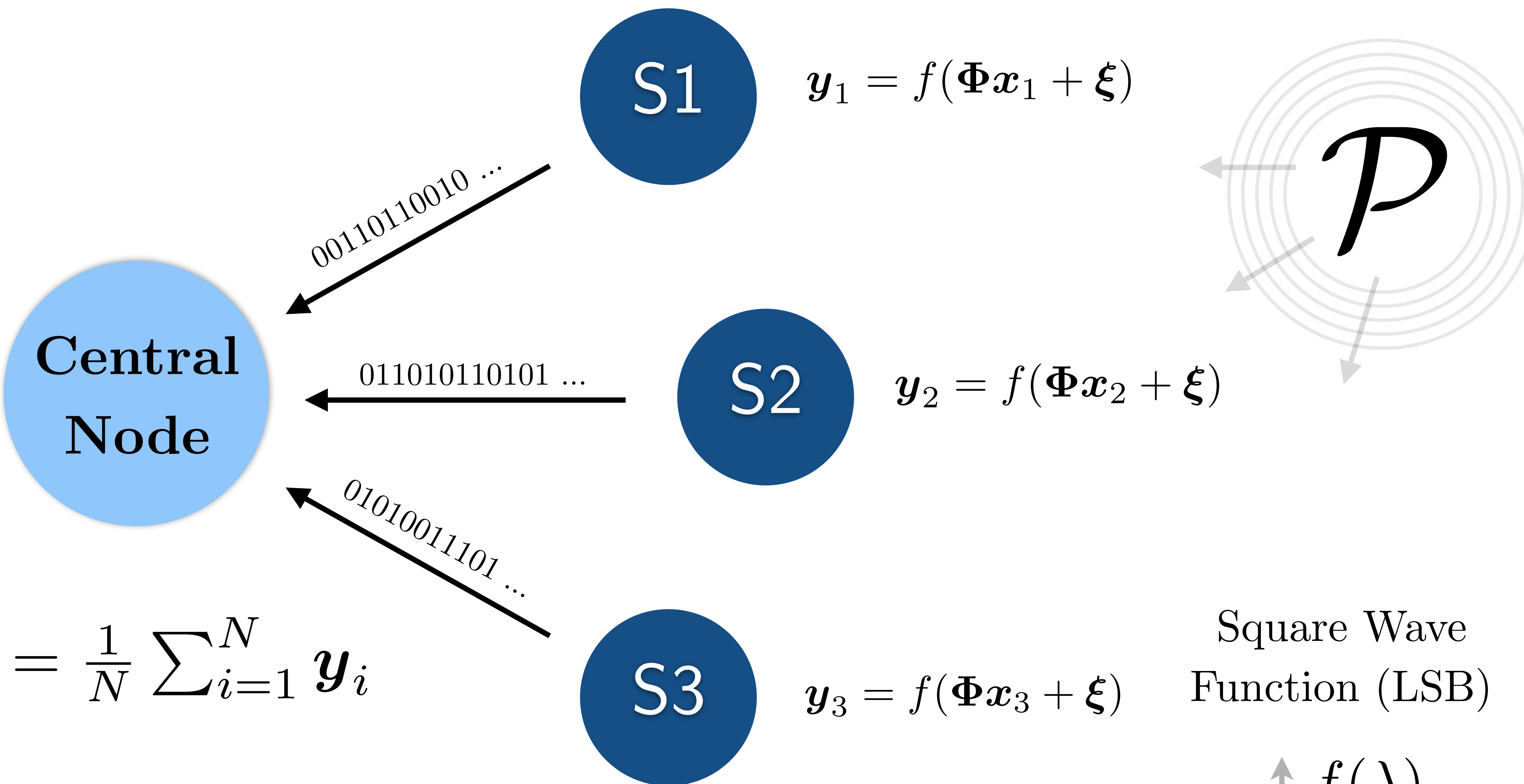
For large N (10^7), CKM up to 150 times faster than KM!

[Keriven, Tremblay, Traonmilin, Gribonval, 17]

Quantizing Sketches?



Quantizing Sketches?

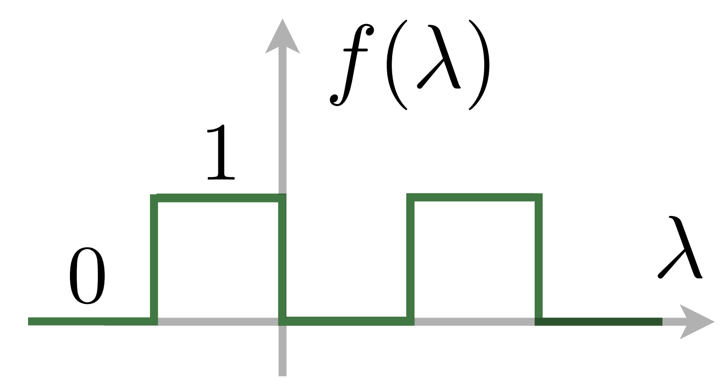


$$\boldsymbol{z}_{\boldsymbol{X}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{y}_i$$

Related to
Universal Quantization
[Boufounos, 12]

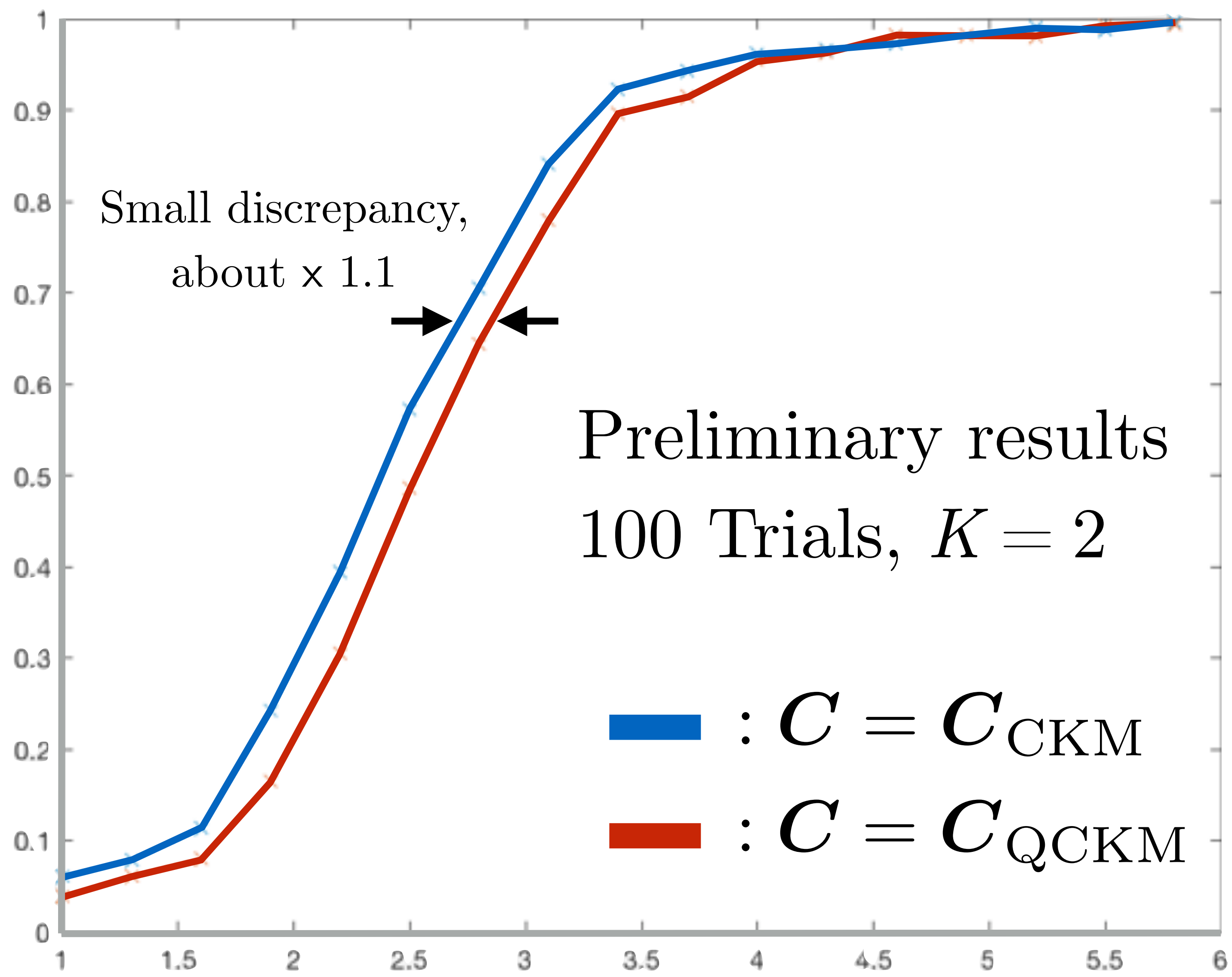
$+$ dithering $\boldsymbol{\xi}$

Square Wave
Function (LSB)



Quantized Compressive K-Means

$$\Pr\left[\mathrm{SSE}(\boldsymbol{X},\boldsymbol{C}) < 1.2\,\mathrm{SSE}(\boldsymbol{X},\boldsymbol{C}_{\mathrm{KM}})\right]$$



$$\log_2(m/nK)$$

Can be proved theoretically, thanks to the dithering
(ongoing work)

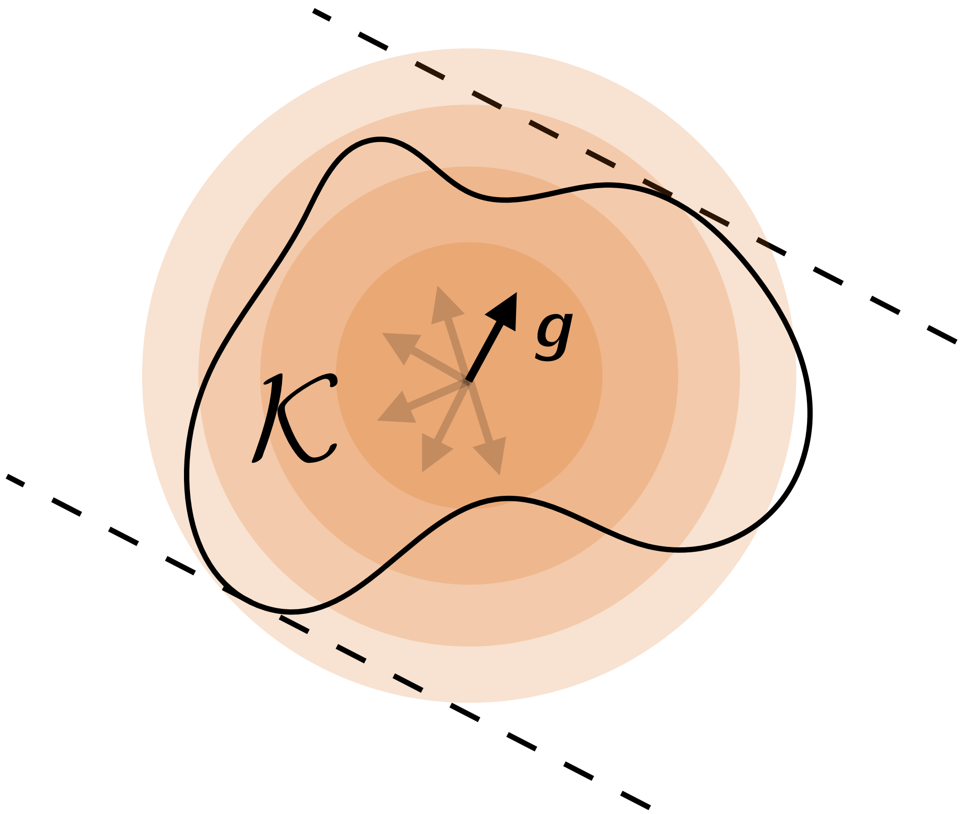
The Linear Case

Bandeira, Mixon, Recht '14 [BMR '14]

Parenthesis: Useful Tool

Gaussian Width (GW):

Let $\mathcal{K} \subset \mathbb{R}^n$

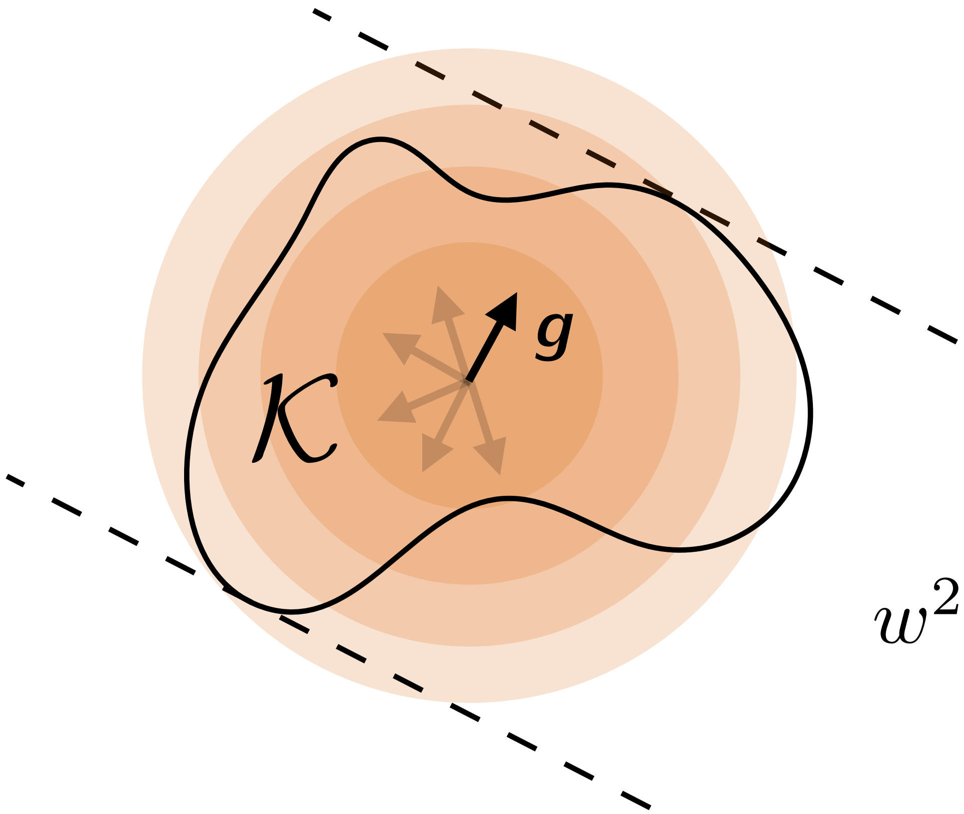


Parenthesis: Useful Tool

Gaussian Width (GW):

Let $\mathcal{K} \subset \mathbb{R}^n$, $\boldsymbol{g} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_n)$,

$$w(\mathcal{K}) = \mathbb{E}_{\boldsymbol{g}} \sup_{\boldsymbol{x} \in \mathcal{K}} |\langle \boldsymbol{x}, \boldsymbol{g} \rangle|$$



Examples:

$$w^2(\mathcal{K}) \lesssim \log |\mathcal{K}|$$
$$w^2(\mathbb{B}^n) \lesssim n$$
$$w^2(\Sigma_k^n \cap \mathbb{B}^n) \lesssim k \log(n/k)$$
$$w^2(\mathcal{M}_r \cap \mathbb{B}_F^{n \times n}) \lesssim rn$$
$$w^2(\cup_{i=1}^T \mathcal{K}_i) \lesssim \log T + \max_i w^2(\mathcal{K}_i)$$
$$\vdots$$

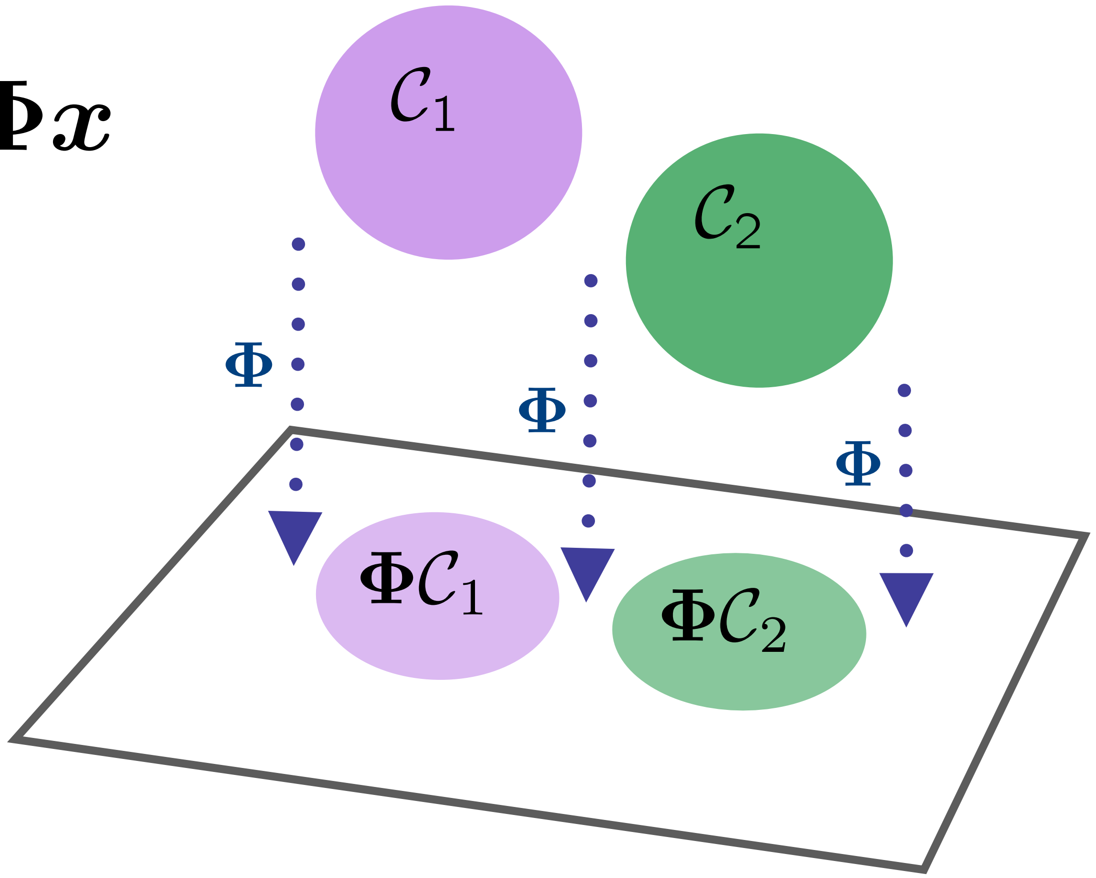
The Rare Eclipse Problem

Problem (Rare Eclipse Problem (Bandeira *et al.* '14))**.**

Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n : \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ be closed convex sets, $\mathbf{\Phi} \sim \mathcal{N}^{m \times n}(0, 1)$. Given $\eta \in (0, 1)$, find the smallest m so that

$$p_0 := \mathbb{P}_{\mathbf{\Phi}}[\mathbf{\Phi}\mathcal{C}_1 \cap \mathbf{\Phi}\mathcal{C}_2 = \emptyset] \geq 1 - \eta.$$

$$\mathsf{A}(\boldsymbol{x}) = \mathbf{\Phi}\boldsymbol{x}$$



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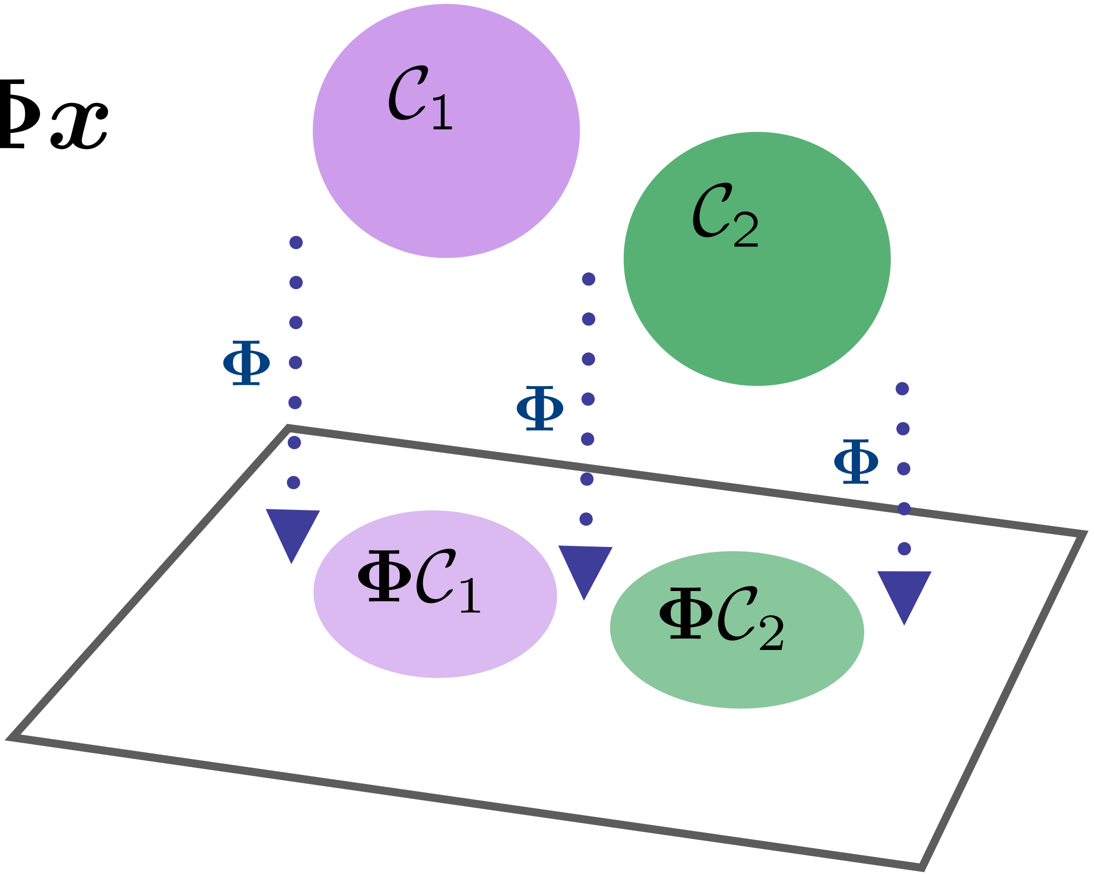
$$p_0 = \mathbb{P}_{\mathbf{\Phi}}[\forall \mathbf{x}_1 \in \mathcal{C}_1, \forall \mathbf{x}_2 \in \mathcal{C}_2, \mathbf{\Phi}(\mathbf{x}_1 - \mathbf{x}_2) \neq 0] \geq 1 - \eta.$$

$$\mathsf{A}(\boldsymbol{x}) = \mathbf{\Phi}\boldsymbol{x}$$



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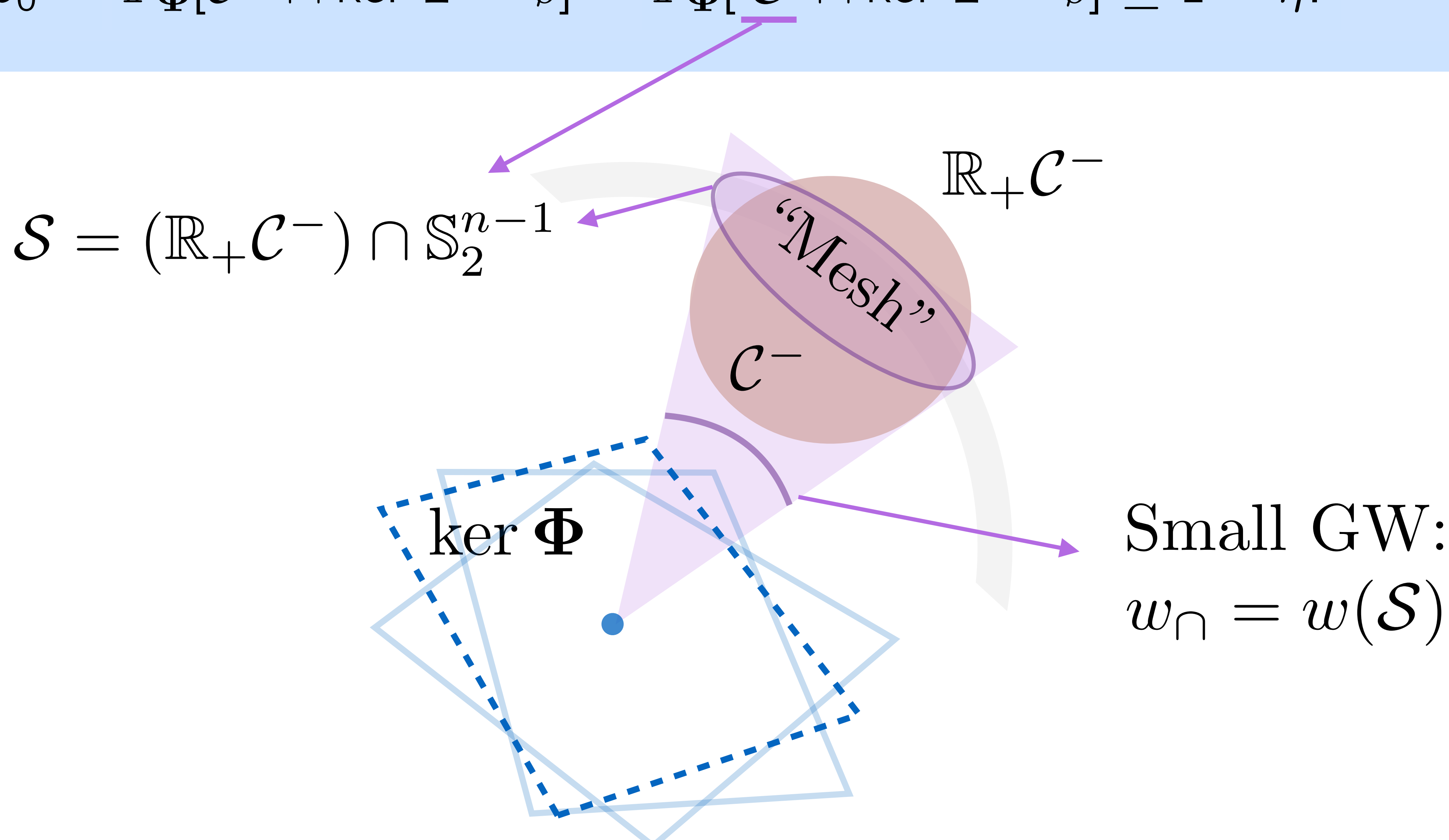
$$p_0 = \mathbb{P}_{\mathbf{\Phi}}[\mathcal{C}^- \cap \ker \mathbf{\Phi} = \emptyset] = \mathbb{P}_{\mathbf{\Phi}}[\mathcal{S} \cap \ker \mathbf{\Phi} = \emptyset] \geq 1 - \eta.$$



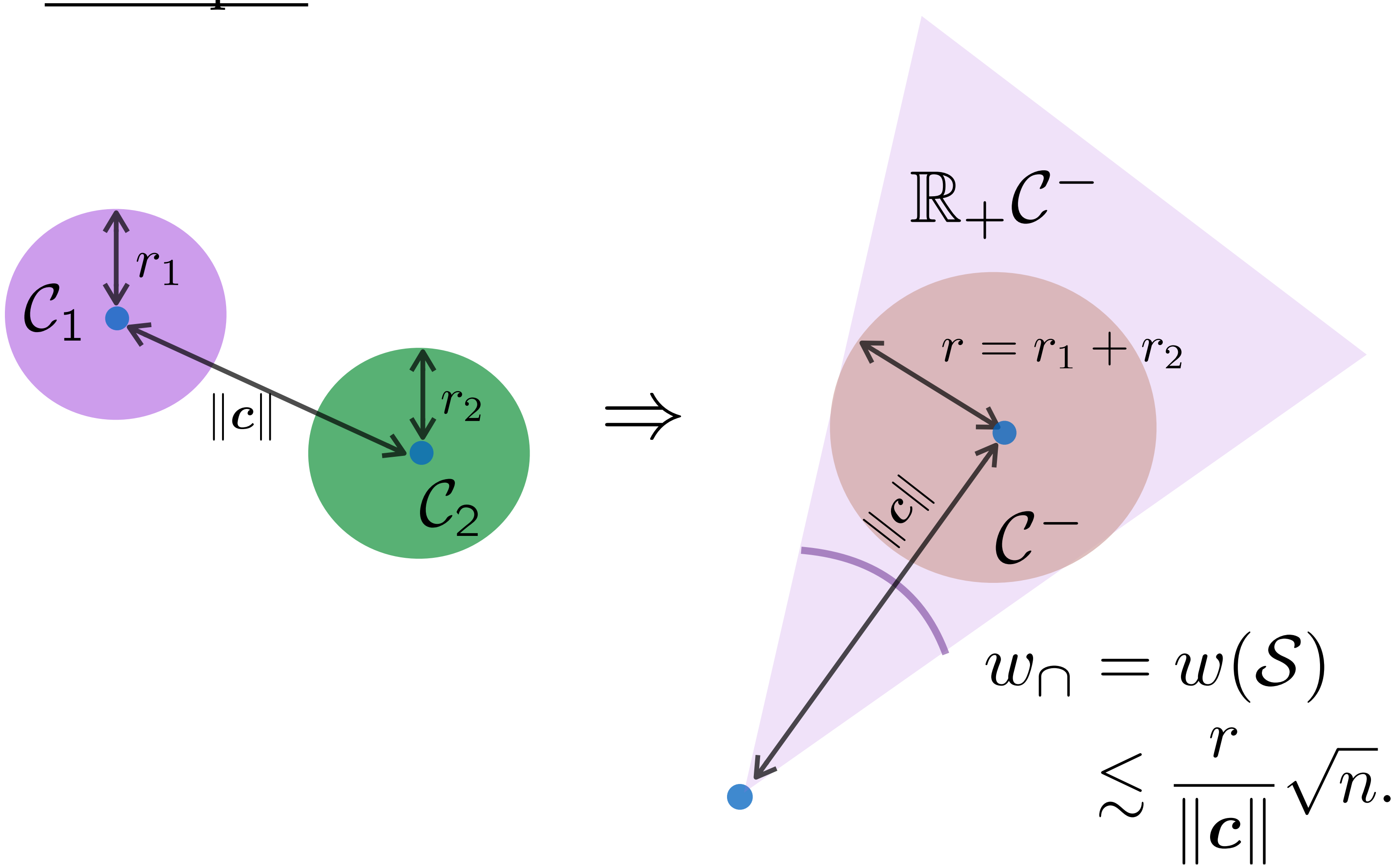
The Rare Eclipse Problem

BMR '14: "Gordon's escape through a mesh" theorem

Proposition (Corollary 3.1 in BMR '14). (& *really* tight [Amelunxen et al, 13])

Given $\eta \in (0,1)$, if $m > \left(w_\cap + \sqrt{2\log\frac{1}{\eta}}\right)^2 + 1$ then $p_0 \geq 1-\eta$.

<u>Example:</u>



$\Rightarrow$

$$m \gtrsim \frac{r^2}{\|c\|^2} n$$

The Rare Eclipse Problem (alternative)

Restricted Isometry Property: (ℓ_1, ℓ_2)-RIP$(\mathcal{K}, \epsilon)$

$$\forall \boldsymbol{x} \in \mathcal{K},\ (1-\epsilon)\,\|\boldsymbol{x}\| \leqslant \|\boldsymbol{\Phi x}\|_1 \leqslant (1+\epsilon)\,\|\boldsymbol{x}\|$$

[Schechtman, 06] [Plan, Vershynin, 14]

If $\mathcal{S} \subset \mathbb{S}^{n-1}$ and $m \gtrsim \epsilon^{-2} w^2(\mathcal{S})$, then, w.h.p.*,

$$(1-\epsilon) \leqslant \sqrt{\tfrac{\pi}{2}}\tfrac{1}{m}\|\boldsymbol{\Phi u}\|_1 \leqslant (1+\epsilon)$$

for $\boldsymbol{\Phi} \in \mathbb{R}^{m\times n}$ and $\Phi_{ij} \sim_{\text{iid}} \mathcal{N}(0,1)$.

*: *i.e.*, $\mathbb{P} \geqslant 1 - C\exp(-c\epsilon^2 m)$.

The Rare Eclipse Problem (alternative)

Restricted Isometry Property: $(\ell_1, \ell_2)\text{-RIP}(\mathcal{K}, \epsilon)$

$$\forall \boldsymbol{x} \in \mathcal{K},\ (1-\epsilon)\,\|\boldsymbol{x}\| \leqslant \|\boldsymbol{\Phi}\boldsymbol{x}\|_1 \leqslant (1+\epsilon)\,\|\boldsymbol{x}\|$$

[Schechtman, 06] [Plan, Vershynin, 14]

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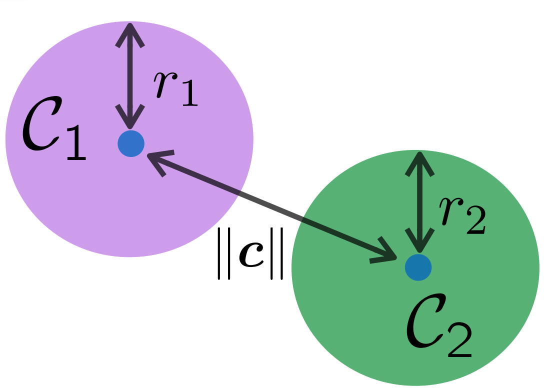
for $\boldsymbol{\Phi} \in \mathbb{R}^{m\times n}$ and $\Phi_{ij} \sim_{\text{iid}} \mathcal{N}(0,1)$.

Therefore:

For $\mathcal{S} = (\mathbb{R}^+\mathcal{C}^-) \cap \mathbb{S}^{n-1}$, if $m \gtrsim \epsilon^{-2} w_\cap^2$, w.h.p*, (P1)

$$(1-\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leqslant \sqrt{\tfrac{\pi}{2}}\,\tfrac{1}{m}\|\boldsymbol{\Phi}\boldsymbol{x}_1 - \boldsymbol{\Phi}\boldsymbol{x}_2\|_1 \leqslant (1+\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|,$$

for all $\boldsymbol{x}_1 \in \mathcal{C}_1$ and all $\boldsymbol{x}_2 \in \mathcal{C}_2$.



This result also explains REP!
(but less sharply)

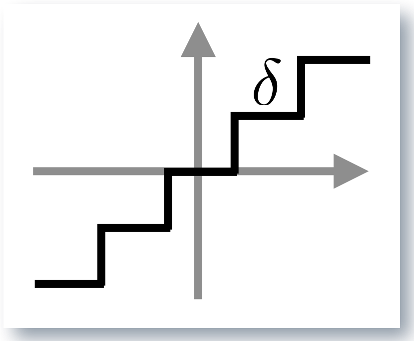
*: *i.e.*, $\mathbb{P} \geqslant 1 - C\exp(-c\epsilon^2 m)$.

The Quantized Case

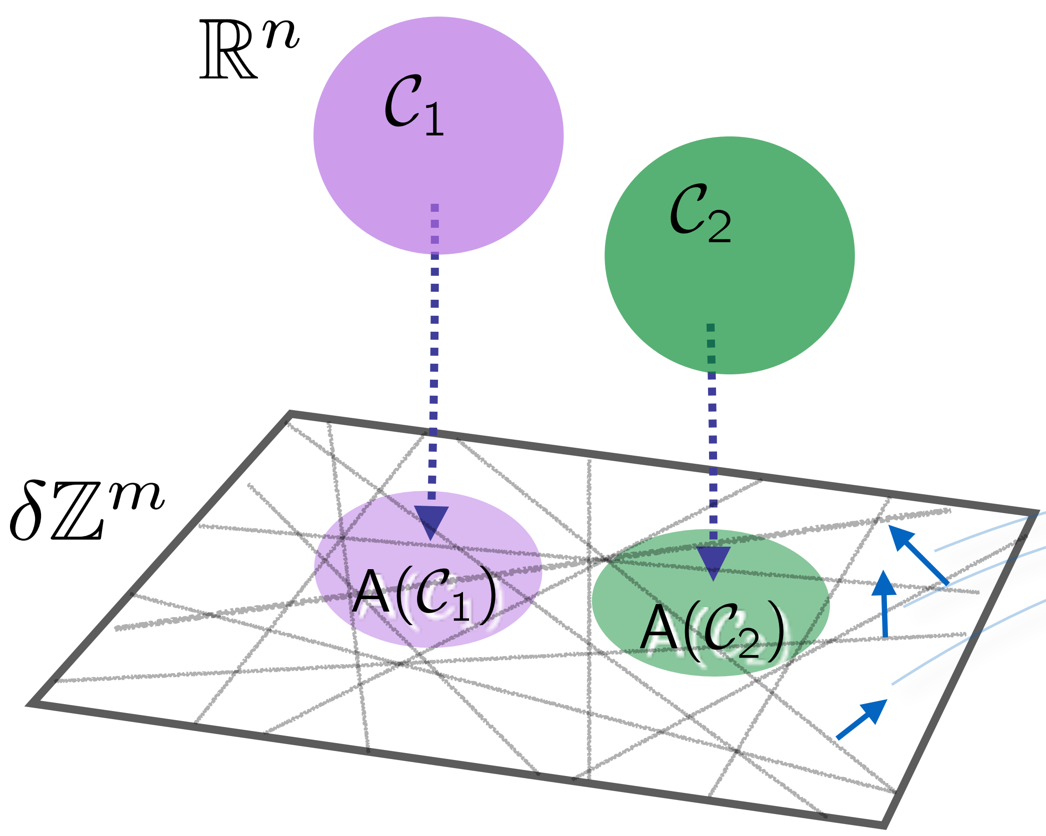
This work

Quantized Dithered Random Mapping

$$\begin{cases} \delta > 0 & \text{(a resolution)}, \\ \mathcal{Q}(\cdot) := \delta\lfloor \cdot/\delta \rfloor \quad \text{(applied componentwise)} & \text{(scalar quantizer)}, \\ \boldsymbol{\Phi} \text{ is } (\ell_1, \ell_2)\text{-RIP}(\mathcal{K}, \epsilon) & \text{(a well-behaved } \boldsymbol{\Phi}), \\ \text{and a } \textit{dithering } \boldsymbol{\xi} \in \mathbb{R}^m \text{ with } \xi_j \sim_{\text{iid}} \mathcal{U}([0,\delta]) & \text{(your friend)} \end{cases}$$



QDRM: $$\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi})$$



$$\mathsf{A}_i(\boldsymbol{x}) = \mathcal{Q}(\boldsymbol{\varphi}_i^\top \boldsymbol{x} + \xi_i) \in \delta\mathbb{Z}$$

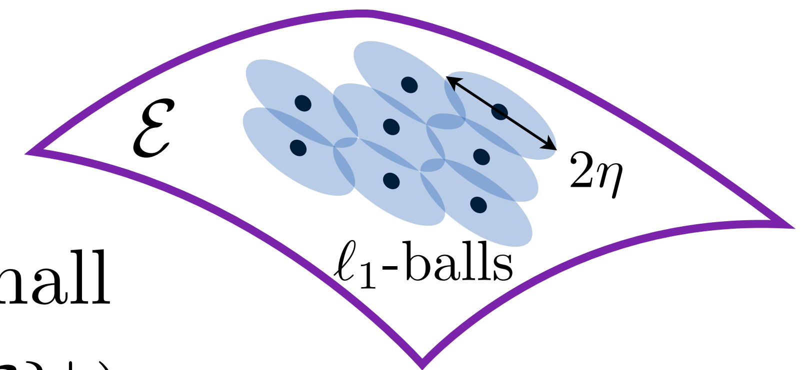
$$\boldsymbol{\Phi} = (\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_m)^\top$$

Quantized Random Embeddings

(i.e., only thanks to the dithering!)

Embedding $\mathcal{E} \subset \mathbb{R}^m$ into $\delta\mathbb{Z}^m$ with

$$\mathsf{A}'(\boldsymbol{y}) := \underset{\delta}{\mathcal{Q}}(\boldsymbol{y} + \underset{\delta}{\boldsymbol{\xi}})$$



If $\mathcal{E}$ has small ℓ_1-Kolmogorov entropy, *i.e.*, small

$$\mathcal{H}_1(\mathcal{E}, \eta) = \log(\, \min_{\mathcal{S}} |\{\mathcal{S} \subset \mathcal{E} : \mathcal{S} + \eta\mathbb{B}_1^n \supset \mathcal{E}\}|\,)$$

*: [LJ, Cambareri, '16] [Cambareri, Xu, LJ, '17]

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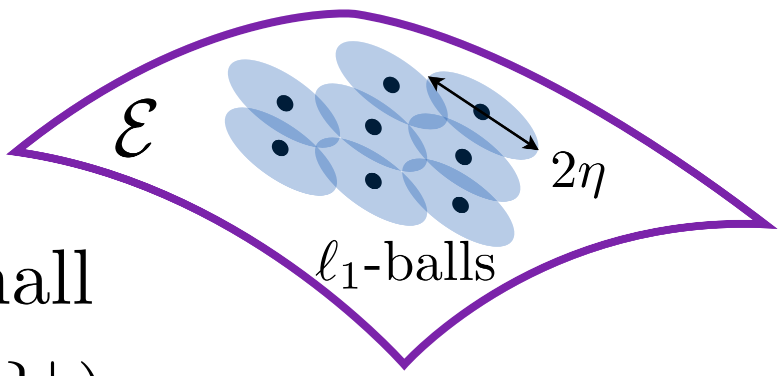

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Given $\epsilon > 0$, if $m \gtrsim \epsilon^{-2}\mathcal{H}_1(\mathcal{E}, \frac{m\delta\epsilon^2}{1+\epsilon})$, then, w.h.p*, for all $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{E}$ and some $c > 0$, (P2)

$$\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_1 - c\delta\epsilon \leqslant \tfrac{1}{m}\|\mathsf{A}'(\boldsymbol{y}_1) - \mathsf{A}'(\boldsymbol{y}_2)\|_1 \leqslant \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_1 + c\delta\epsilon.$$

Remarks:

- For $\mathcal{E} = \boldsymbol{\Phi}\mathcal{K}$ and $\boldsymbol{\Phi}$ an (ℓ_1, ℓ_2)-RIP$(\mathcal{K} - \mathcal{K}, \epsilon' < 1)$,

$$\mathcal{H}_1(\mathcal{E}, 2m\eta) \leqslant \mathcal{H}_2(\mathcal{K}, \eta) \quad \text{(P3)}$$

- $\mathcal{H}_2$ is bounded for sets (cones, convex spaces, ...)

*: [LJ, Cambareri, '16] [Cambareri, Xu, LJ, '17]

Conclusion

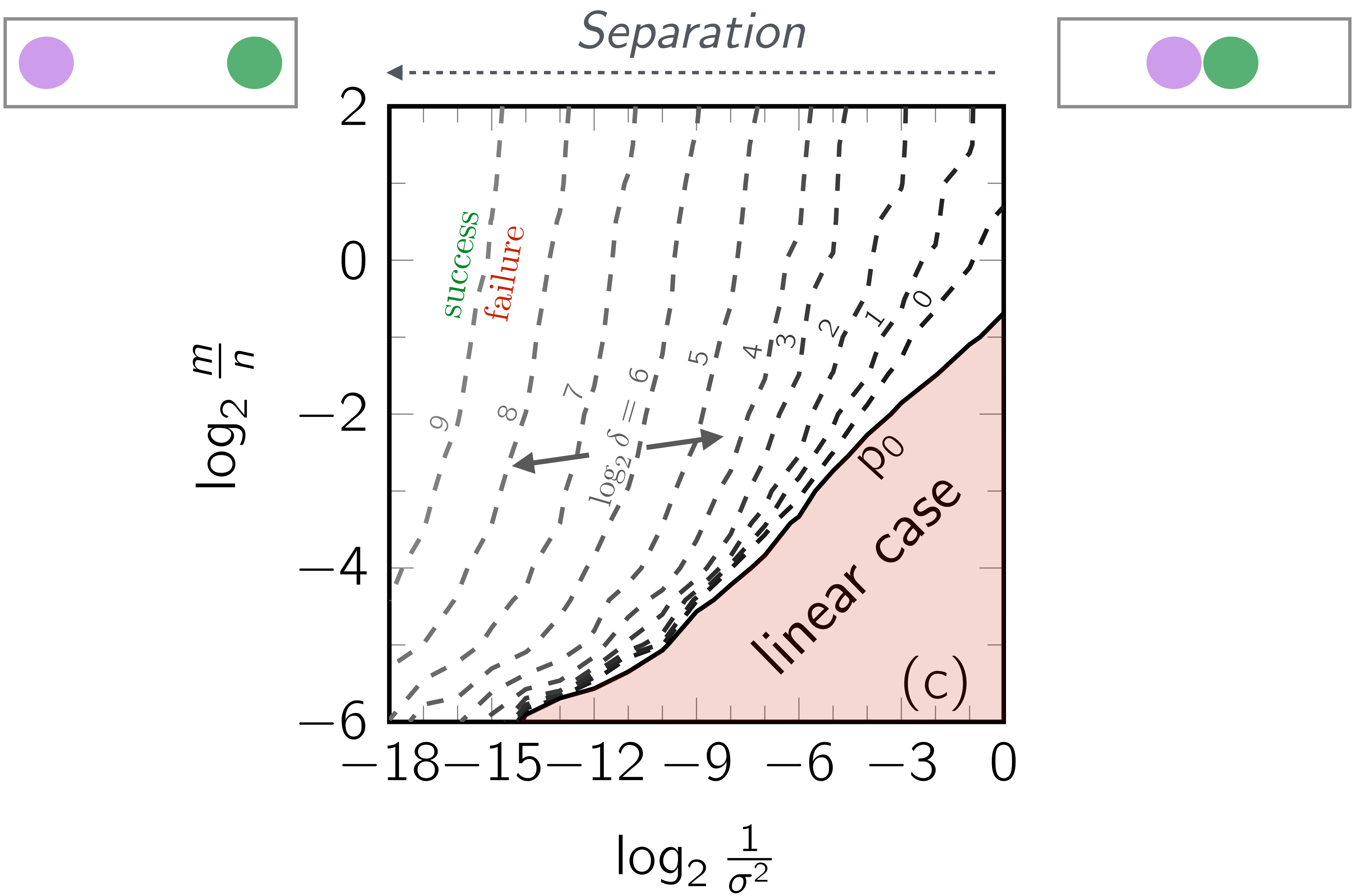
- **Take-away messages:**
  - QDRMs: non-adaptive dimensionality reduction,
    preserve geometry of datasets
  - Extension of BMR '14 to QDRMs,
    with sample complexity loss in $\frac{\delta^2}{\sigma^2} n$ (quantiz. impact)
- **Future work:**
  - Better bound and testable conditions for empirical tests.
  - Extension of result to $\text{RIP}_{2,2}$ matrices, "fast" random matrices
  - Extension of framework to other non-linear maps (ReLU?)
    $\rightarrow$ applicability to D/CNN with random weights [Giryes et al. 15]?

Simulations: Phase transition study

Empirical evaluation, crude lower bound (with no dithering),

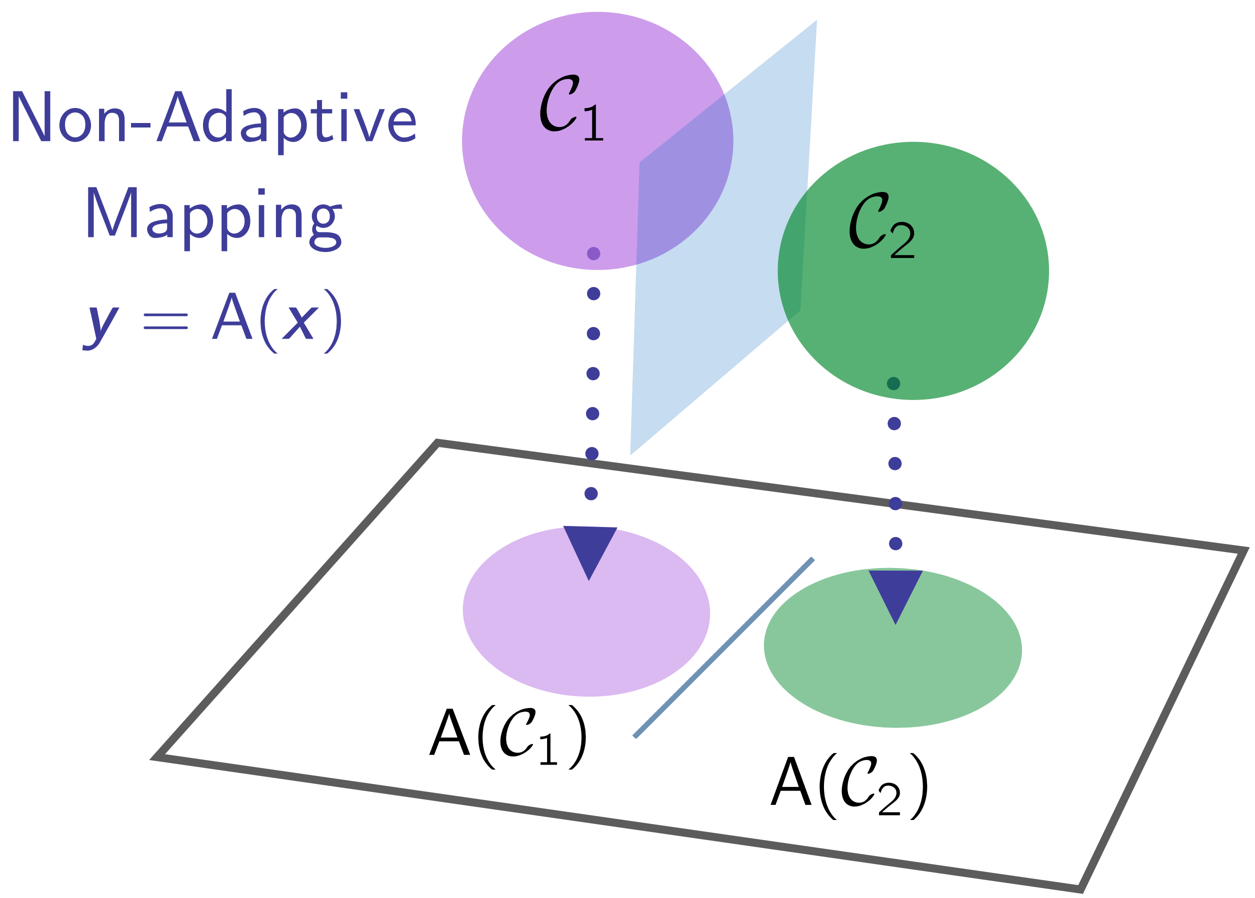
128 trials, randomly drawn, n = 64, fixed r = 1, δ = {1, 2, ..., 512}.

Represented: Phase transition level-curves (at 0.9).



The Big Picture (an easy classification problem)

How does the *probability of error* of a (generic) learning task depend on m, A, $\mathcal{K}$? What if A is (*mildly*) non-linear?



Related works on "(linear) compressive classification":

Davenport *et al.* '07-'10, Haupt *et al.* '06,

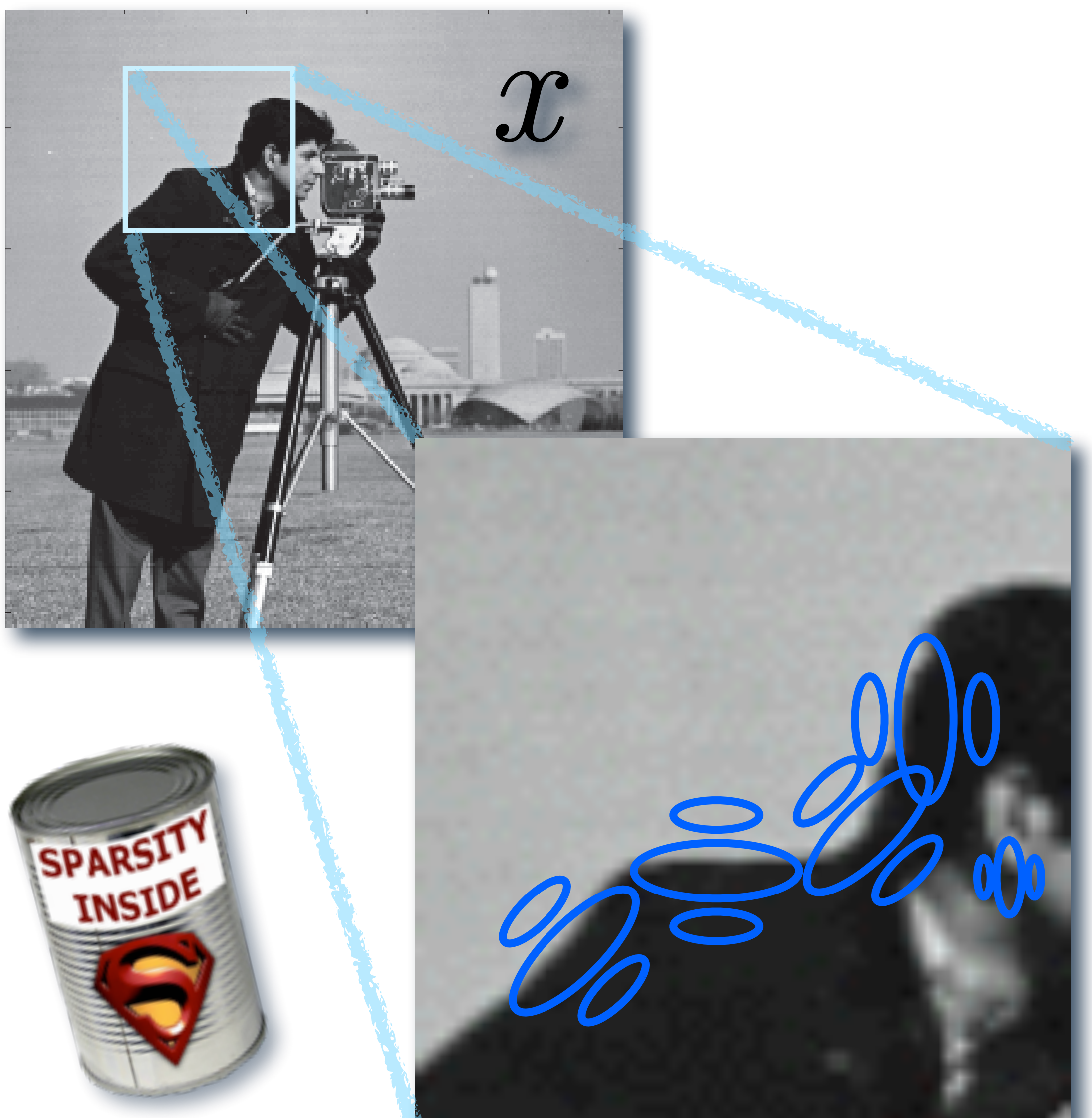
Reboredo *et al.* '13-'16,

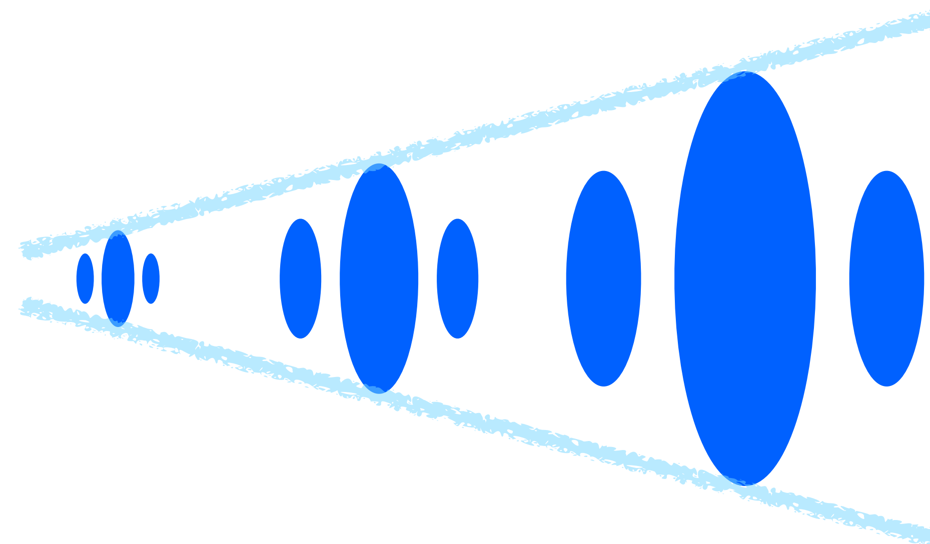
Bandeira, Mixon, Recht '14 [BMR '14]

Prior information?

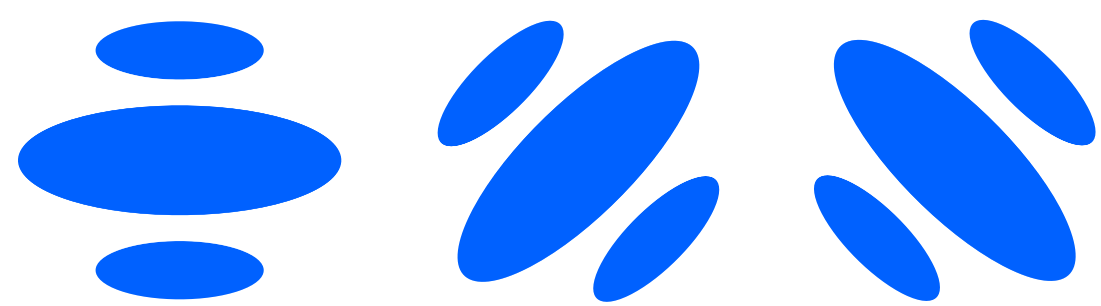
Example: sparse images in wavelets

Representing this image ...



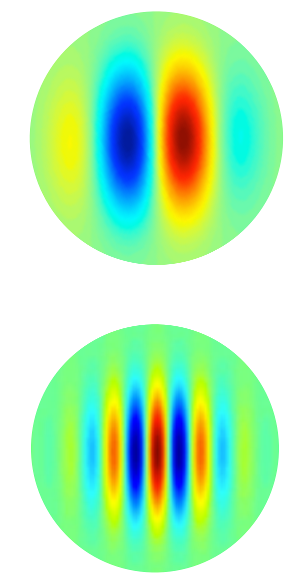


different sizes, scales



different orientations

e.g.,



Compressed Sensing...

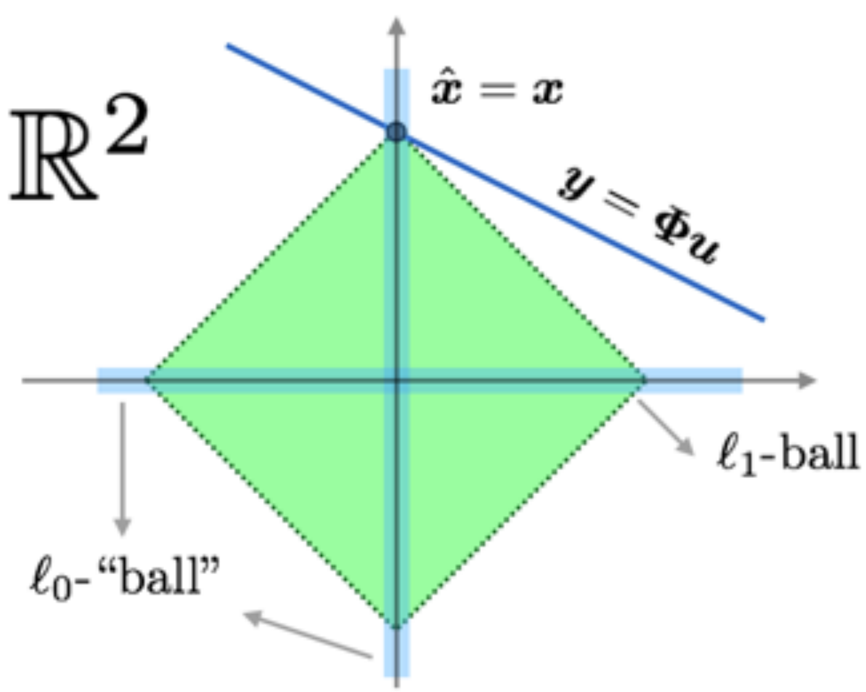
Use non-linear reconstruction methods:

Basis Pursuit DeNoise [Chen, Donoho, Saunders, 98]

$$\hat{\boldsymbol{x}} \in \underset{\boldsymbol{u} \in \mathbb{R}^N}{\arg\min} \ \|\boldsymbol{u}\|_1 \ \text{s.t.} \ \|\boldsymbol{y} - \boldsymbol{\Phi u}\| \leq \epsilon$$

Sparsity promotion
$\|\boldsymbol{u}\|_1 = \sum_j |u_j|$

Level of "noise"
$\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{n}, \ \|\boldsymbol{n}\| \leqslant \epsilon$



Many toolboxes:
SPGL1, Numerical Tours, ...

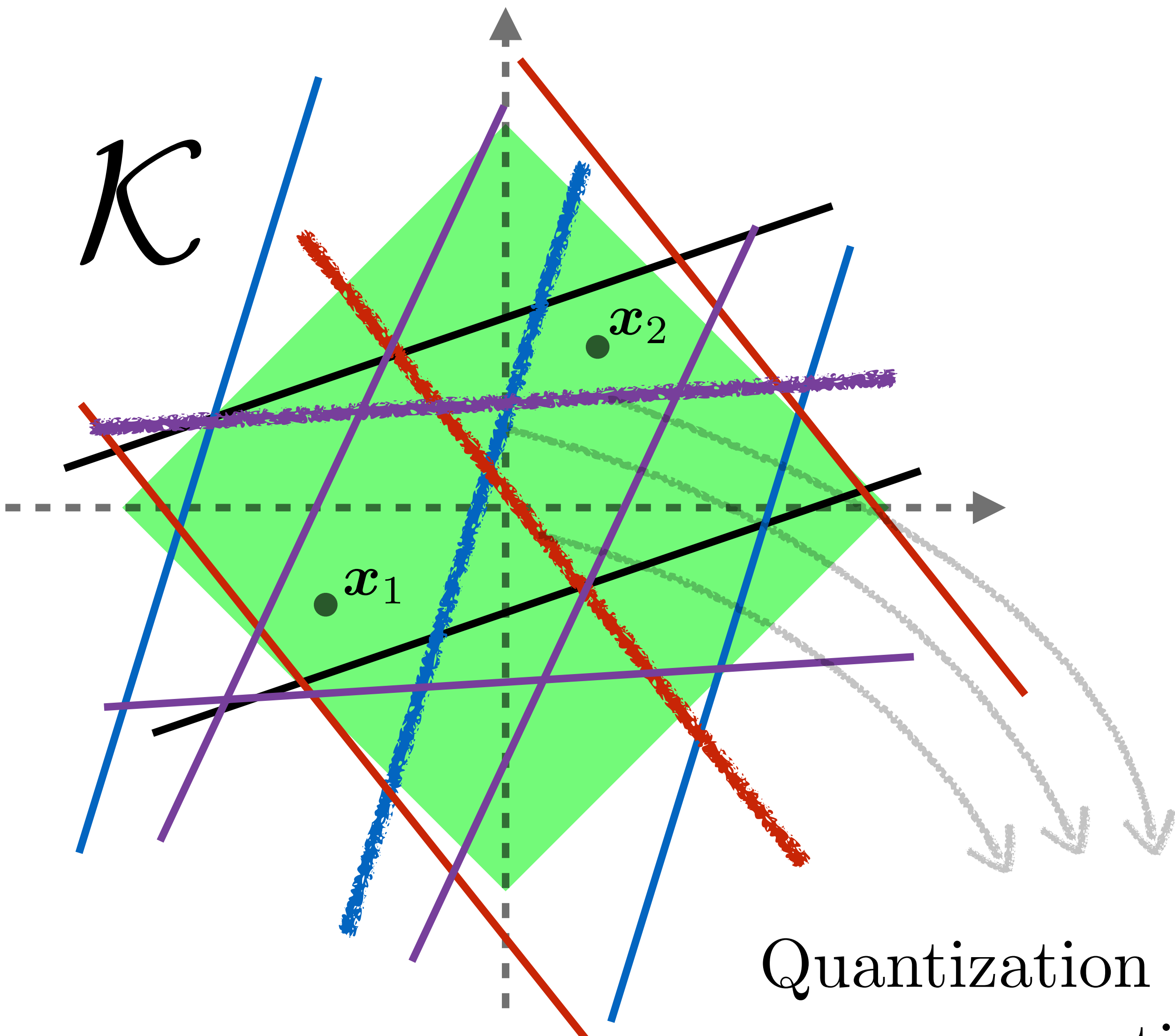
+ Many other algorithms
(ex. greedy algorithms)

ℓ_1-ball in high dimension
$\approx$ set of bounded sparse signals

*: with other norms for other low-complexity sets (e.g., nuclear norm)

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$

- 3. Quantizing the RIP (approximate consistency)



Quantization frontiers separating $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$
= separating random hyperplanes oriented and positioned according to $(\boldsymbol{\Phi}, \boldsymbol{\xi})$

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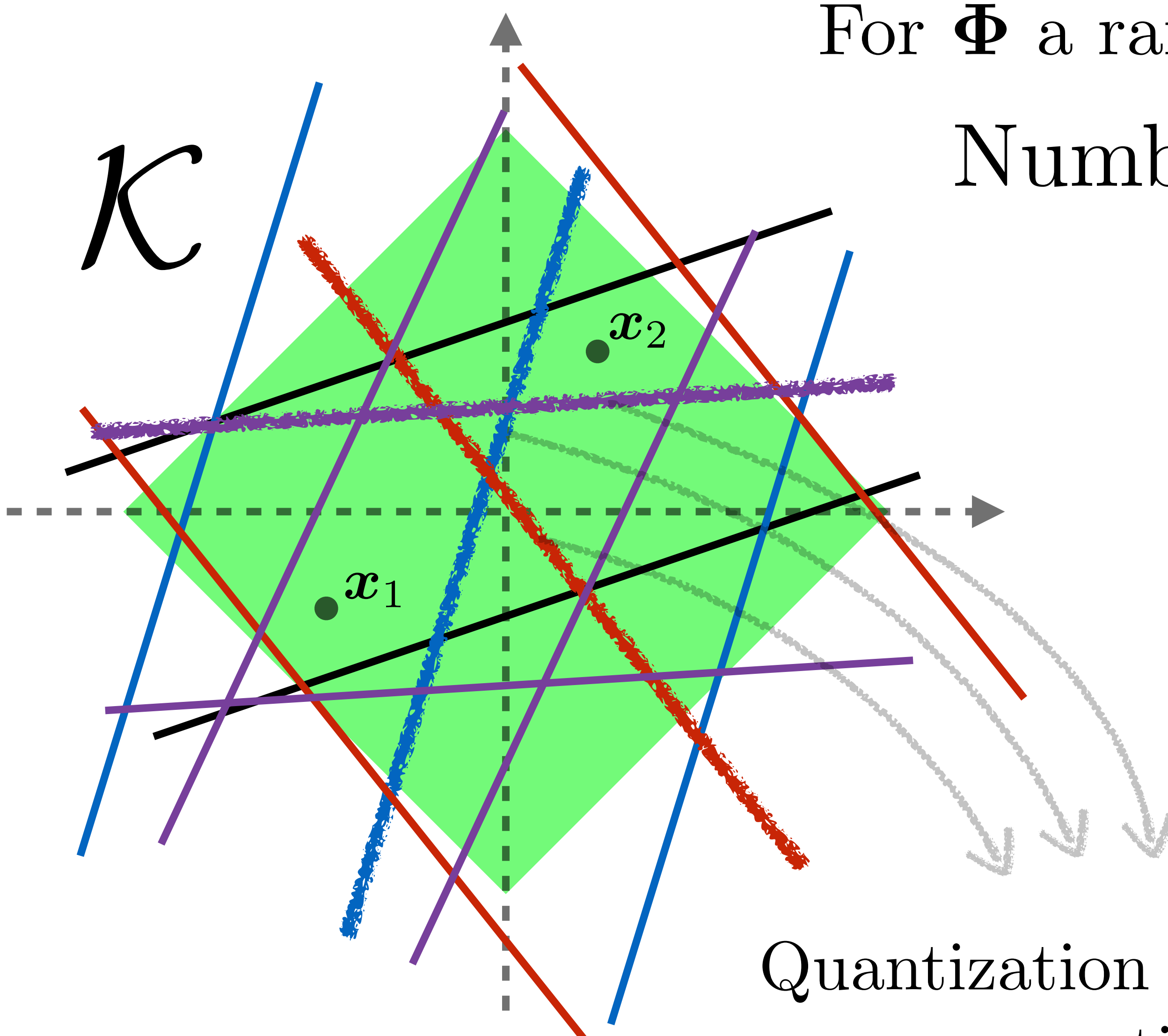

For $\boldsymbol{\Phi}$ a random Gaussian matrix,

Number of such hyperplanes
$\approx \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$
with high probability
provided $M \geqslant C_{\mathcal{K}}\, \epsilon^{-2} \log(1/\delta\epsilon^2)$
(e.g., for sparse signals, low-rank matrices)

Quantization frontiers separating $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$
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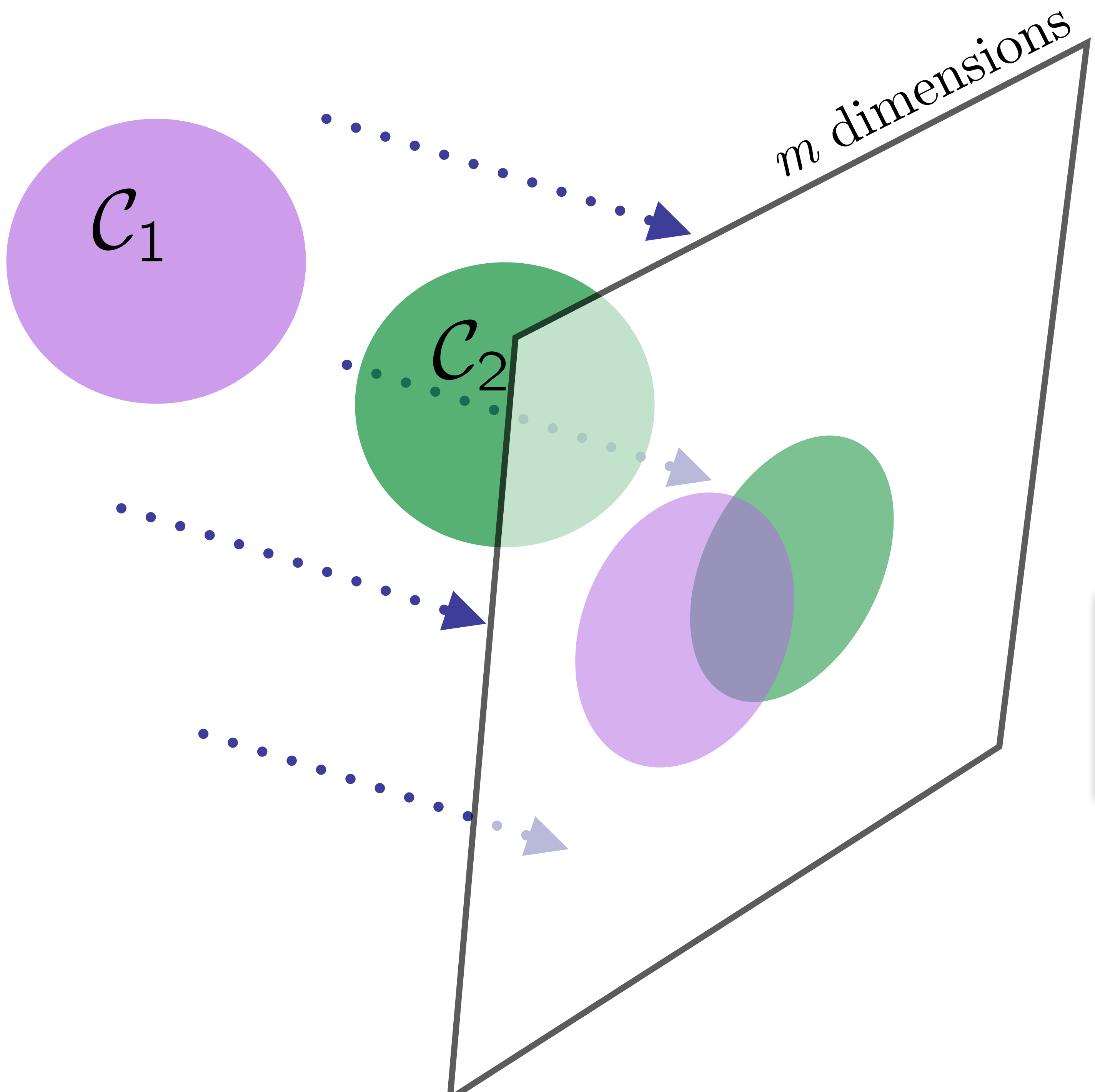
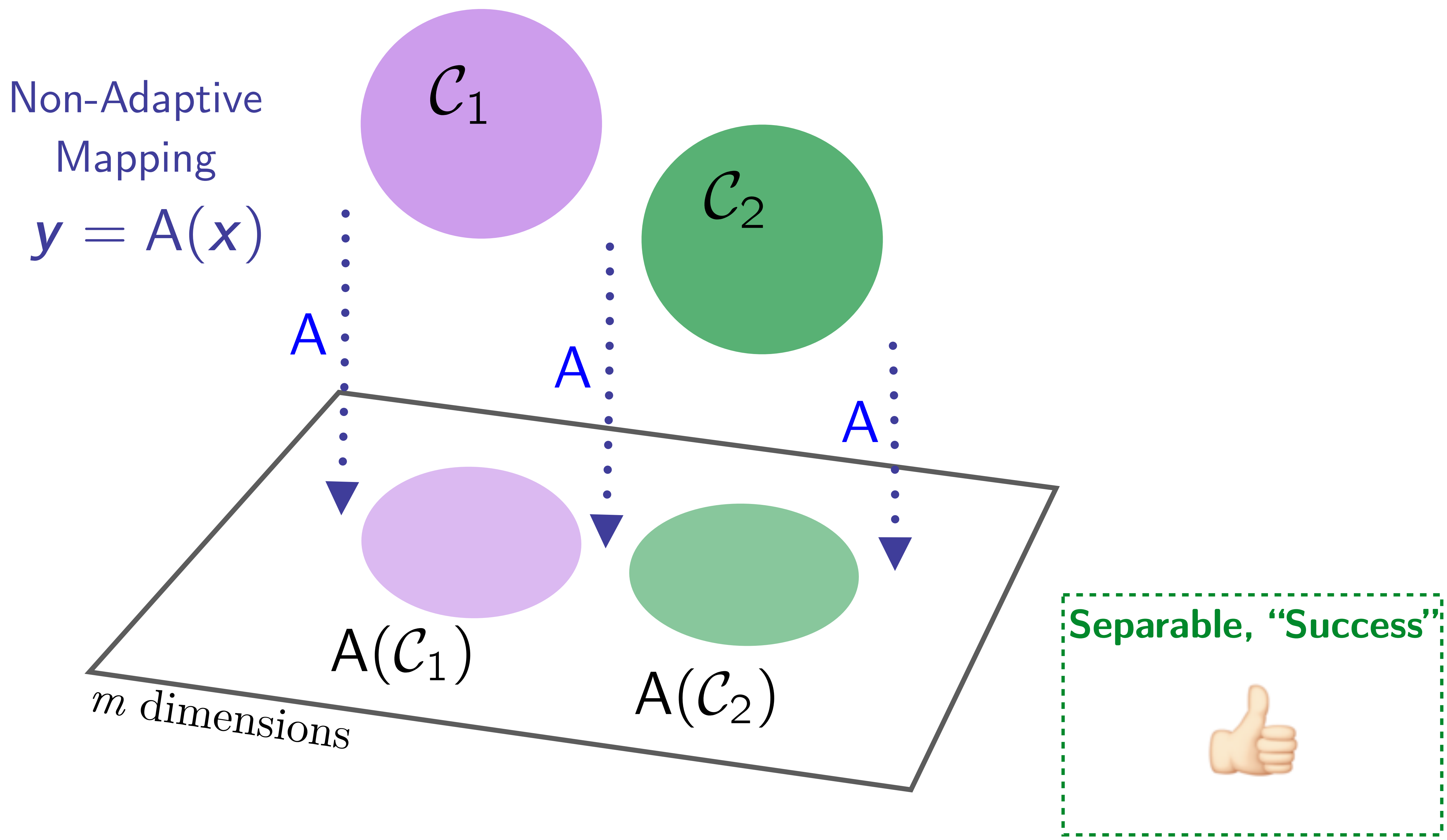
The Big Picture (an easy classification problem)



The Big Picture (an easy classification problem)

$\mathcal{K} \subset \mathbb{R}^n$ dataset
$\mathcal{C}_i \subset \mathcal{K}$ classes, $i = 1, 2, \ldots$



First attempts [Candès, Tao, 04]

- Quantization is like a noise! (e.g., for $\mathcal{Q}[\lambda] = \delta\lfloor \frac{\lambda}{\delta} \rfloor \in \delta\mathbb{Z}$)

$$\boldsymbol{y} = \mathcal{Q}(\boldsymbol{\Phi x}) = \boldsymbol{\Phi x} + \boldsymbol{n}, \quad \text{with } \boldsymbol{n} = \mathcal{Q}(\boldsymbol{\Phi x}) - \boldsymbol{\Phi x}.$$

$$\text{and } \|\boldsymbol{n}\|^2 = O(m\,\delta^2)$$

- Problem: (e.g., for $\hat{\boldsymbol{x}} \in \underset{\boldsymbol{u} \in \mathbb{R}^N}{\arg\min} \|\boldsymbol{u}\|_1 \text{ s.t. } \|\boldsymbol{y} - \boldsymbol{\Phi u}\| \leq \epsilon$, aka BPDN)

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \lesssim \frac{\epsilon}{\sqrt{m}} = O(\delta) \text{ does not decay if } m \text{ increases!}$$

Remark: r-order $\Sigma\Delta$ quantizer + CS achieves anyway

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = O(M^{-r+1/2}),$$

and, in specific cases, exponentially decaying rate.

(see the works of, e.g., [Gunturk, Lammers, Powell, Saab, Yilmaz])

but, more complex implementation (memory)

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$

- 2. For a non-consistent reconstruction method:

Given $\frac{1}{\sqrt{m}}\boldsymbol{\Phi}$ RIP over Σ_{4K}, and $\xi_i \sim_{\rm iid} \mathcal{U}([0,\delta])$,
for all $\boldsymbol{x} \in \Sigma_K$ observed through $\boldsymbol{y} = \mathsf{A}(\boldsymbol{x}) = \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$,

> The Projected Back Projection estimate
>
> $$\hat{\boldsymbol{x}} = \mathcal{H}_K(\tfrac{1}{m}\boldsymbol{\Phi}^\top \boldsymbol{y}),$$
>
> is such that
>
> $$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \leqslant C_K(1+\delta)M^{-1/2},$$
>
> with very high probability.

Really simple (almost dumb) reconstruction

This extends to other low-complexity sets and it works for any RIP matrix!

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$

- 2. For a non-consistent reconstruction method:

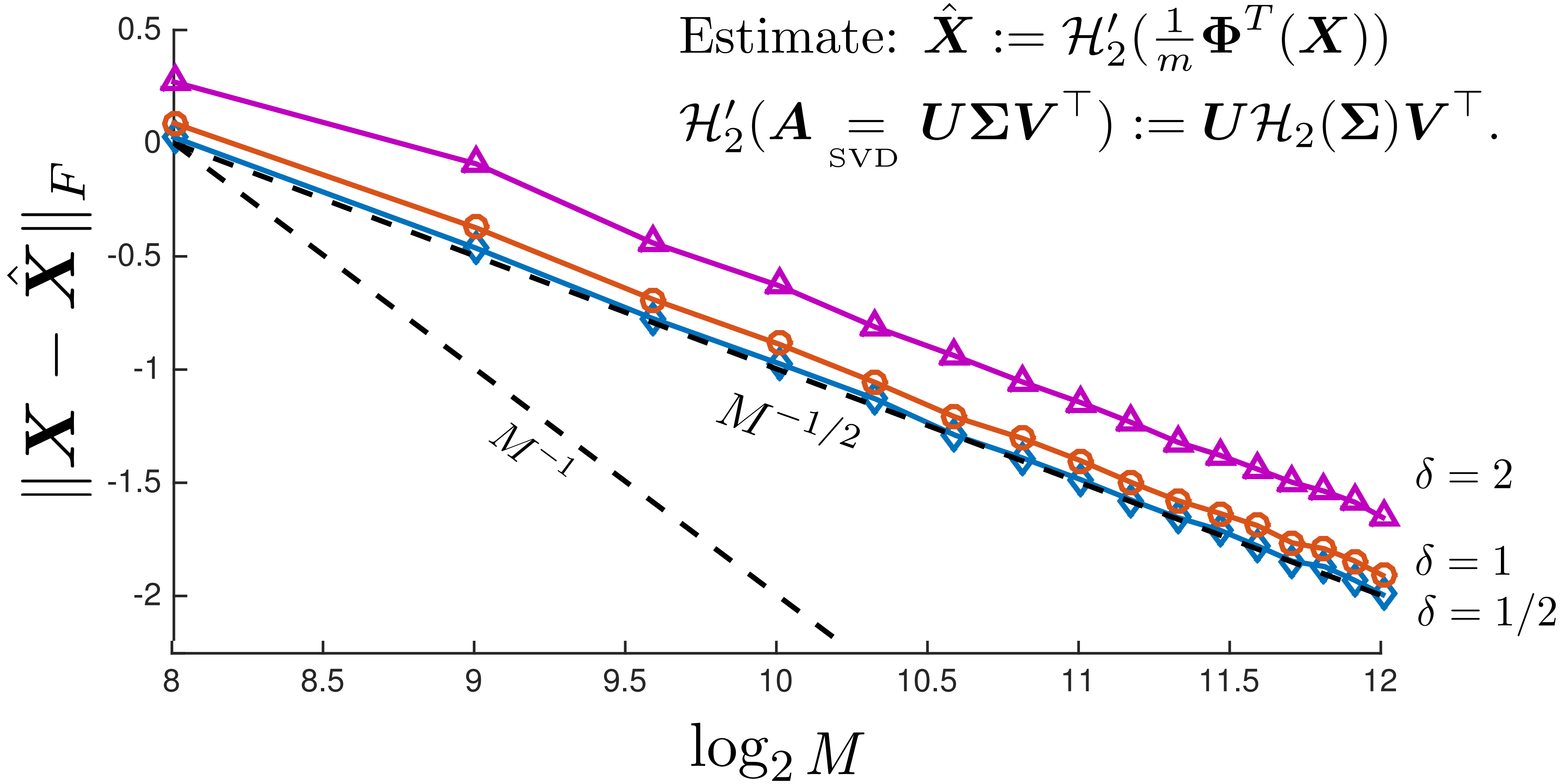
<u>Example:</u>

$\boldsymbol{X}$: Rank-2 64×64-matrix ($n = 64^2$, vectorized)

$\boldsymbol{\Phi}$: Gaussian random matrix

Estimate: $\hat{\boldsymbol{X}} := \mathcal{H}_2'(\frac{1}{m}\boldsymbol{\Phi}^T(\boldsymbol{X}))$

$\mathcal{H}_2'(\boldsymbol{A} \underset{\text{SVD}}{=} \boldsymbol{U\Sigma V}^\top) := \boldsymbol{U}\mathcal{H}_2(\boldsymbol{\Sigma})\boldsymbol{V}^\top.$



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$\|\boldsymbol{u}\|_1 = \sum_j |u_j|$

Sparsity promotion

Level of "noise"

$\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{n}, \ \|\boldsymbol{n}\| \leqslant \epsilon$

Solved by many toolboxes:
SPGL1, Numerical Tours, CVX ...

+ Many other algorithms
(ex. greedy algorithms)

ℓ_1-ball in high dimension

Compressive Sampling and Quantization

Compressed sensing theorist says:

"Linearly sample a signal

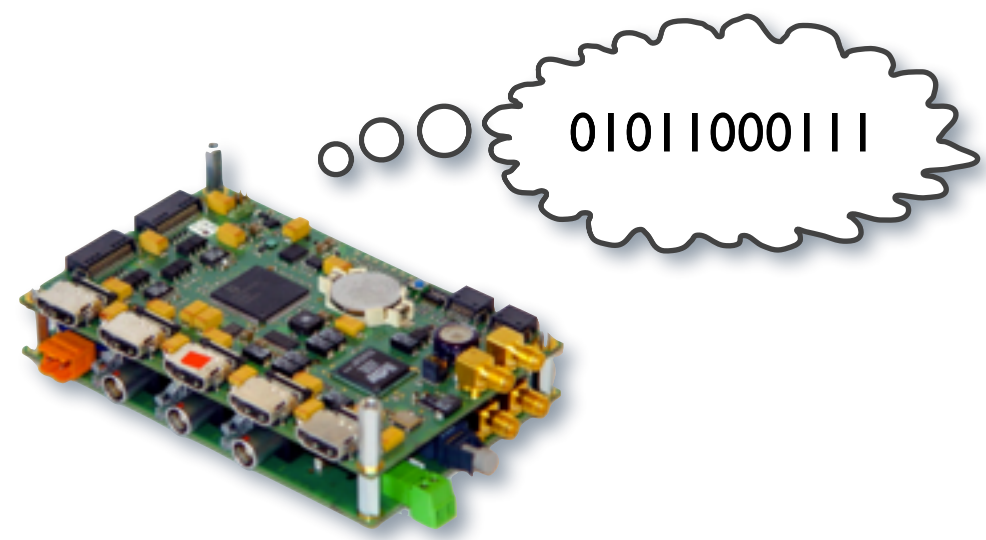
at a rate function of

its intrinsic dimensionality"

Information theorist and sensor designer say:

"Okay, but I need to quantize/digitize my measurements!"

(e.g., in ADC)



Integration?

QCS theory?

Theoretical Bounds

The Rare Eclipse Problem (Linear case)

Adapted* from **[Bandeira, Mixon, Recht 14]**

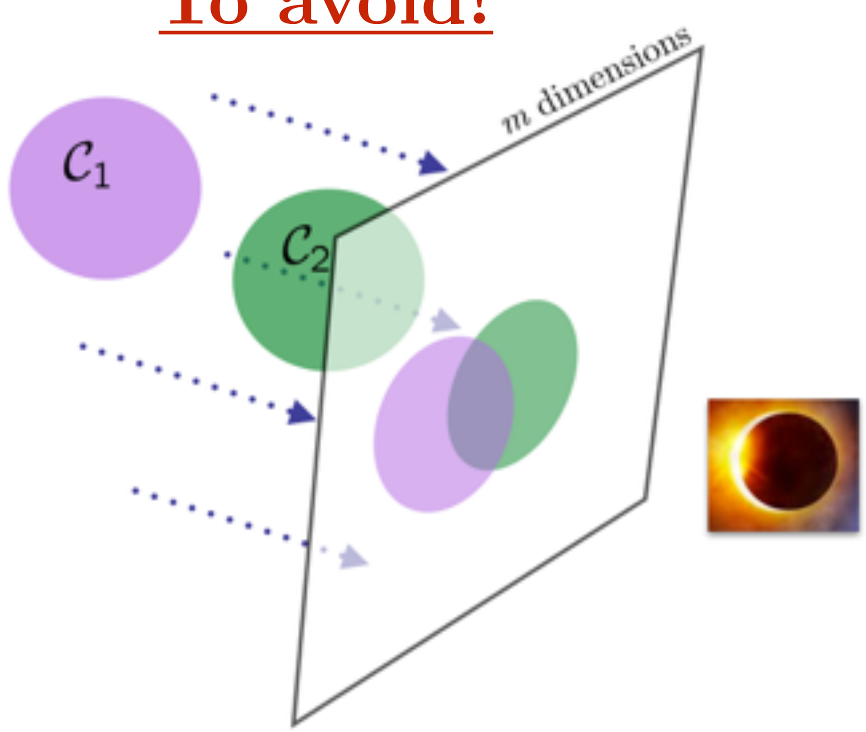
Given $0 < \eta < 1$ and $\mathbf{\Phi} \in \mathbb{R}^{m \times n}$ a random Gaussian matrix, if

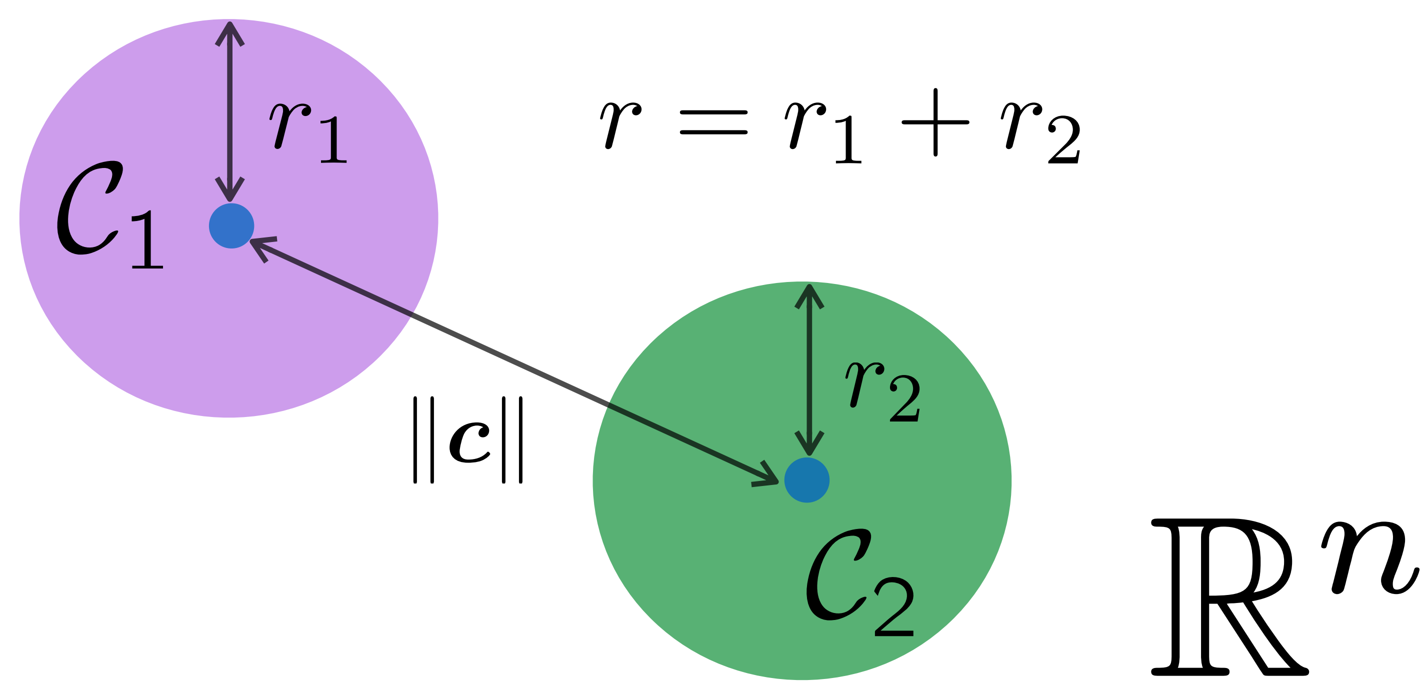
$$m \gtrsim \frac{r^2}{\|\boldsymbol{c}\|^2} n + \log \frac{1}{\eta},$$

then

$$\mathbb{P}\big[\, \mathbf{\Phi}\mathcal{C}_1 \cap \mathbf{\Phi}\mathcal{C}_2 \;=\; \emptyset \,\big] \geqslant 1 - \eta.$$

To avoid!





*: what is presented here is a special case.