Time for dithering! Quantized random embeddings with RIP random matrices

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Outline

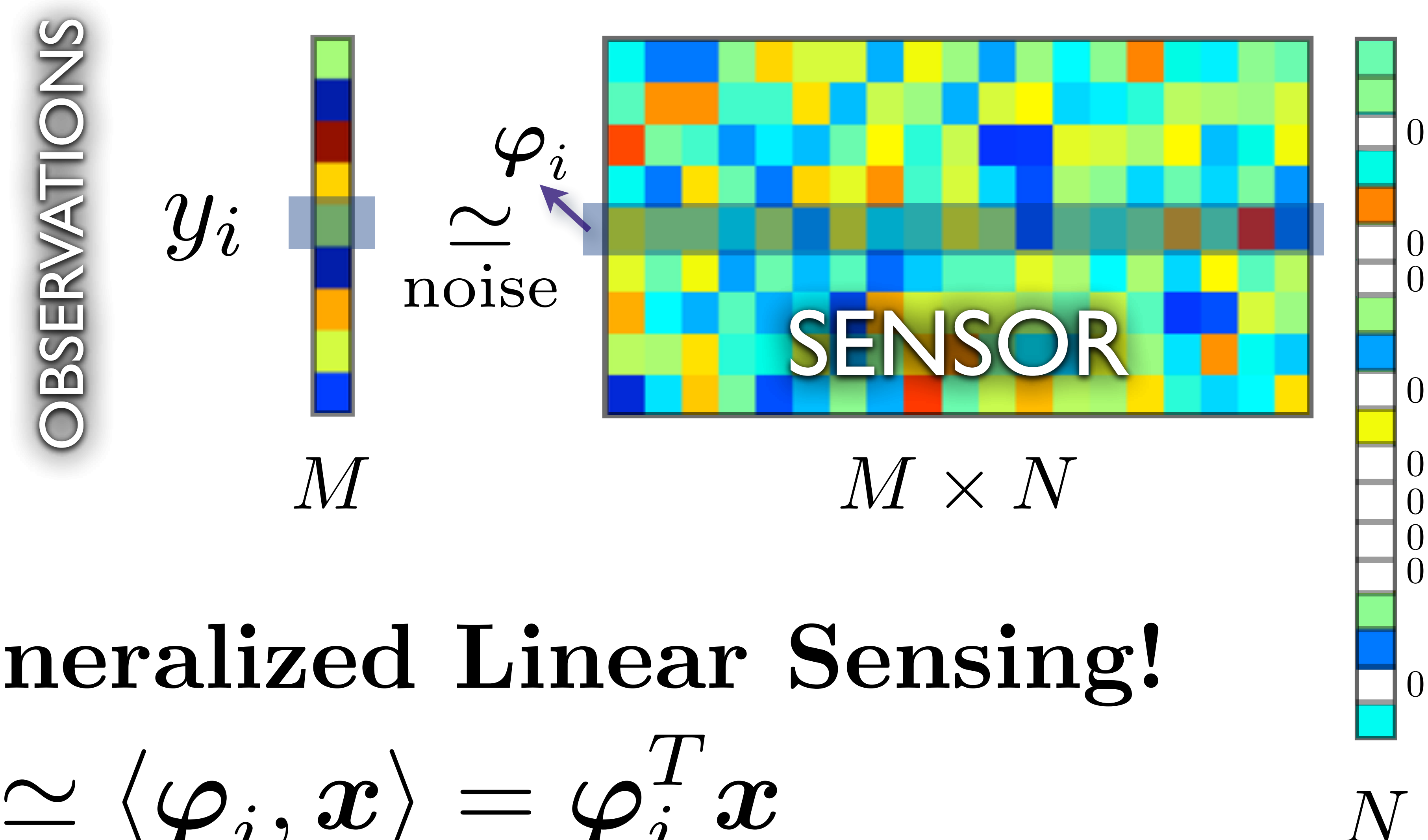
- 1. Brief introduction to
  - compressive sensing (CS)
  - & quantized CS (QCS)
- 2. Quantized dithered random mapping
  - Dimensionality reduction
  - Recovering low-complexity vectors in QCS with any RIP matrix
  - Classification in a quantized world
- Conclusion

1. Brief introduction to CS & QCS

Compressive sensing...

M questions | Sensing method | Signal

$\boldsymbol{y}$ $\quad \boldsymbol{\Phi}$ $\quad \boldsymbol{x}$



Sparsity Prior ($\Psi = \mathrm{Id}$)

A signal in this discrete world

Generalized Linear Sensing!

$$y_i \simeq \langle \boldsymbol{\varphi}_i, \boldsymbol{x} \rangle = \boldsymbol{\varphi}_i^T \boldsymbol{x}$$

$$1 \leq i \leq M$$

e.g., to be realized optically/analogically

Compressive sensing...

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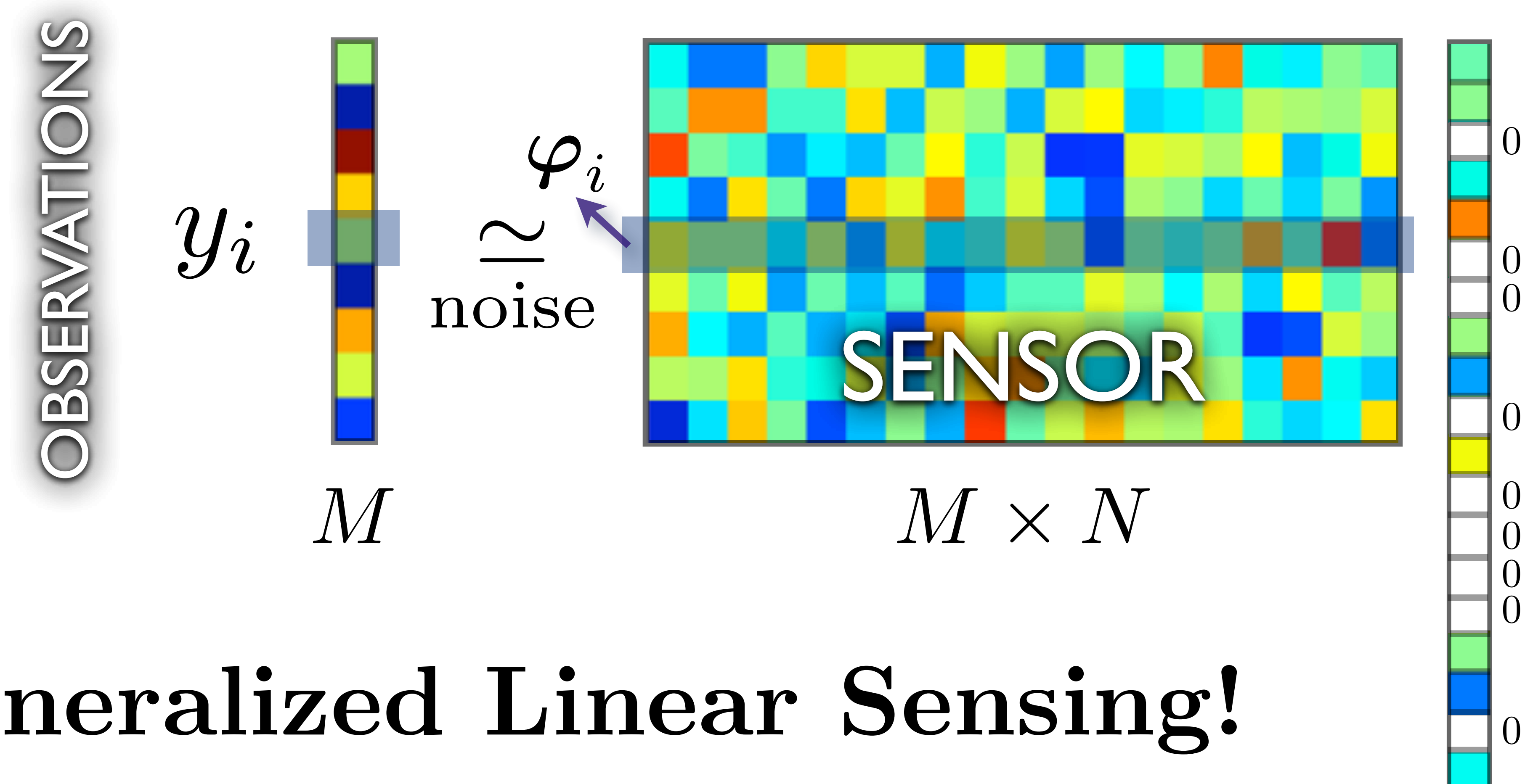

Sparsity Prior ($\Psi = \mathrm{Id}$)

N

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$$1 \leq i \leq M$$

e.g., to be realized optically/analogically

Identifiability of $\boldsymbol{x}$ from $\boldsymbol{\Phi x}$?

Compressive sensing of sparse signals

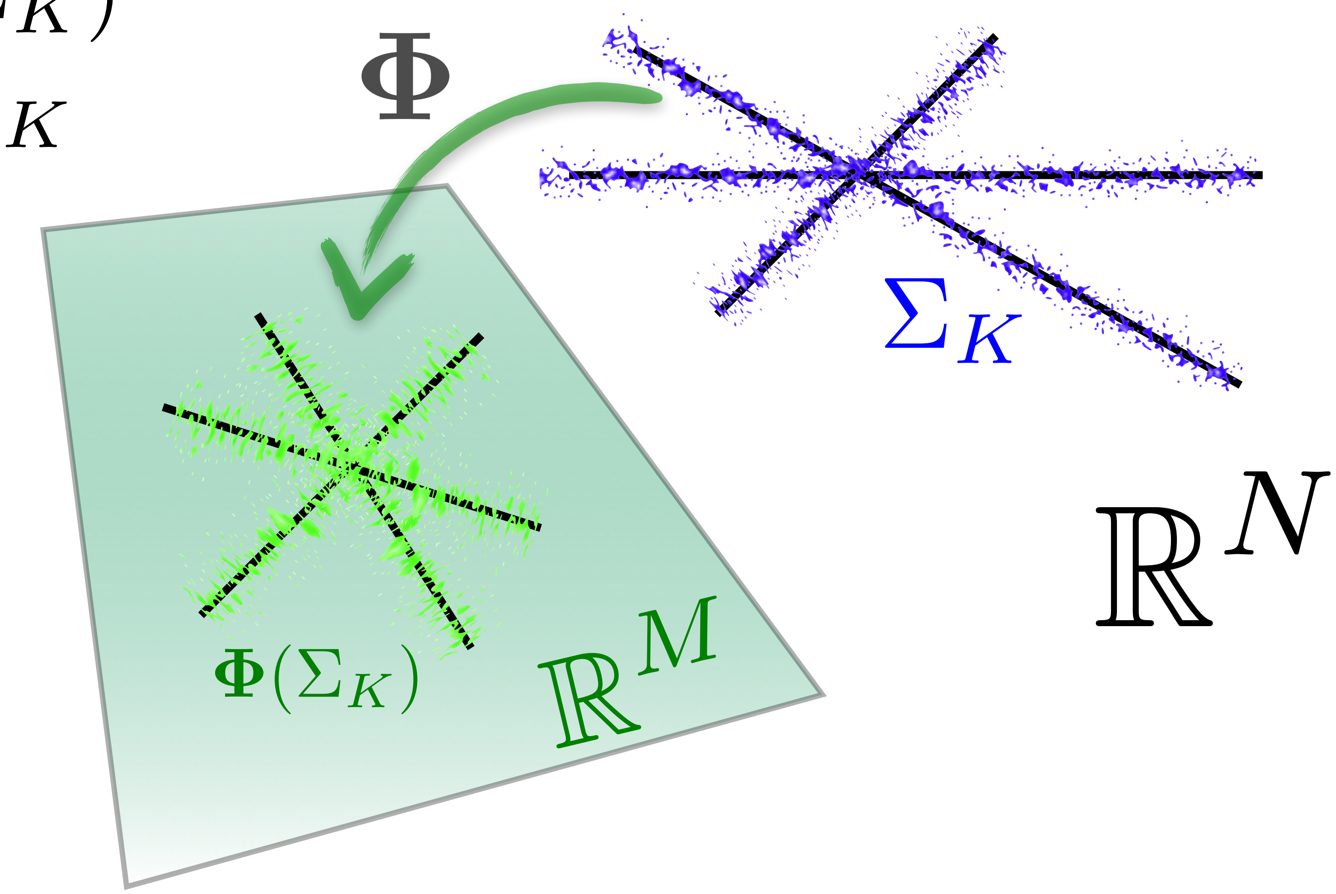
Two K-sparse signals $\boldsymbol{x}, \boldsymbol{x}' \in \Sigma_K := \{\boldsymbol{u} : \|\boldsymbol{u}\|_0 := |\operatorname{supp} \boldsymbol{u}| \leqslant K\}$

For many random constructions of $\boldsymbol{\Phi}$ (e.g., Gaussian, Bernoulli, structured) and "$M \gtrsim K \log(N/K)$", with high probability,

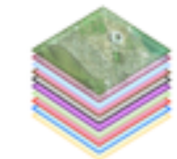
Geometry of $\boldsymbol{\Phi}(\Sigma_K)$
$\approx$ Geometry of Σ_K

$$\boldsymbol{\Phi x} \approx \boldsymbol{\Phi x}' \Leftrightarrow \boldsymbol{x} \approx \boldsymbol{x}'$$

observations — true signals



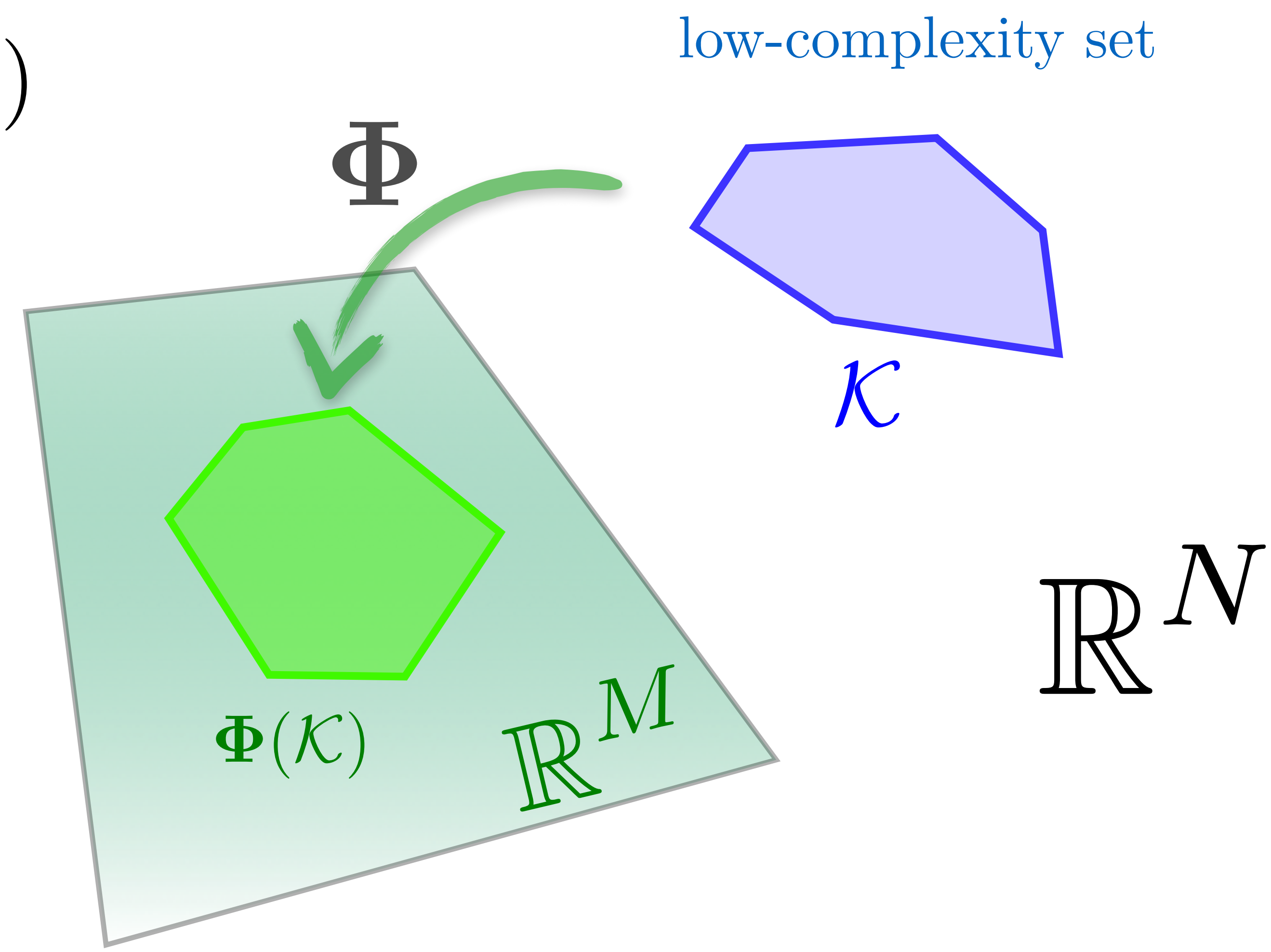
Compressive sensing of *l.c.* signals

Two low-complexity signals $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{K}$ (e.g., low-rank data )

For many random constructions of $\boldsymbol{\Phi}$ (e.g., Gaussian, Bernoulli, structured) and "$M \gtrsim C_{\mathcal{K}}$", with high probability,

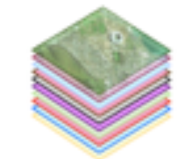
Geometry of $\boldsymbol{\Phi}(\mathcal{K})$
$\approx$ Geometry of $\mathcal{K}$

$$\boldsymbol{\Phi x} \approx \boldsymbol{\Phi x}' \;\Leftrightarrow\; \boldsymbol{x} \approx \boldsymbol{x}'$$



For instance: $C_{\mathcal{K}} = w^2(\mathcal{K})$, the Gaussian mean width, *i.e.*, $C_{\mathcal{K}} \lesssim K \log N/K$ for k-sparse vectors, or $C_{\mathcal{K}} \lesssim rn$ for rank-r $n \times n$ matrices.

Compressive sensing of *l.c.* signals

Two low-complexity signals $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{K}$ (e.g., low-rank data )

For many random constructions of $\boldsymbol{\Phi}$ (e.g., Gaussian, Bernoulli, structured) and "$M \gtrsim C_{\mathcal{K}}$", with high probability,

Restricted **I**sometry **P**roperty (RIP)

For all $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{K}$ and $0 < \rho < 1$,

$$(1-\rho)\|\boldsymbol{x}-\boldsymbol{x}'\|^2 \leq \tfrac{1}{M}\|\boldsymbol{\Phi}\boldsymbol{x}-\boldsymbol{\Phi}\boldsymbol{x}'\|^2 \leq (1+\rho)\|\boldsymbol{x}-\boldsymbol{x}'\|^2$$

Signal reconstruction: ensured with if RIP matrix and with non-linear methods, e.g., Basis Pursuit DeNoise (BPDN), greedy methods (MP, OMP, ...).

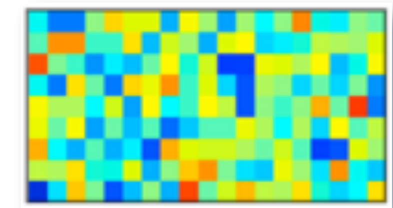
Random sensing matrix market?

Dense & unstructured sensing matrices:

- random sub-Gaussian ensembles (e.g., Gaussian, Bernoulli)

  e.g., Gaussian: $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$, with $\Phi_{ij} \sim_{\rm iid} \mathcal{N}(0,1)$

  or $\Phi_{ij} \sim_{\rm iid} \pm 1$ (eq. prob), $\cdots$

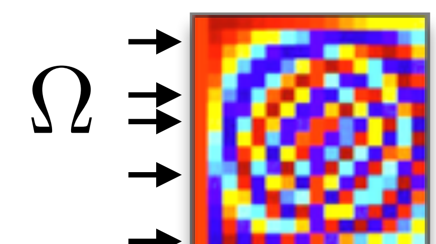


Structured sensing matrices (less memory, fast computations):

- random Fourier/Hadamard ensembles (e.g., for CT, MRI);

  e.g., $\boldsymbol{\Phi} = \boldsymbol{F}_\Omega$, with $\boldsymbol{F} \in \mathbb{C}^{n \times n}$

  and random $\Omega \subset \{1, \cdots, n\}$, $|\Omega| = m$

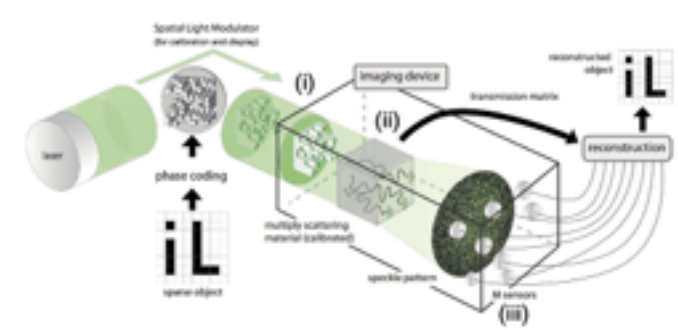


- random convolutions, spread-spectrum (e.g., for imaging), …

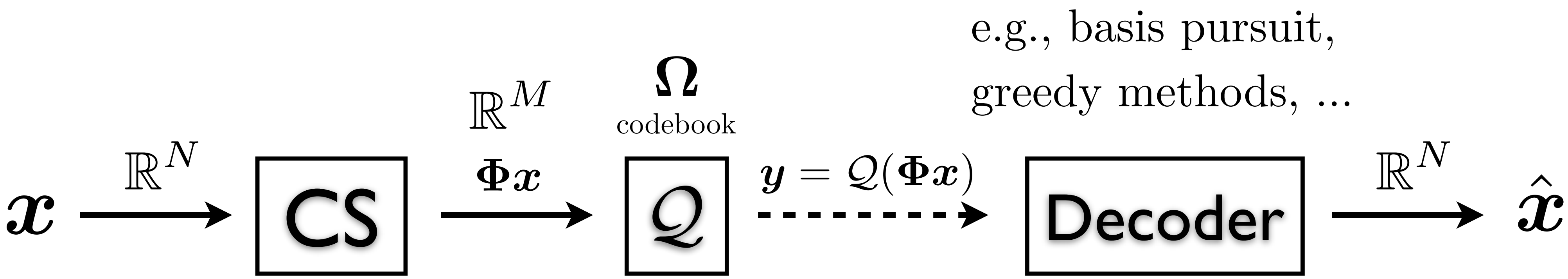
  (see, *e.g.*, **[Foucart, Rauhut, 2013]**)

- or random sensing with natural processes

  (e.g., LightOn!)



Quantizing CS?

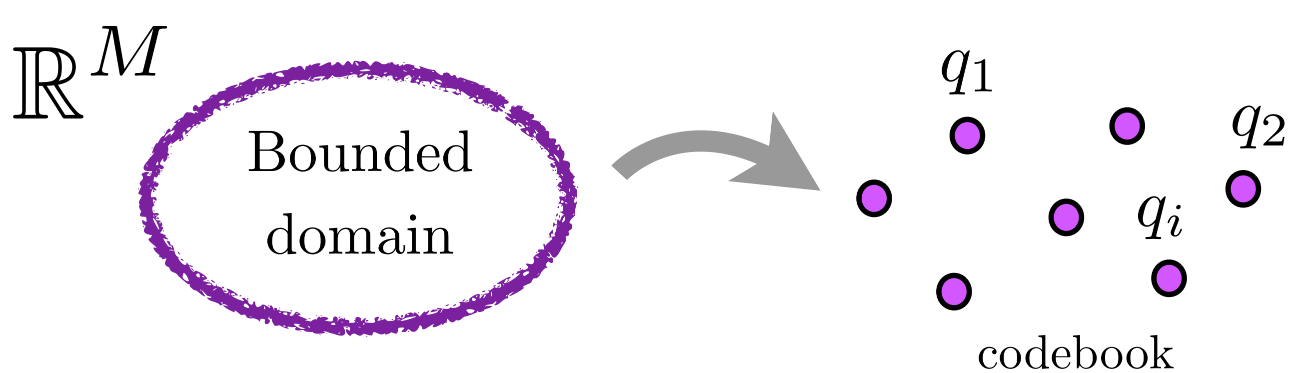


Finite codebook $\Rightarrow \hat{\boldsymbol{x}} \neq \boldsymbol{x}$

i.e., impossibility to encode continuous domain in a finite number of elements.

Objective: Minimize $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|$

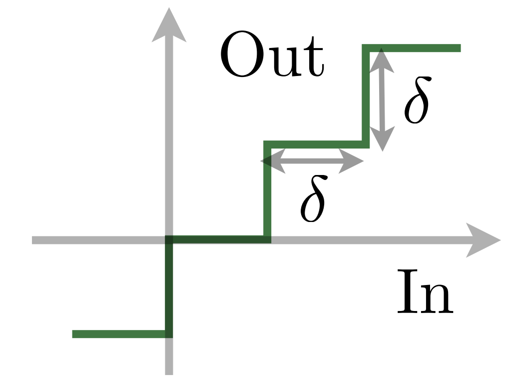
given a certain number of bits, measurements, or bits per meas.



$$\mathcal{Q}[\boldsymbol{u}] \in \{\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots\}$$

Examples of quantization

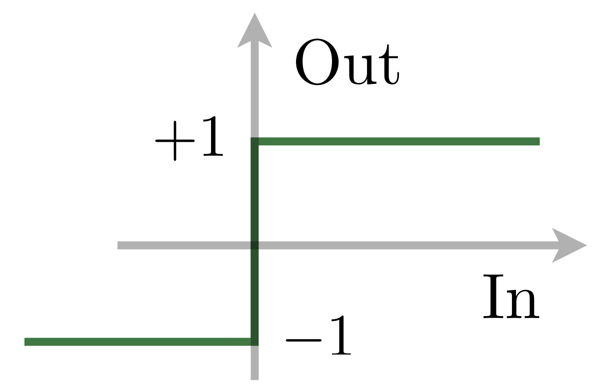
- Simple example: rounding/flooring*



$$\mathcal{Q}[\lambda] = \delta \lfloor \tfrac{\lambda}{\delta} \rfloor \in \delta\mathbb{Z}$$

for some resolution $\delta > 0$ and $\mathcal{Q}(\boldsymbol{u}) = (\mathcal{Q}(u_1), \mathcal{Q}(u_2), \cdots)$.

- Even simpler: 1-bit quantizer



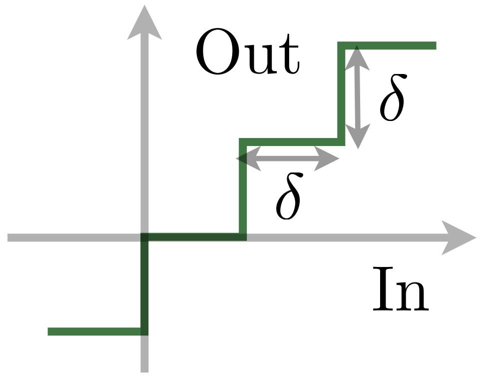
$$\mathcal{Q}[\lambda] = \operatorname{sign} \lambda \in \pm 1$$

(with lost of the global measurement amplitude)

*: Also known as a special case of Pulse Code Modulation - PCM, or Memoryless Scalar Quantization - MSQ

Examples of quantization

- Simple example: rounding/flooring



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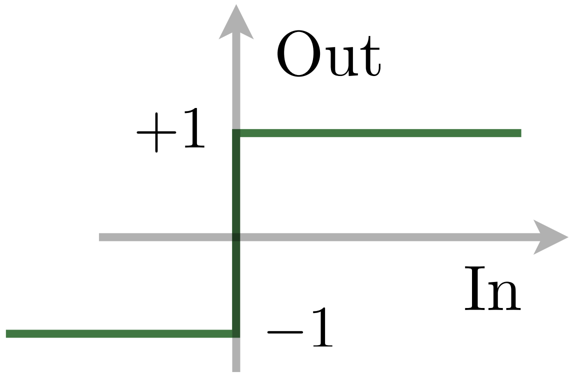
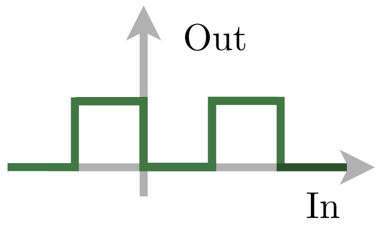


$$\mathcal{Q}[\lambda] = \operatorname{sign} \lambda \in \pm 1$$

<u>Other examples (not covered here):</u>

Non-regular, e.g., square wave (or LSB)

$$\mathcal{Q}[\lambda] := \delta(\lfloor \tfrac{\lambda}{\delta} \rfloor \bmod 2)$$



Non-uniform scalar quantizer, vector quantizer, $\Sigma\Delta$ quantizer/noise shaping, ... (see the works of, e.g., **[Gunturk, Lammers, Powell, Saab, Yilmaz, Goyal]**)

QCS, first attempt [Candès, Tao, 04]

- Quantization is like a noise! (e.g., for $\mathcal{Q}[\lambda] = \delta\lfloor \frac{\lambda}{\delta} \rfloor \in \delta\mathbb{Z}$)

$$\boldsymbol{y} = \mathcal{Q}(\boldsymbol{\Phi x}) = \boldsymbol{\Phi x} + \boldsymbol{n}, \quad \text{with } \boldsymbol{n} = \mathcal{Q}(\boldsymbol{\Phi x}) - \boldsymbol{\Phi x}.$$

$$\text{and } \|\boldsymbol{n}\|^2 = O(m\,\delta^2)$$

- **Problem**: e.g., for BPDN,

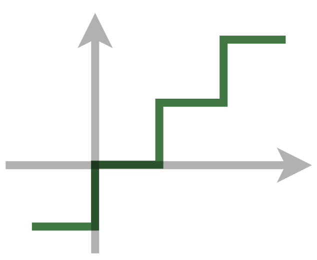
$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \lesssim \tfrac{\epsilon}{\sqrt{m}} = O(\delta) \text{ does not decay if } m \text{ increases!}$$

counterintuitive?

QCS, first attempt [Candès, Tao, 04]

Cause of the problem:

Quantization is discontinuous (it does not "**dither**")



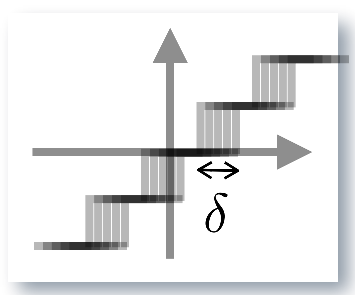
$$\exists\, \boldsymbol{x} \neq \boldsymbol{x}' : \mathcal{Q}(\boldsymbol{\Phi x}) = \mathcal{Q}(\boldsymbol{\Phi x}') \Rightarrow \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| > C$$

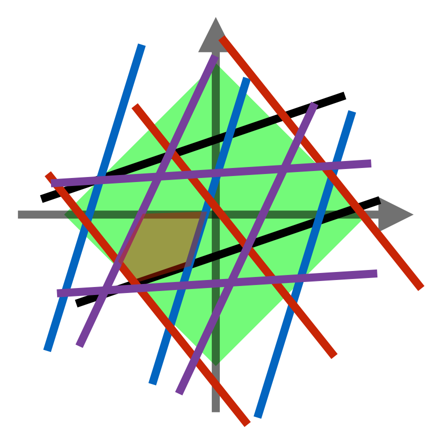
(e.g., take $\Phi_{ij} \in \{\pm 1\}$ and $\mathcal{Q} = \mathrm{sign}$)!

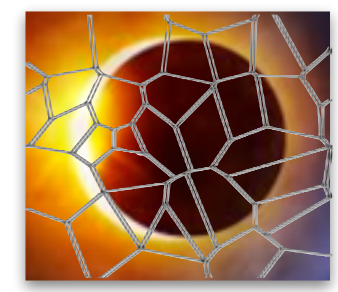
[Plan, Vershynin, 13] [LJ, 17]

Problem with too sparse signals!

2. Quantized dithered random mapping





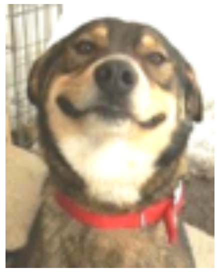


The power of dithering (an old trick, revisited)

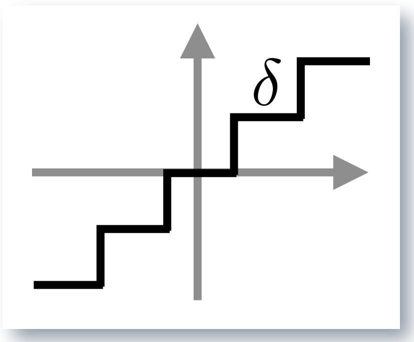
- Inject a pre-quantization, uniform "noise":

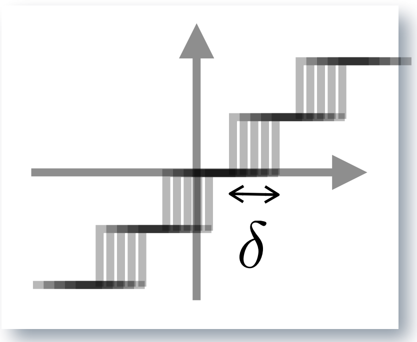
  i.e., a *dithering* $\boldsymbol{\xi} \in \mathbb{R}^m$ with $\xi_j \sim_{\text{iid}} \mathcal{U}([0, \delta])$ (your friend)

The good boy!



$$\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi})$$ (QDRM)





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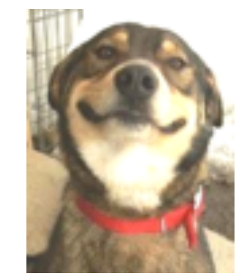
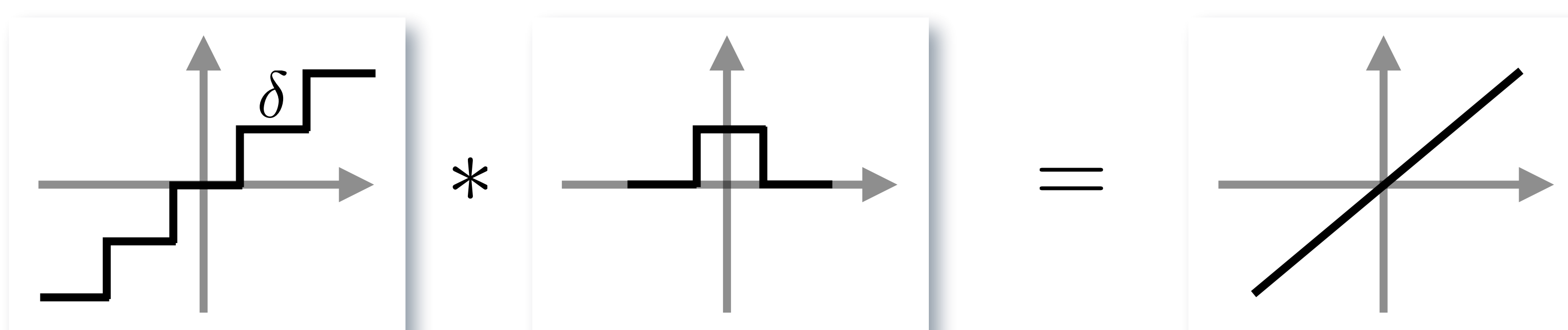
  The good boy!

$$\boxed{\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi})} \quad \text{(QDRM)}$$



- Motivation? $\mathbb{E}_{\boldsymbol{\xi}} \mathcal{Q}(\boldsymbol{u} + \boldsymbol{\xi}) = \boldsymbol{u}$
  $\Rightarrow \mathsf{A}(\boldsymbol{x}) \approx \boldsymbol{\Phi}\boldsymbol{x}$ if M large



The power of dithering (an old trick, revisited)

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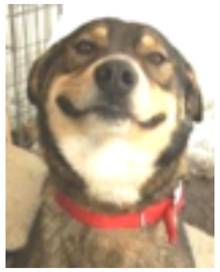
  i.e., a *dithering* $\boldsymbol{\xi} \in \mathbb{R}^m$ with $\xi_j \sim_{\rm iid} \mathcal{U}([0, \delta])$ (your friend)

  → **The good boy!**

$$\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi}) \quad \text{(QDRM)}$$



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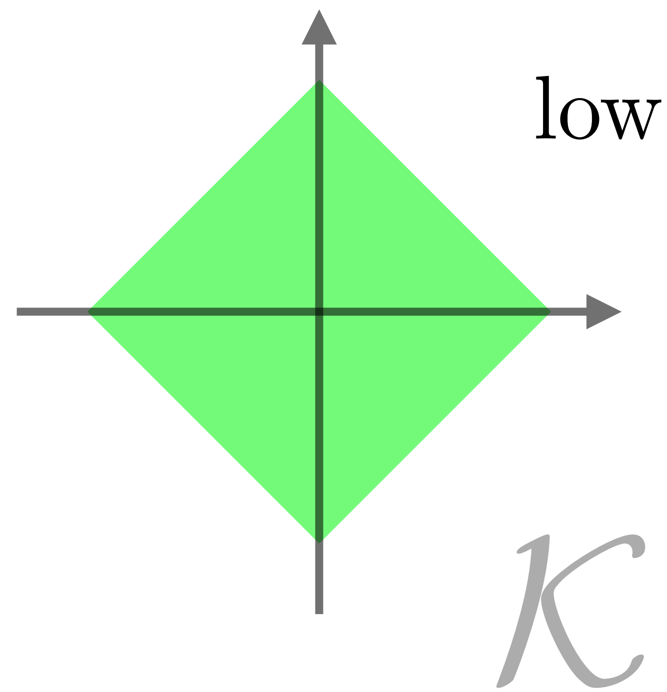
  $\Rightarrow \mathsf{A}(\boldsymbol{x}) \approx \boldsymbol{\Phi}\boldsymbol{x}$ if M large

- Possibility to define quantized dimensionality reduction/embedding!

2.1. Quantized Dimensionality Reduction

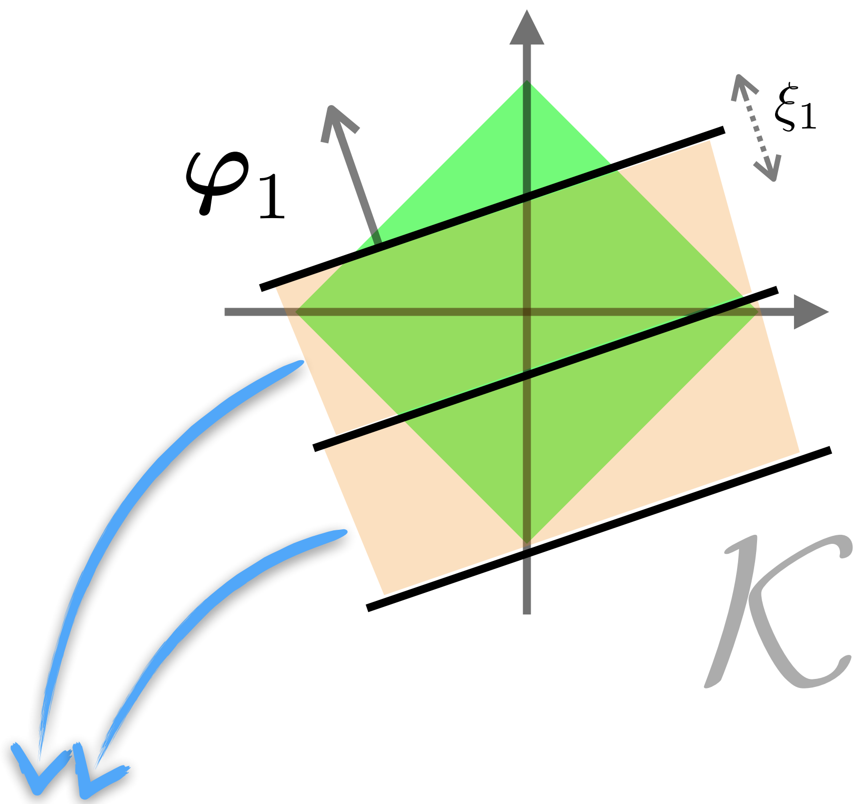
Control of the "consistency width"



low complexity set $\mathcal{K}$

(e.g., sparse signals,
low-rank matrix,
compressible signals, ...)

Control of the "consistency width"

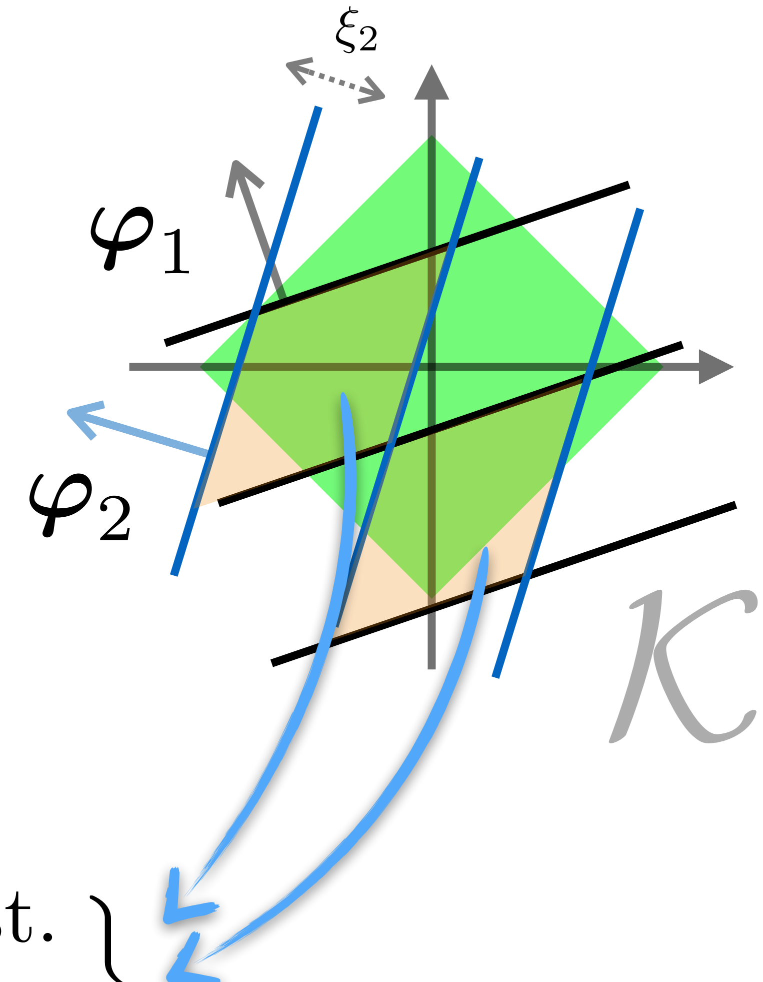


$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\underbrace{\mathcal{Q}(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1)}_{\delta\lfloor(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1)/\delta\rfloor} = \text{cst.}$$

Control of the "consistency width"

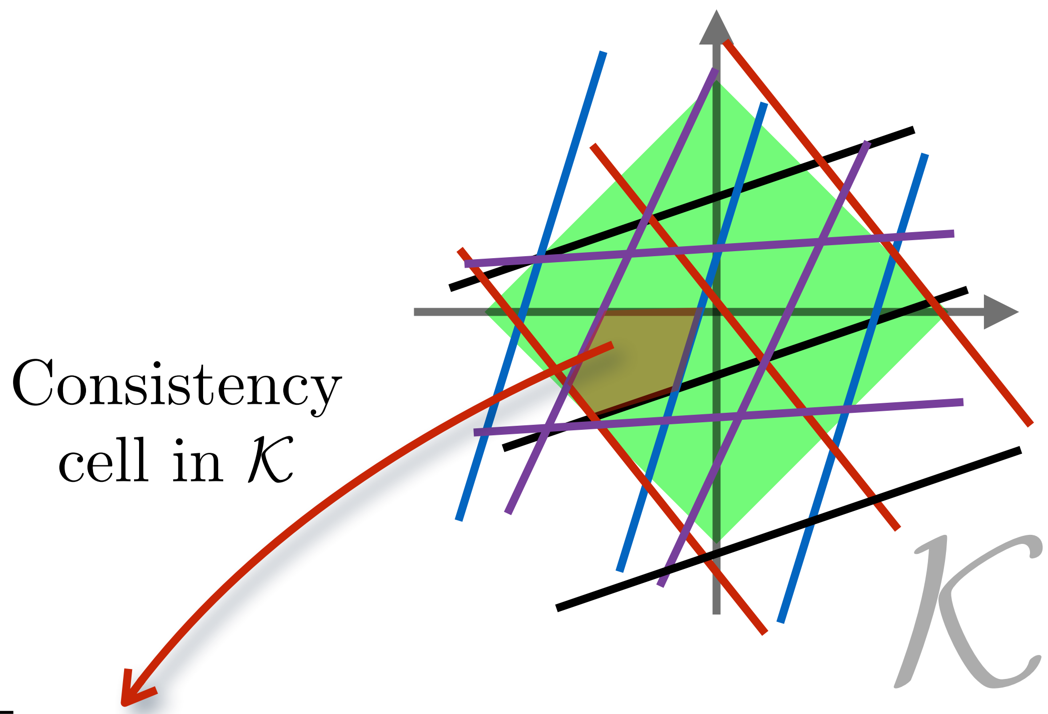


$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\left.\begin{aligned} \mathcal{Q}(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1) &= \text{cst.} \\ \mathcal{Q}(\boldsymbol{\varphi}_2^\top \boldsymbol{u} + \xi_2) &= \text{cst.} \end{aligned}\right\}$$

Control of the "consistency width"



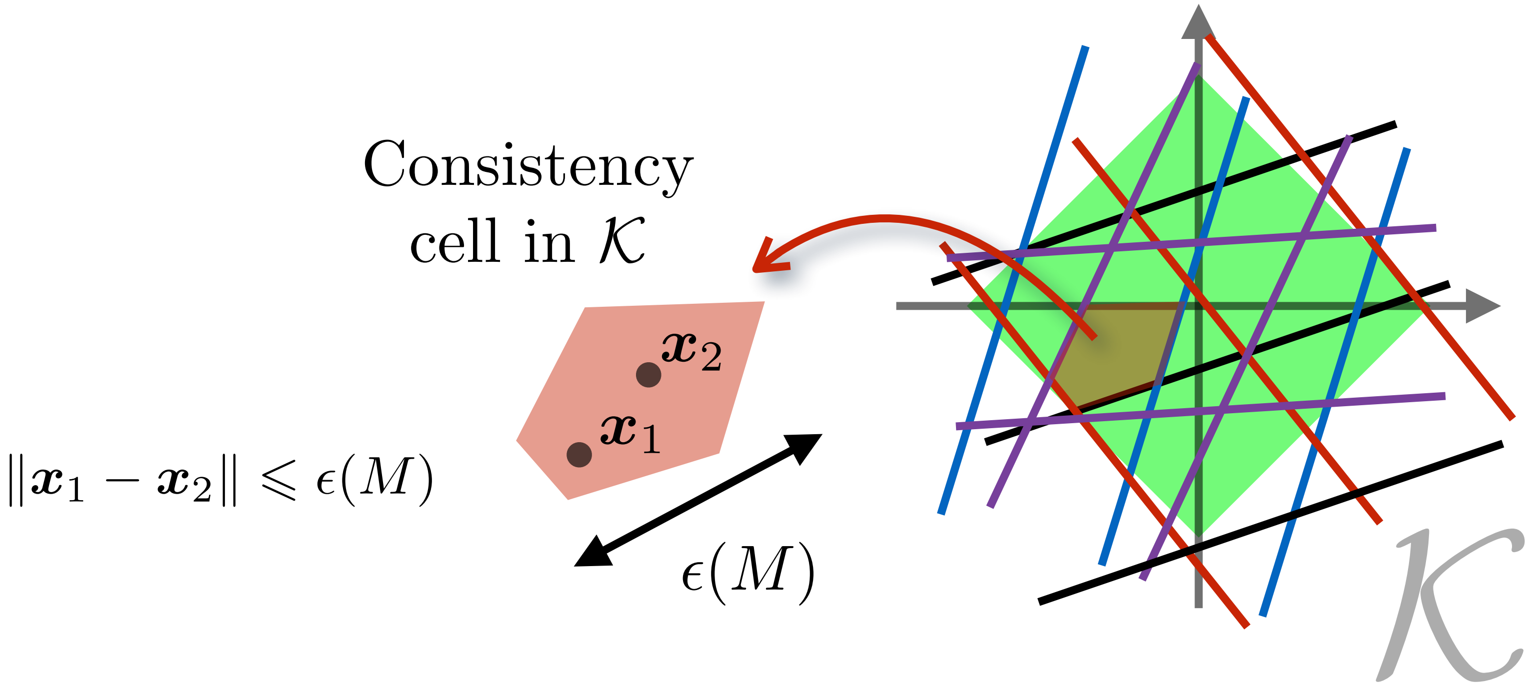
$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\mathsf{A}(\boldsymbol{u}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{u} + \boldsymbol{\xi}) = \boldsymbol{y}$$

for some $\boldsymbol{y} \in \delta\mathbb{Z}^M$

Control of the "consistency width"



$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

For $\boldsymbol{\Phi}$ a random Gaussian matrix, with high probability,

[LJ, 16], [LJ, 17]

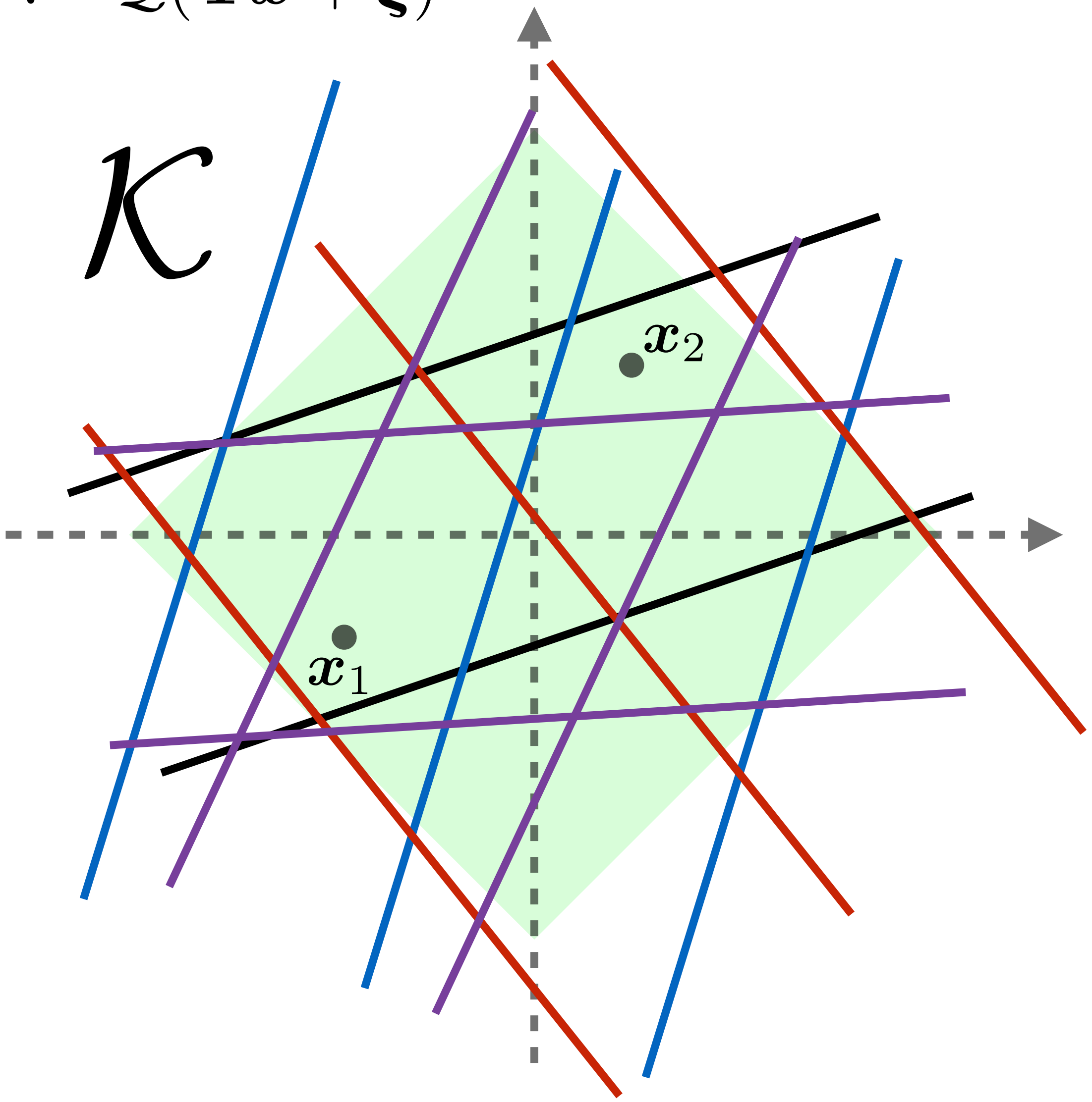
$$\epsilon(M) \leqslant C_{\mathcal{K},\delta}\, M^{-1/q}$$

with $q = 1$ (for, *e.g.*, sparse signals, low-rank matrices), or $q = 4$ for convex sets.

Open problem:
Extension to RIP matrices?

Quantizing the RIP (approximate consistency)

$\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$



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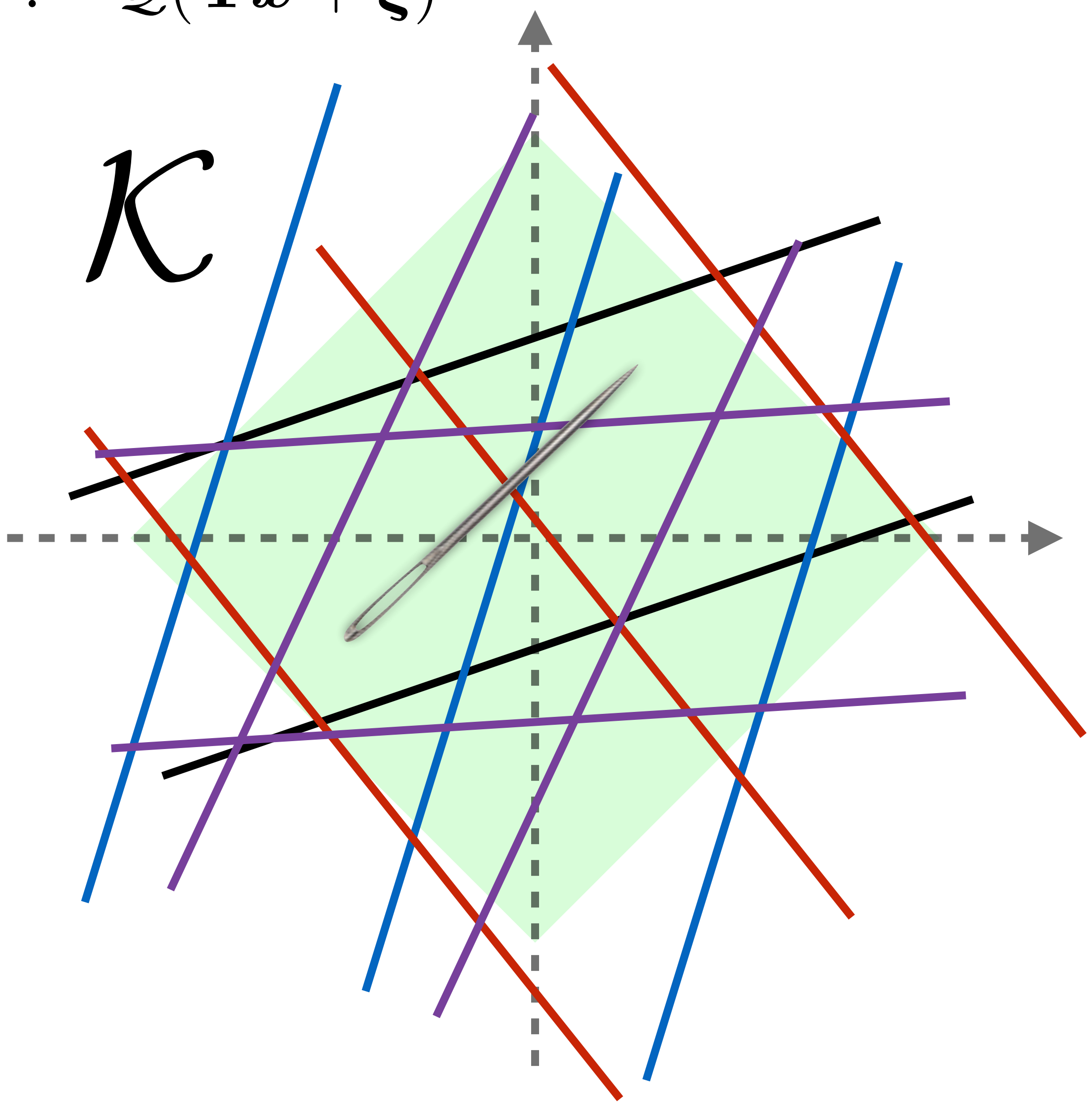
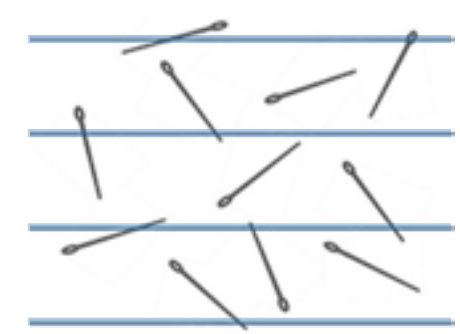


Length of  $= \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$

(thanks to the dithering)

Buffon's needle problem



$\mathbb{E}(\text{intersections}) \propto \text{length}$

http://www.buffon.cnrs.fr

(In 1733)

Quantizing the RIP (approximate consistency)

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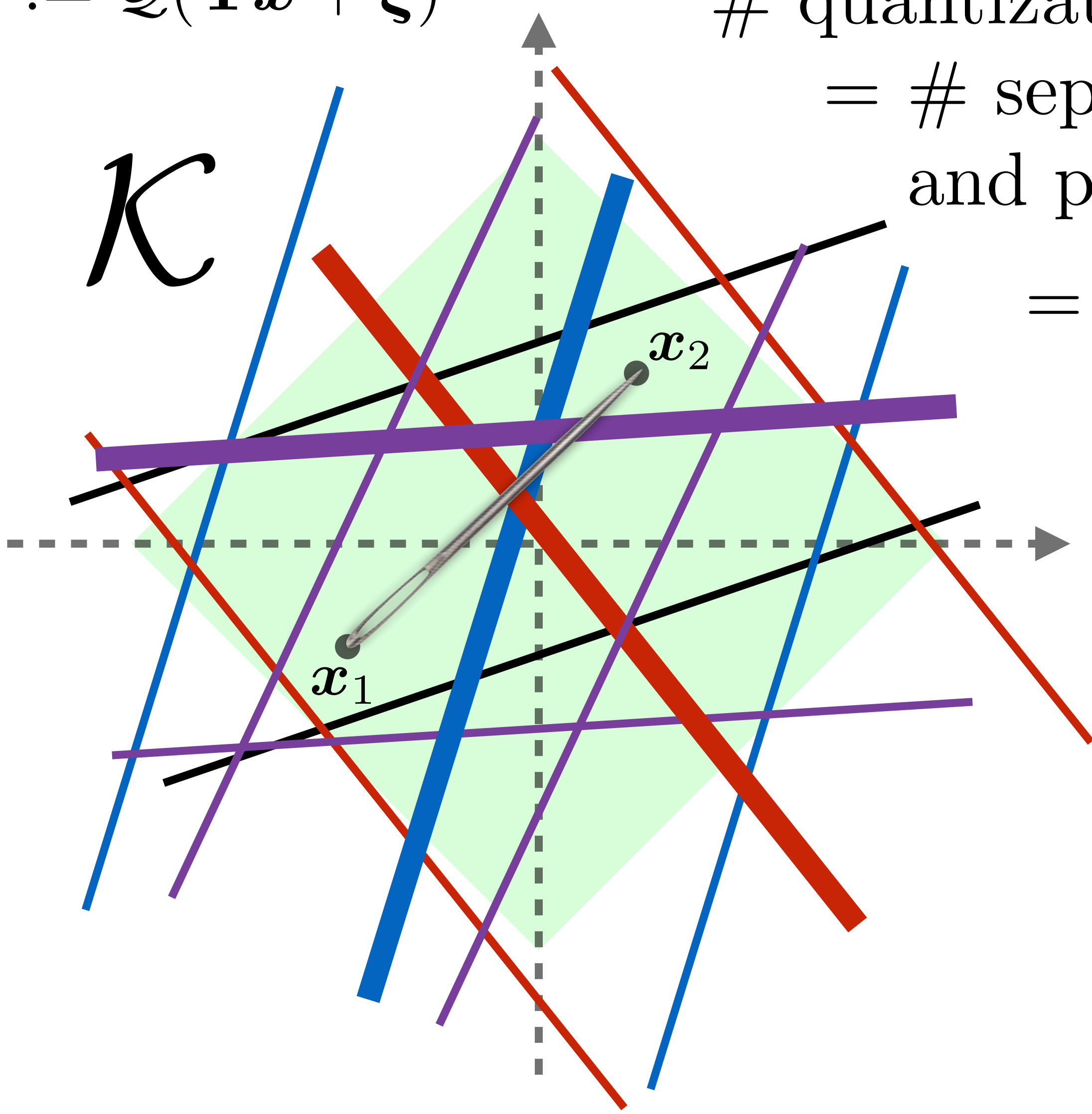
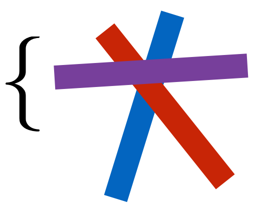


\# quantization frontiers separating $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$
= # separating random hyperplanes oriented and positioned according to $(\boldsymbol{\Phi}, \boldsymbol{\xi})$

$$= \tfrac{1}{M\delta}\|\mathsf{A}(\boldsymbol{x}_1) - \mathsf{A}(\boldsymbol{x}_2)\|_1 \approx \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \;??$$

$|||$

\#{  }

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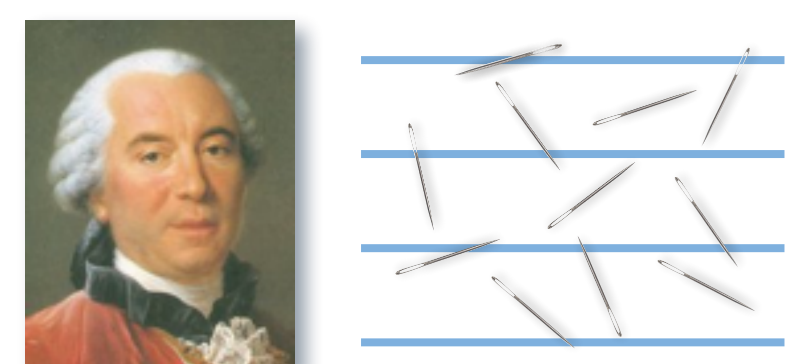

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http://www.buffon.cnrs.fr
(In 1733)

Hope: dithering sufficiently smoothen discontinuities to allow for RIP matrices.

Quantizing the RIP (approximate consistency)

Let $\mathcal{K} \subset \mathbb{R}^N$ be a structured set (*e.g.*, sparse signals, low-rank matrices).

Let $\mathbf{\Phi}$ be a (ℓ_1, ℓ_2)-RIP$(\epsilon, \mathcal{K} - \mathcal{K})$ matrix, *i.e.*,

$$(1-\epsilon)\|\boldsymbol{x}\|^2 \leqslant \tfrac{c_{\mathbf{\Phi}}}{m}\|\mathbf{\Phi}\boldsymbol{x}\|_1^2 \leqslant (1+\epsilon)\|\boldsymbol{x}\|^2, \forall \boldsymbol{x} \in \mathcal{K} - \mathcal{K},$$

(*e.g.*, Gaussian random matrix, circulant Gaussian random matrix for $\mathcal{K} = \Sigma_K$)

[Dirksen, Jung, Rauhut, 17]

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(*e.g.*, Gaussian random matrix, circulant Gaussian random matrix for $\mathcal{K} = \Sigma_K$)
[Dirksen, Jung, Rauhut, 17]

Provided that $M \gtrsim \epsilon^{-2}\, C_{\mathcal{K}} \log(1 + \frac{1}{\delta\epsilon})$, (with $C_{\mathcal{K}} > 0$ an upper bound on $w(\mathcal{K})^2$)
with probability exceeding $1 - C\exp(-\epsilon^2 m)$,

$$(1-\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| - c'\epsilon\delta \leqslant \tfrac{1}{m}\|\mathsf{A}(\boldsymbol{x}_1) - \mathsf{A}(\boldsymbol{x}_2)\|_1 \leqslant (1+\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + c'\epsilon\delta,$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{K} \cap \mathbb{B}^N$.

($\exists$ other variants with ℓ_2/ℓ_2 and standard RIP)

[LJ, Cambareri, 17]

Quantizing the RIP (approximate consistency)

Let $\mathcal{K} \subset \mathbb{R}^N$ be a structured set ($e.g.$, sparse signals, low-rank matrices).

Let $\boldsymbol{\Phi}$ be a (ℓ_1, ℓ_2)-RIP$(\epsilon, \mathcal{K} - \mathcal{K})$ matrix, $i.e.$,

$$(1-\epsilon)\|\boldsymbol{x}\|^2 \leqslant \tfrac{c_{\boldsymbol{\Phi}}}{m}\|\boldsymbol{\Phi x}\|_1^2 \leqslant (1+\epsilon)\|\boldsymbol{x}\|^2, \forall \boldsymbol{x} \in \mathcal{K} - \mathcal{K},$$

($e.g.$, Gaussian random matrix, circulant Gaussian random matrix for $\mathcal{K} = \Sigma_K$)

[Dirksen, Jung, Rauhut, 17]

Provided that $M \gtrsim \epsilon^{-2} C_{\mathcal{K}} \log(1 + \frac{1}{\delta\epsilon})$, (with $C_{\mathcal{K}} > 0$ an upper bound on $w(\mathcal{K})^2$)
with probability exceeding $1 - C\exp(-\epsilon^2 m)$,

$$(1-\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| - c'\epsilon\delta \leqslant \tfrac{1}{m}\|\mathsf{A}(\boldsymbol{x}_1) - \mathsf{A}(\boldsymbol{x}_2)\|_1 \leqslant (1+\epsilon)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + c'\epsilon\delta,$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{K} \cap \mathbb{B}^N$.

($\exists$ other variants with ℓ_2/ℓ_2 and standard RIP)

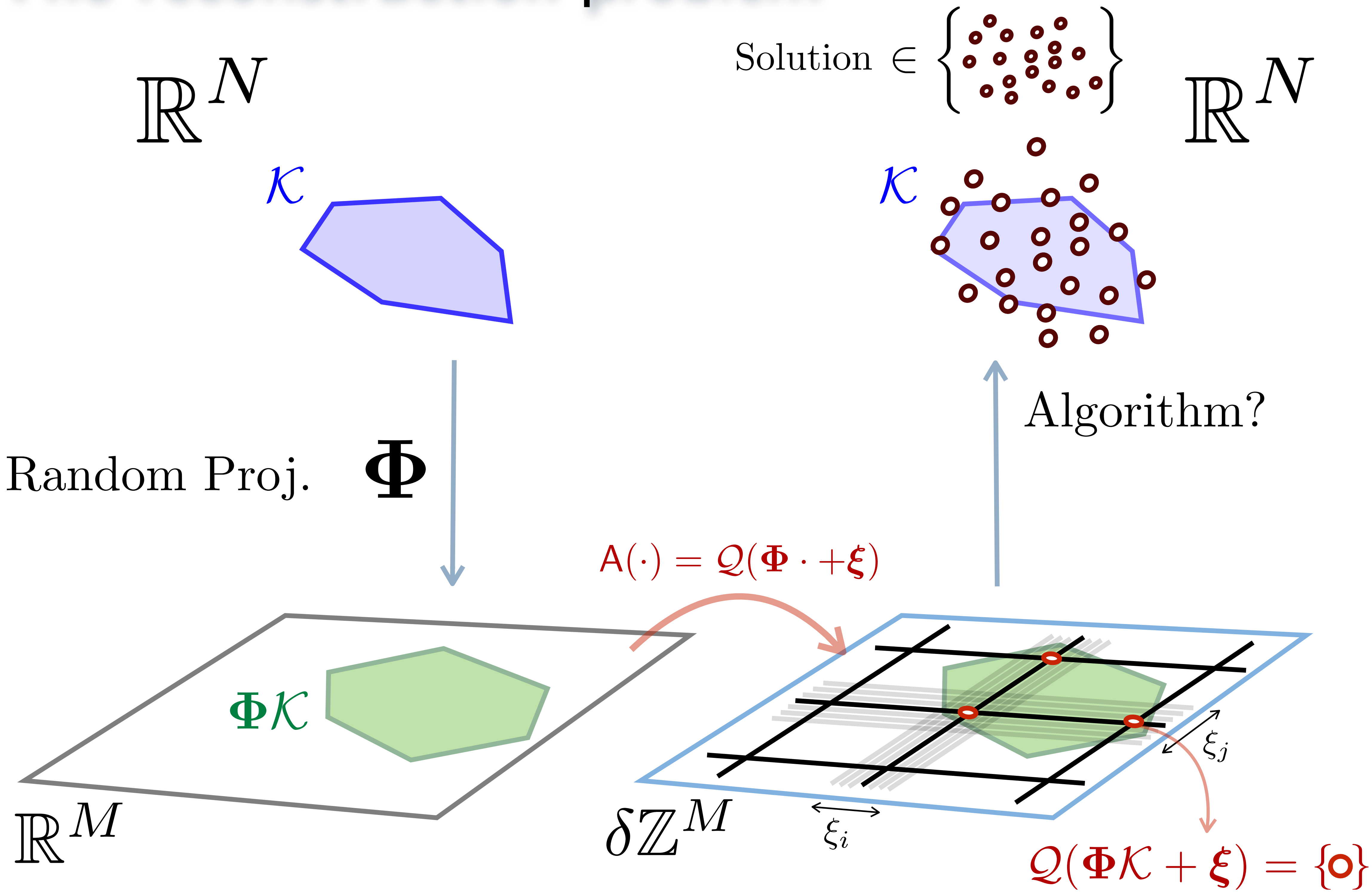
Dimensionality reduction!

Classification?

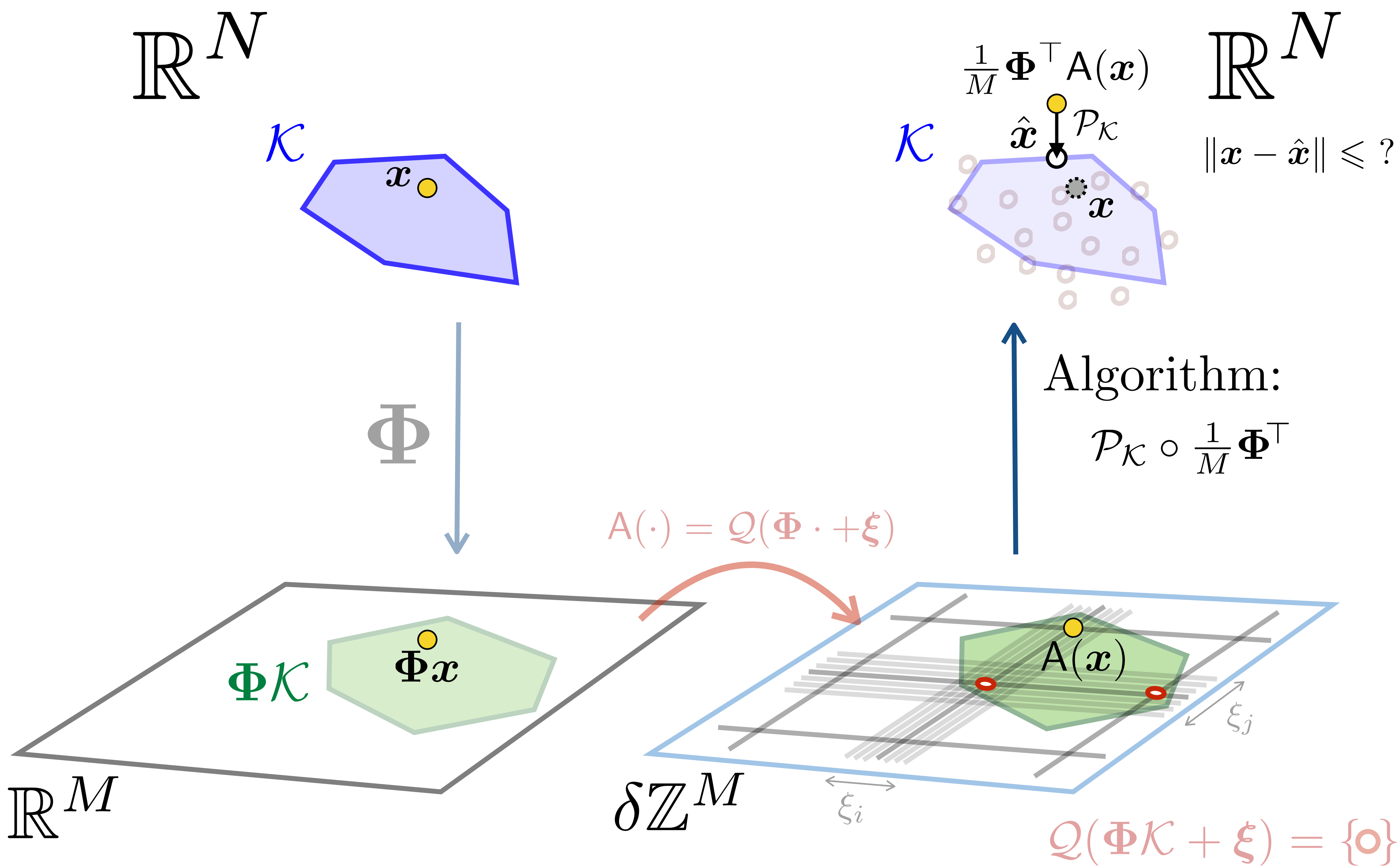
[LJ, Cambareri, 17]

2.2. Recovering low-complexity vectors in QCS with any RIP matrix

The reconstruction problem



Projected Back Projection (PBP)



$$\hat{\boldsymbol{x}} = \mathcal{P}_\mathcal{K}\big(\tfrac{1}{M}\boldsymbol{\Phi}^\top \mathsf{A}(\boldsymbol{x})\big)$$

(1^{st} iteration of other potential methods)

PBP Error Analysis

Limited projection property (LPD):

A respect the LPD$(\mathcal{K}, \boldsymbol{\Phi}, \nu)$ if

$$\frac{1}{M}\big|\langle \mathsf{A}(\boldsymbol{u}), \boldsymbol{\Phi}\boldsymbol{v}\rangle - \langle \boldsymbol{\Phi}\boldsymbol{u}, \boldsymbol{\Phi}\boldsymbol{v}\rangle\big| \leqslant \nu,\ \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{K} \cap \mathbb{B}^N.$$

$\equiv$ How close A is from $\boldsymbol{\Phi}$

If $\frac{1}{\sqrt{M}}\boldsymbol{\Phi}$ is RIP(Σ_{2K}, ϵ) & A is LPD$(\Sigma_{2K}, \boldsymbol{\Phi}, \nu)$, then

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \leqslant 2(\epsilon + \nu).$$

(same result for, e.g., union of low-dimensional spaces, low-rank matrices, and convex sets with square rooted error)

PBP Error Analysis

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If $\frac{1}{\sqrt{M}}\boldsymbol{\Phi}$ is RIP(Σ_{2K}, ϵ) & A is LPD$(\Sigma_{2K}, \boldsymbol{\Phi}, \nu)$, then

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \leqslant 2(\epsilon + \nu).$$

(same result for, e.g., union of low-dimensional spaces, low-rank matrices, and convex sets with square rooted error)

Question: Which matrices do satisfy the LPD? All RIP ones!

If $\boldsymbol{\Phi}$ is RIP(Σ_{2K}, ϵ) and

$$M \gtrsim \underline{\epsilon^{-2} K \log(N/K)} \log(1 + \epsilon^{-3}) \log(1/\zeta),$$

then A respects LPD$(\Sigma_{2k}, \boldsymbol{\Phi}, \epsilon)$ with Pr $\geqslant 1 - \zeta$.

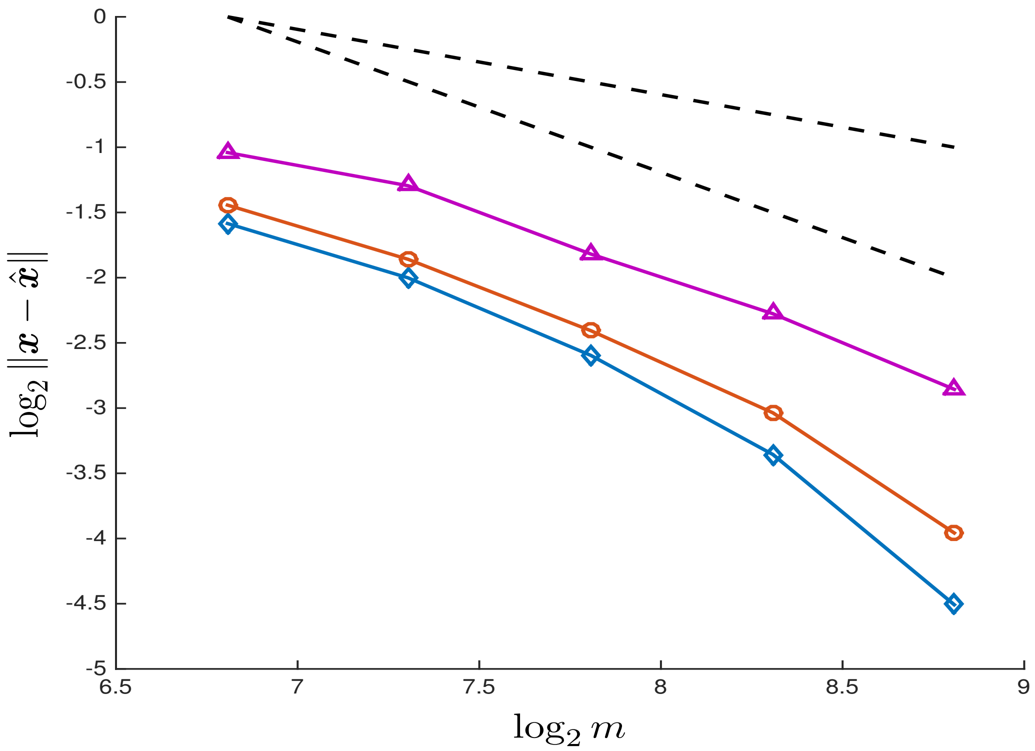
Error: $O(\frac{\sqrt{K}}{\sqrt{M}})$

(extends to union of low-dimensional spaces, low-rank matrices, & convex sets)

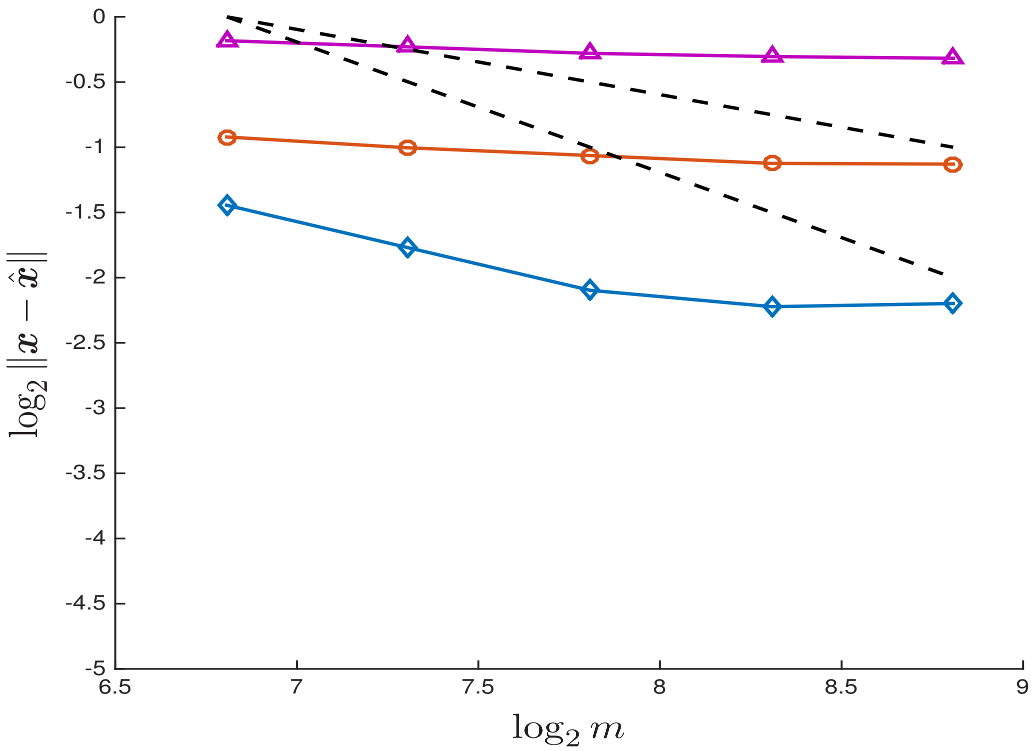
Error bound of PBP: benefit of dithering

Example: PBP reconstruction over $\mathcal{K} = \Sigma_k^n$ with partial DCT from dithered and non-dithered quantized measurements

$$\begin{cases} \boldsymbol{x} \in \mathcal{K} \cap \mathbb{B}^n, \ \ n = 512, k = 4 \\ \boldsymbol{\Phi} \text{ is a random partial DCT, } \xi_i \sim_{iid} U([0,\delta]) \\ \delta = 0.5 \ (\textit{diamond}), \delta = 1 \ (\textit{circle}), \delta = 2 \ (\textit{triangle}) \end{cases}$$



dithered quan. measurements



non-dithered quan. measurements

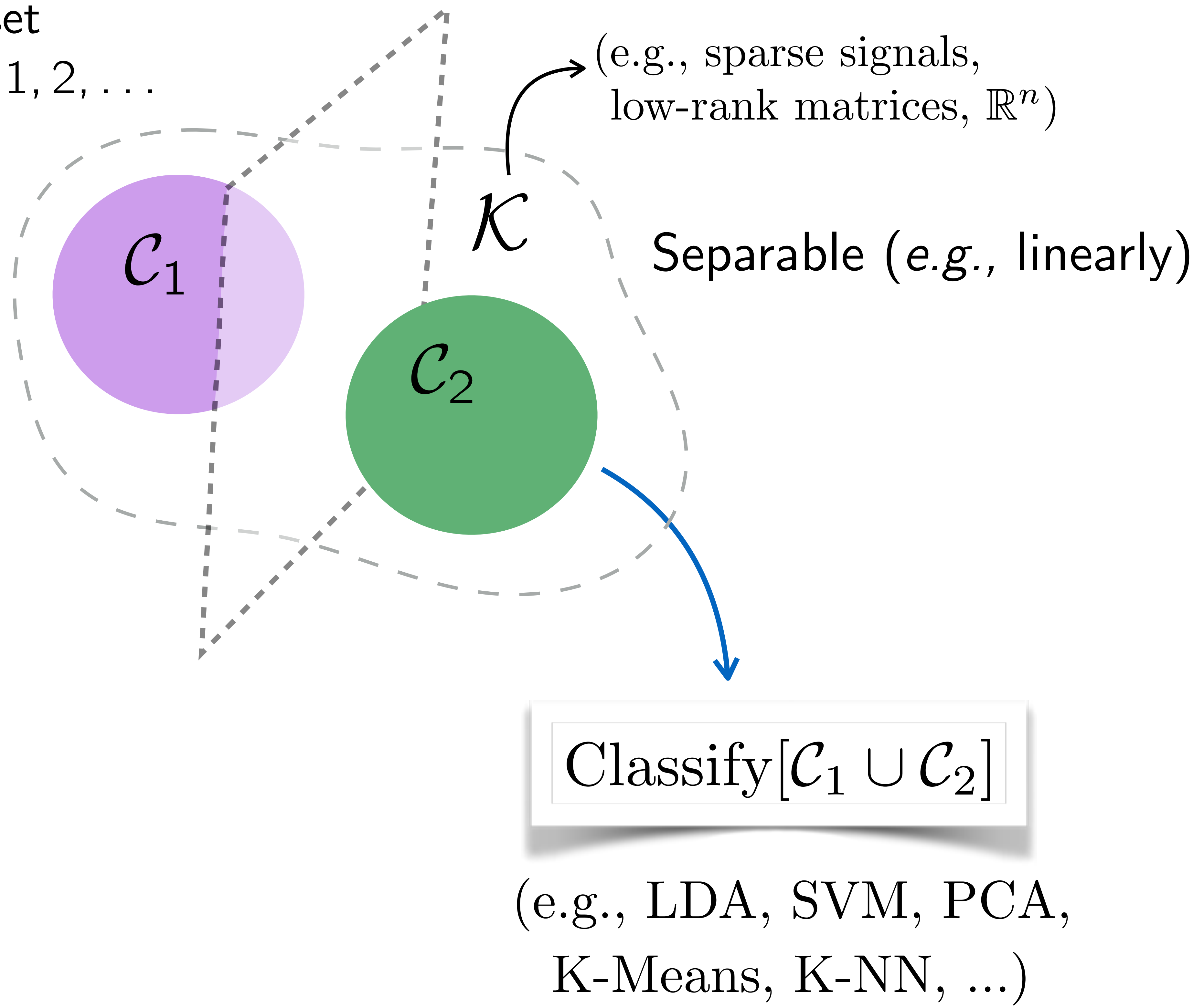
Remark : thanks to dithering, reconstruction error decays as m increases.

2.3. Classification in a quantized world

(caveat: in this part, $M \to m$, $N \to n$)

The Big Picture (an easy classification problem)

$\mathcal{K} \subset \mathbb{R}^n$ dataset
$\mathcal{C}_i \subset \mathcal{K}$ classes, $i = 1, 2, \ldots$



The Big Picture (an easy classification problem)

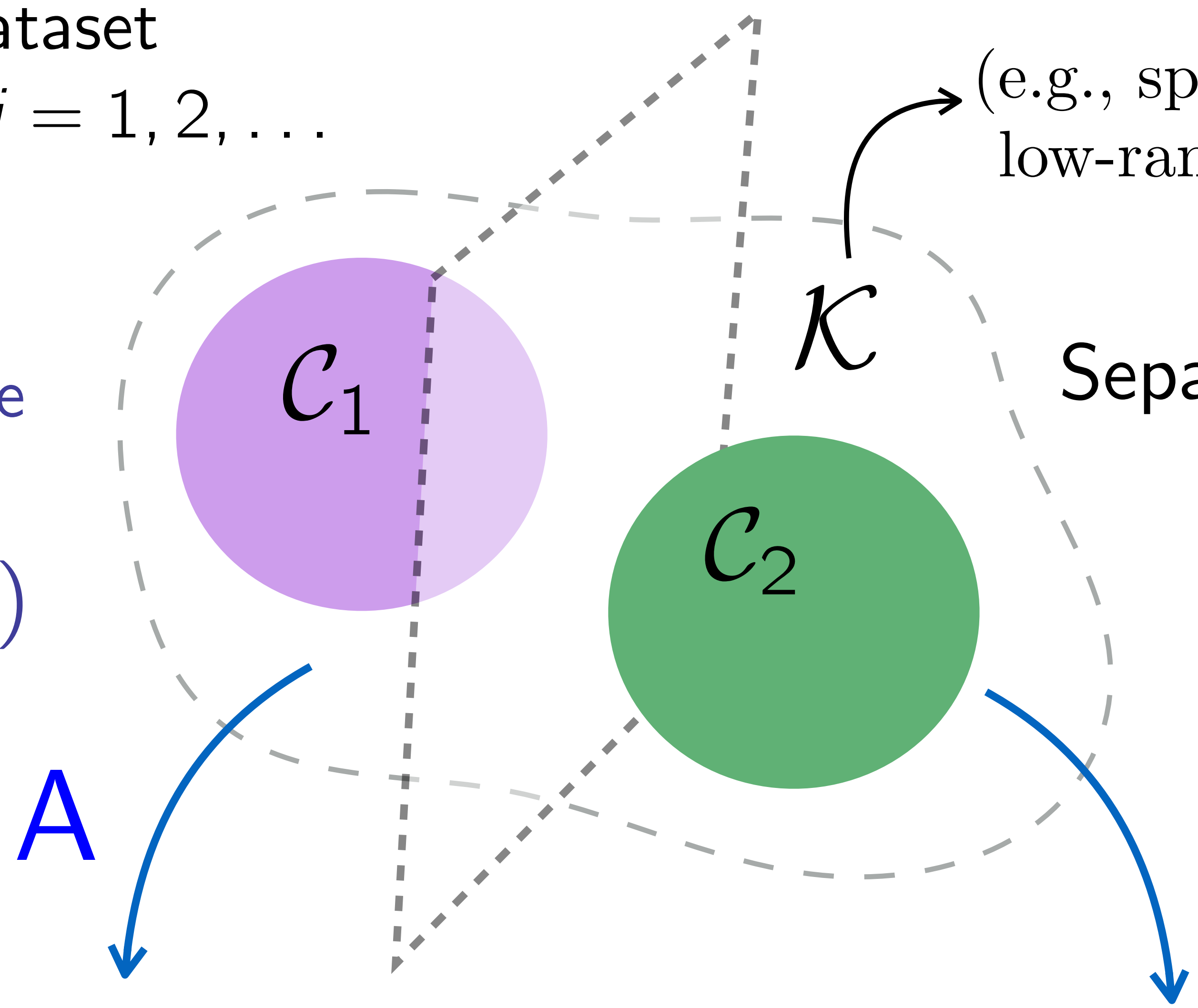
$\mathcal{K} \subset \mathbb{R}^n$ dataset

$\mathcal{C}_i \subset \mathcal{K}$ classes, $i = 1, 2, \ldots$

(e.g., sparse signals, low-rank matrices, $\mathbb{R}^n$)

Non-Adaptive Mapping

$y = \mathrm{A}(x)$



Separable (*e.g.*, linearly)

$\text{Classify}[\mathrm{A}(\mathcal{C}_1) \cup \mathrm{A}(\mathcal{C}_2)]$ $\approx$??? $\text{Classify}[\mathcal{C}_1 \cup \mathcal{C}_2]$

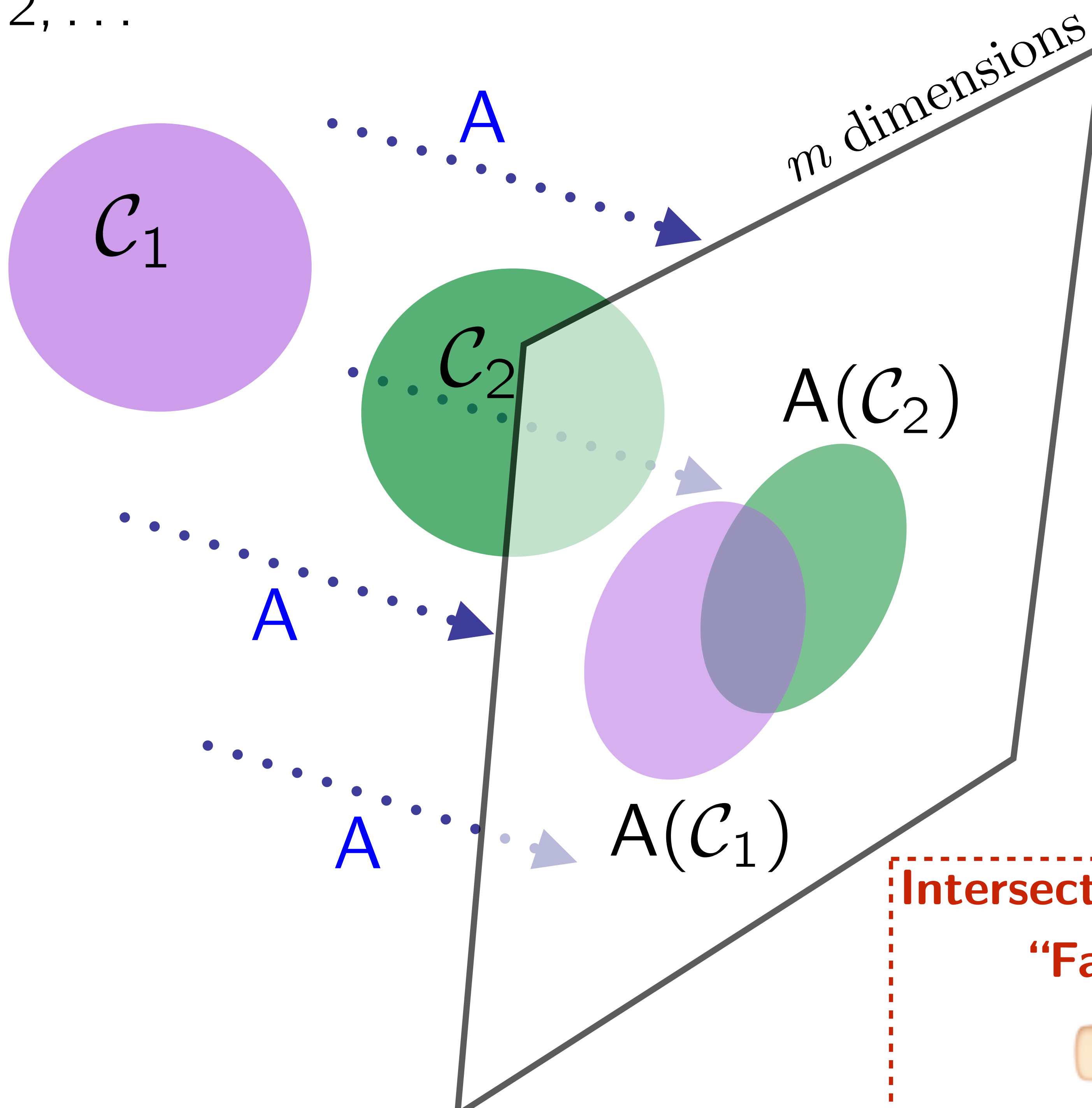
(e.g., LDA, SVM, PCA, K-Means, K-NN, ...)

(e.g., LDA, SVM, PCA, K-Means, K-NN, ...)

The Big Picture

$\mathcal{K} \subset \mathbb{R}^n$ dataset
$\mathcal{C}_i \subset \mathcal{K}$ classes, $i = 1, 2, \ldots$

Non-Adaptive
Mapping
$\boldsymbol{y} = \mathrm{A}(\boldsymbol{x})$



Intersecting/Eclipse
"Failure"

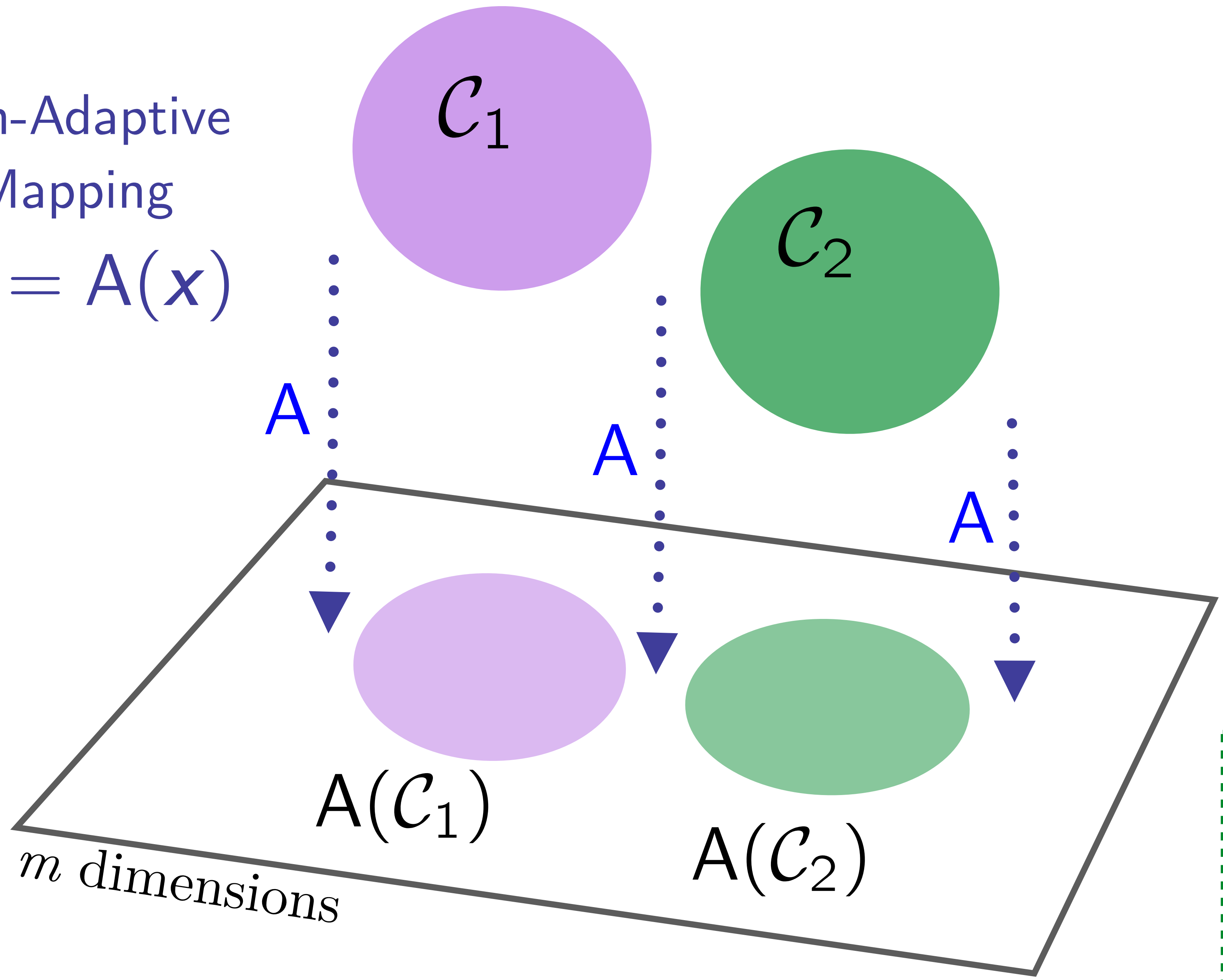
The Big Picture

$\mathcal{K} \subset \mathbb{R}^n$ dataset

$\mathcal{C}_i \subset \mathcal{K}$ classes, $i = 1, 2, \ldots$

Non-Adaptive Mapping

$\boldsymbol{y} = \mathsf{A}(\boldsymbol{x})$



Separable, "Success"

The Rare Eclipse Problem (Linear case)

Problem (Rare Eclipse Problem (Bandeira *et al.* '14)).

Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n : \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ be closed convex sets, $\mathbf{\Phi} \sim \mathcal{N}^{m \times n}(0,1)$. Given $\eta \in (0,1)$, find the smallest m so that

$$p_0 := \mathbb{P}_{\mathbf{\Phi}}[\mathbf{\Phi}\mathcal{C}_1 \cap \mathbf{\Phi}\mathcal{C}_2 = \emptyset] \geq 1 - \eta.$$

Bandeira, Mixon, Recht '14 [BMR '14]

The Rare Eclipse Problem (Linear case)

BMR '14: "*Gordon's escape through a mesh*" theorem

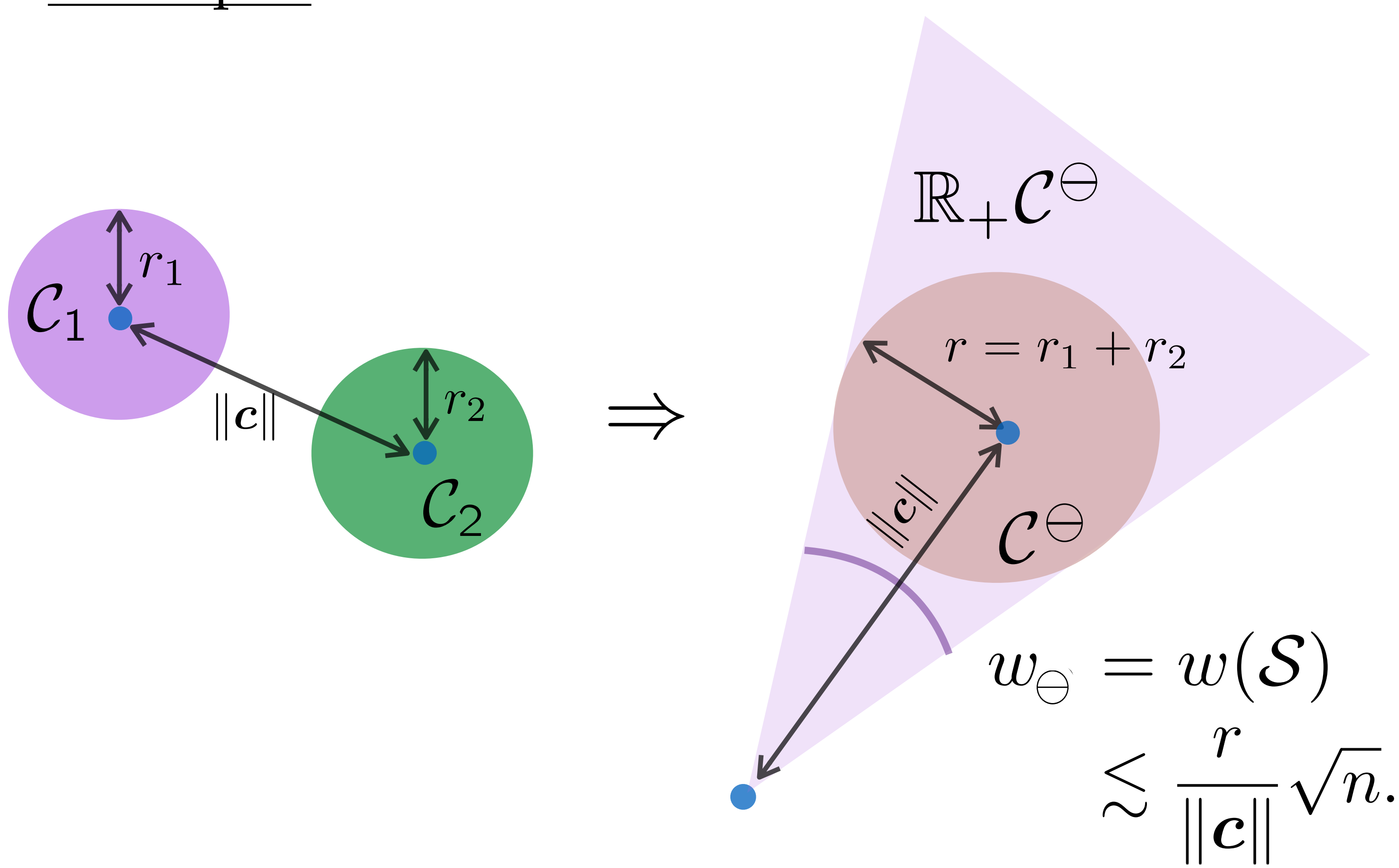
Proposition (Corollary 3.1 in BMR '14).

(& *really* tight [Amelunxen et al, 13])

Given $\eta \in (0,1)$, if $m > \left(w_{\ominus} + \sqrt{2\log\frac{1}{\eta}}\right)^2 + 1$ then $p_0 \geq 1 - \eta$.

Bandeira, Mixon, Recht '14 [BMR '14]

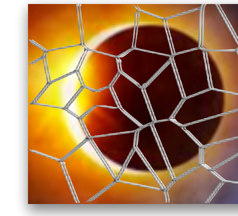
Example:



$\Rightarrow$

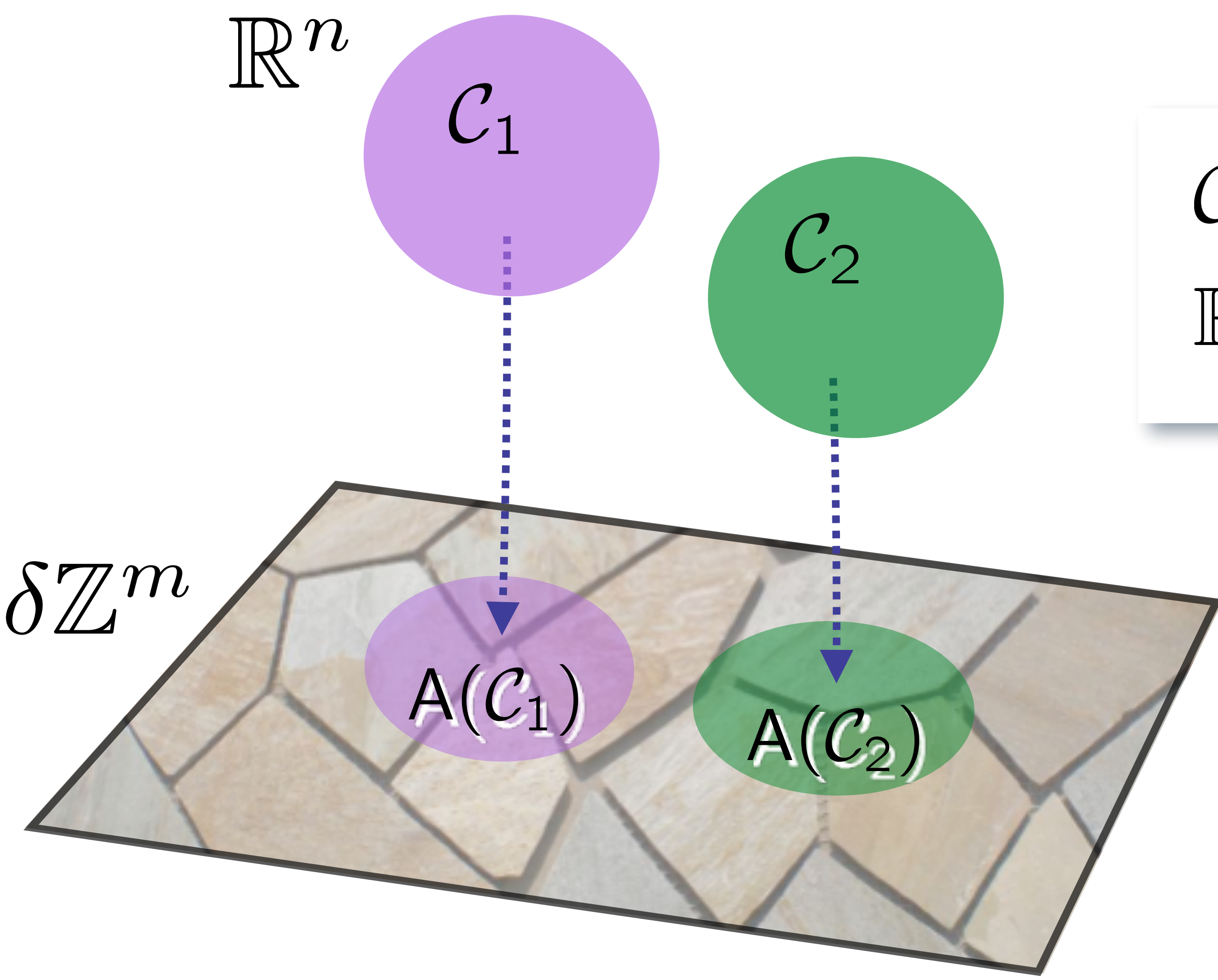
$$m \gtrsim \frac{r^2}{\|\boldsymbol{c}\|^2} n$$

The Rare Eclipse Problem "on Tiles"



$$\boxed{\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})}$$

with $\boldsymbol{\Phi}$ Gaussian random matrix,
$\mathcal{Q}(\lambda) = \delta\lfloor \frac{\lambda}{\delta} \rfloor$, $\xi_i \sim \mathcal{U}([0, \delta])$.

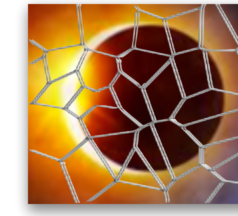


$\mathcal{C}_1$, $\mathcal{C}_2$, m and δ such that
$\mathbb{P}[\mathsf{A}(\mathcal{C}_1) \cap \mathsf{A}(\mathcal{C}_2) = \emptyset] \geqslant 1 - \eta$?

<u>Idea</u>: use the QRIP, i.e.,

$$\tfrac{1}{M\delta}\|\mathsf{A}(\boldsymbol{x}_1) - \mathsf{A}(\boldsymbol{x}_2)\|_1 \approx \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$$

w.h.p.



The Rare Eclipse Problem “on Tiles”

Given $\sigma := \min_{\boldsymbol{z}\in\mathcal{C}^{\ominus}} \|\boldsymbol{z}\|$ and $w_{\cap} = w((\mathbb{R}_+\mathcal{C}^{\ominus}) \cap \mathbb{S}^{n-1})$.

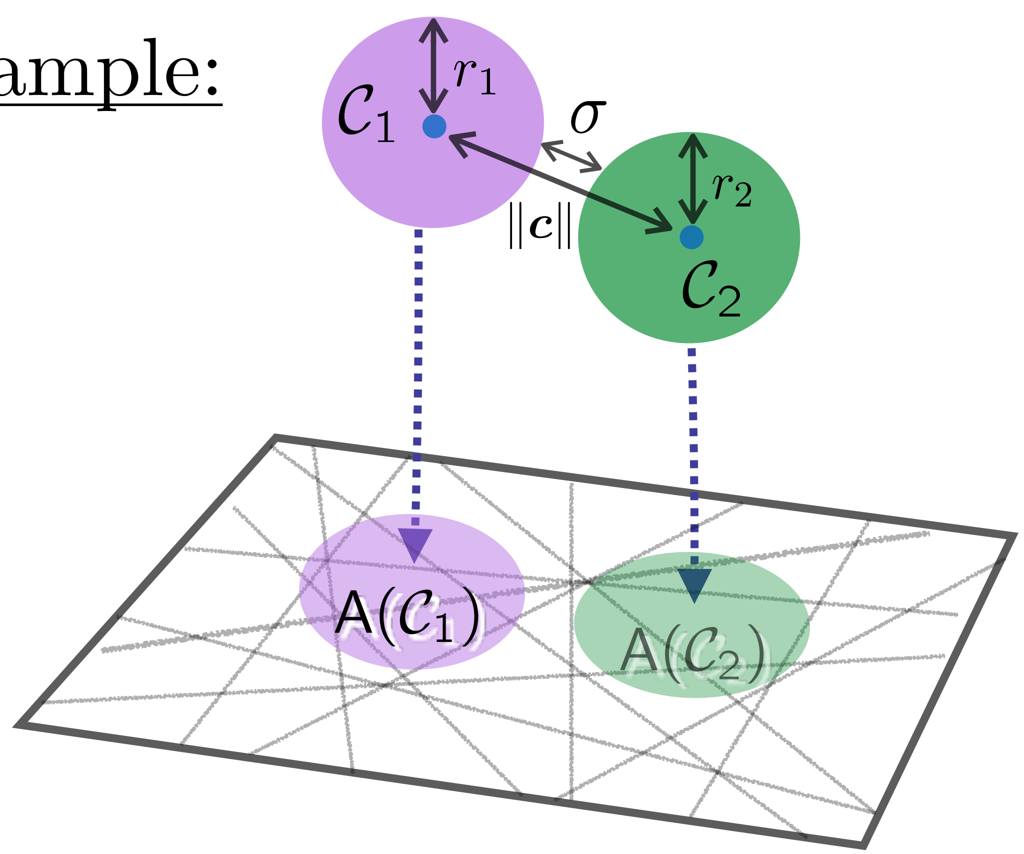
Provided

$$m \gtrsim \Big(\underbrace{w_{\ominus}^2}_{\text{linear}} + \underbrace{n\tfrac{\delta^2}{\sigma^2}}_{\text{quantiz.}}\Big)\ \big(1 + \underbrace{\log\big(1+\tfrac{rm}{\delta n}\big)}_{\text{proof artifact?}} + \underbrace{w_{\ominus}^{-2}\log\tfrac{1}{\eta}}_{\text{linear}}\big),$$

we have

$$\mathbb{P}[\mathsf{A}(\mathcal{C}_1) \cap \mathsf{A}(\mathcal{C}_2) = \emptyset] \geqslant 1 - \eta.$$

Example:



$$\Rightarrow \quad m \gtrsim \Big(\frac{r^2}{\|\boldsymbol{c}\|^2} + \frac{\delta^2}{(\|\boldsymbol{c}\|-r)^2}\Big)\, n$$

Note: $\delta > \sigma$ is allowed (dithering effect!)

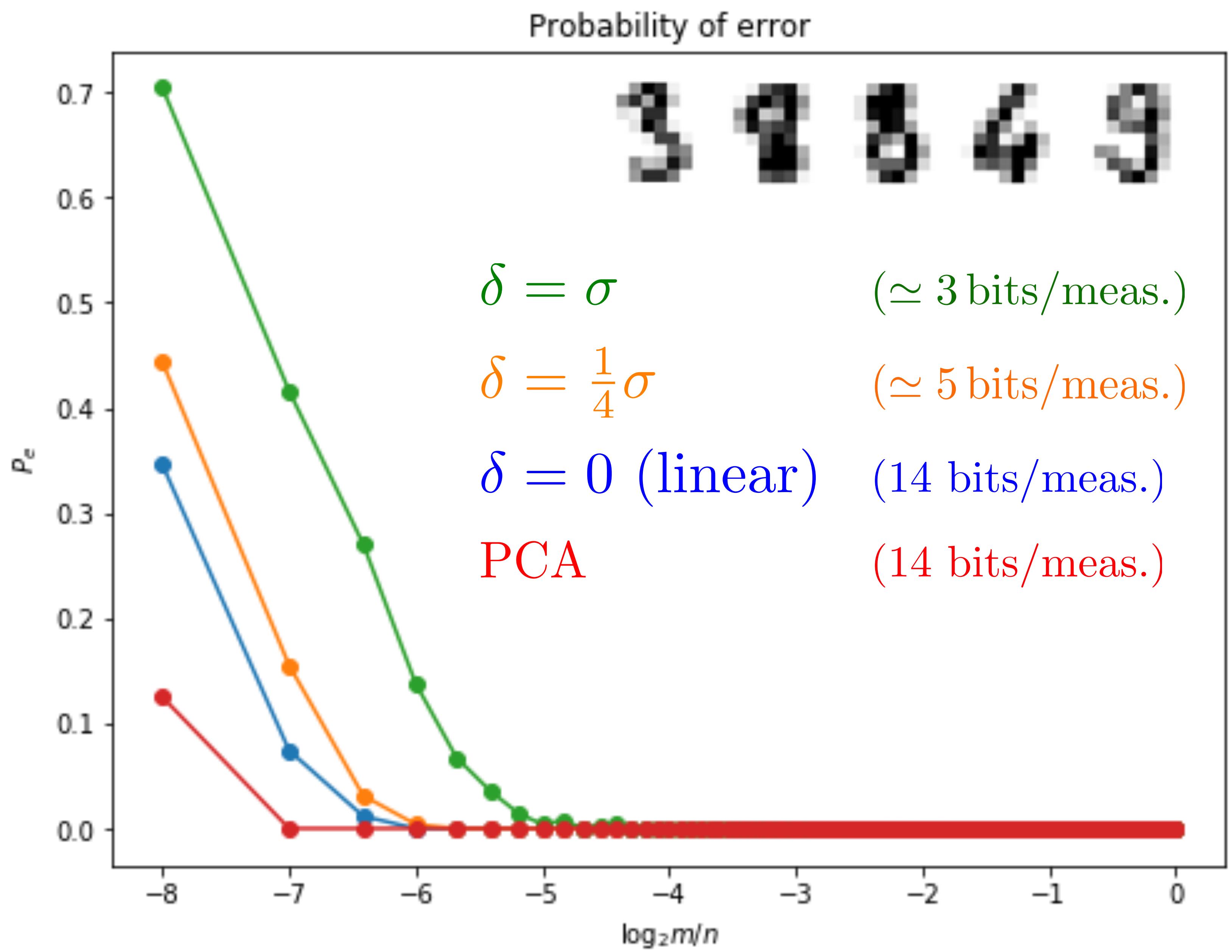
Note bis: $m > n$ not specially bad ($\delta\mathbb{Z}^m$).

Simulations: Digit dataset (from scikit learn)

10 handwritten digits, 8x8 pixels (n=64), samples/class ≃ 12.

Training/Test sets = 50%/50%. $\sigma = \min_{i,j:i\neq j} \min_{\boldsymbol{u}\in\mathcal{C}_i,\boldsymbol{v}\in\mathcal{C}_j} \|\boldsymbol{u}-\boldsymbol{v}\|$

Classification: 5-NN Classifier.



Try some code out here: **github.com/VC86/MLSPbox**

Take-away messages

- From CS to QCS (for scalar quantizers)
  - Importance of consistency and dithering
- Reconstruction still possible in QCS with decaying error as m increases
- Learning/Classification possible in QCS domain

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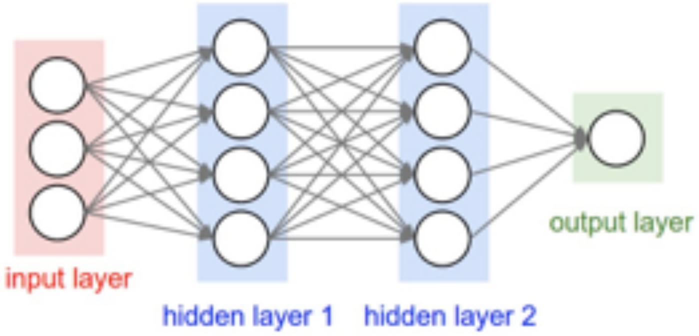
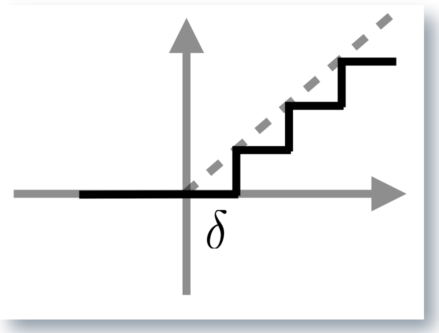
- From CS to QCS (for scalar quantizers)
  - Importance of consistency and dithering
- Reconstruction still possible in QCS with decaying error as m increases
- Learning/Classification possible in QCS domain
- <u>Open problems</u>
  - CW for (other/all?) RIP matrices?
  - Quantizing non-linear embedding (clipping, ReLU)?
  - ...





Thank you for your attention!

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