Keep the phase!

Signal recovery in phase-only compressive sensing

Laurent Jacques* and Thomas Feuillen+

*: INMA, UCLouvain, Belgium. +: Uni Lu, Luxembourg

October 31st, 2023

Amplitude and Phase of Image Frequencies

A bit of history . . . Oppenheim and Lim, 1981:

"*What's the most important information between the spectral amplitude and phase of signals?*"

Amplitude and Phase of Image Frequencies

A bit of history ... Oppenheim and Lim, 1981:

"*What's the most important information between the spectral amplitude and phase of signals?*"

A simple experiment: Let $\mathcal{F}$ the (2-D) discrete Fourier transform (DFT)

Original image $f \in \mathbb{R}^{N_x \times N_y}$;
& we compute $\mathcal{F}f \in \mathbb{C}^{N_x \times N_y}$

Amplitude and Phase of Image Frequencies

A bit of history ... Oppenheim and Lim, 1981:

"*What's the most important information between the spectral amplitude and phase of signals?*"

A simple experiment: Let $\mathcal{F}$ the (2-D) discrete Fourier transform (DFT)

Original image $f \in \mathbb{R}^{N_x \times N_y}$;
& we compute $\mathcal{F}f \in \mathbb{C}^{N_x \times N_y}$

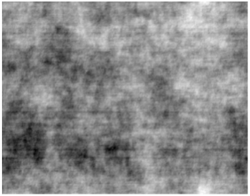


Image reconstructed with spectral amplitude

$$f' = \mathcal{F}^{-1}(\underbrace{|\mathcal{F}f|}_{*})$$

*: applied component-wise

Amplitude and Phase of Image Frequencies

A bit of history . . . Oppenheim and Lim, 1981:

"*What's the most important information between the spectral amplitude and phase of signals?*"

A simple experiment: Let $\mathcal{F}$ the (2-D) discrete Fourier transform (DFT)

Original image $f \in \mathbb{R}^{N_x \times N_y}$;
& we compute $\mathcal{F}f \in \mathbb{C}^{N_x \times N_y}$

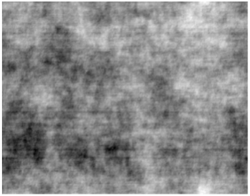


Image reconstructed with spectral amplitude

$$f' = \mathcal{F}^{-1}(\underbrace{|\mathcal{F}f|}_{*})$$

*: applied component-wise

Image reconstructed with spectral phase

$$f' = \mathcal{F}^{-1}(\underbrace{\frac{\mathcal{F}f}{|\mathcal{F}f|}}_{*})$$

Reconstructing an image from its spectral phases (1/2)

Fact: $\exists$ algorithm to recover *certain* images from their spectral phase (up to a global amplitude).

Reconstructing an image from its spectral phases (1/2)

Fact: $\exists$ algorithm to recover *certain* images from their spectral phase (up to a global amplitude).

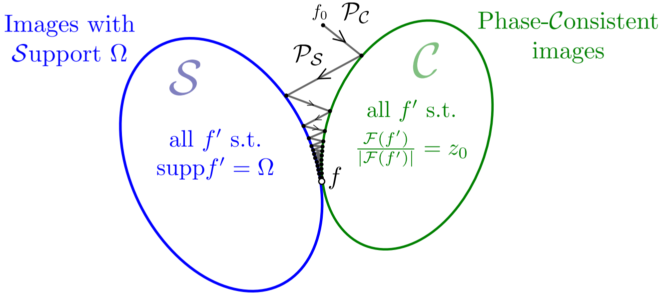
$\Rightarrow$ Use *alternate projections onto convex sets*, *i.e.*,

- given $z_0 = \mathcal{F}(f)/|\mathcal{F}(f)|$, the observed spectral phases,
- assuming $f \in \mathcal{S} :=$ set of images supported on $\Omega \subset \mathbb{R}^2$ (with $|\Omega| \leqslant N_x N_y$).

Fact: $\exists$ algorithm to recover *certain* images from their spectral phase (up to a global amplitude).

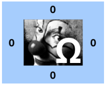
$\Rightarrow$ Use *alternate projections onto convex sets*, *i.e.*,

- given $z_0 = \mathcal{F}(f)/|\mathcal{F}(f)|$, the observed spectral phases,
- assuming $f \in \mathcal{S} :=$ set of images supported on $\Omega \subset \mathbb{R}^2$ (with $|\Omega| \leqslant N_x N_y$).

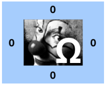


$$f^{(n+1)} = \mathcal{P}_\mathcal{B}\mathcal{P}_\mathcal{C} f^{(n)}, \;\; f^{(0)} = f_0 \in \mathbb{C}^n, \qquad \lim_{n\to+\infty} f^{(n)} = c\, f.$$

Reconstructing an image from its spectral phases (2/2)

Applying this method to our example ... (*i.e.*, with Ω set from  *)

***: This is equivalent to oversampling the Fourier domain of f.**

Applying this method to our example . . . (*i.e.*, with Ω set from 

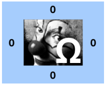
*))

1^{st} iteration (init with ones)
(normalized SNR: 6.6 dB)

10 iterations
(normalized SNR: 11.6 dB)

***: This is equivalent to oversampling the Fourier domain of f.**

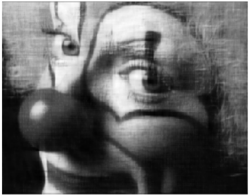
Reconstructing an image from its spectral phases (2/2)

Applying this method to our example . . . (*i.e.*, with Ω set from 

*)

1^{st} iteration (init with ones)
(normalized SNR: 6.6 dB)

10 iterations
(normalized SNR: 11.6 dB)

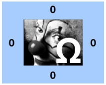


100 iterations
(normalized SNR: 19.6 dB)

1 000 iterations
(normalized SNR: 41 dB)

original

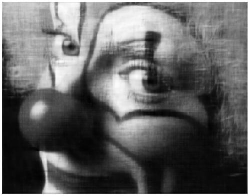
*: This is equivalent to oversampling the Fourier domain of f.

Applying this method to our example . . . (*i.e.*, with Ω set from 

*)

1st iteration (init with ones)
(normalized SNR: 6.6 dB)

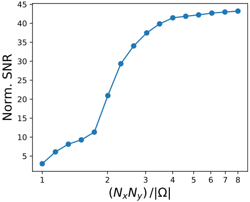
10 iterations
(normalized SNR: 11.6 dB)



100 iterations
(normalized SNR: 19.6 dB)

1 000 iterations
(normalized SNR: 41 dB)

original



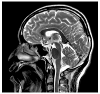
SNR vs oversampling
(1 000 iter. per point)

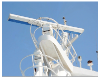
***: This is equivalent to oversampling the Fourier domain of f.**

Why could it be useful?

Numerous Fourier/spectral sensing applications:

- Magnetic resonance imaging (MRI);
- Radar systems;
- Michelson interferometry / Fourier transform imaging;
- Aperture synthesis by radio interferometry.

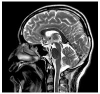


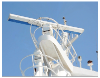


Why could it be useful?

Numerous Fourier/spectral sensing applications:

- Magnetic resonance imaging (MRI);
- Radar systems;
- Michelson interferometry / Fourier transform imaging;
- Aperture synthesis by radio interferometry.





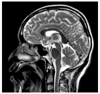
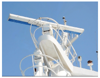
Challenges:

- Massive data stream imposes new data compression strategies.
- Compress but keep useful information (*e.g.*, for subsequent imaging).
- Large magnitude variations $\Rightarrow$ different compression impact.

Why could it be useful?

Numerous Fourier/spectral sensing applications:

- Magnetic resonance imaging (MRI);
- Radar systems;
- Michelson interferometry / Fourier transform imaging;
- Aperture synthesis by radio interferometry.


Challenges:

- Massive data stream imposes new data compression strategies.
- Compress but keep useful information (*e.g.*, for subsequent imaging).
- Large magnitude variations $\Rightarrow$ different compression impact.

Questions: Which systems are compatible with phase-only signal estimation?

$\triangleright$ (this talk) Is complex compressive sensing compatible?

Why asking?

- If compatible, insensitive to large amplitudes variations (by definition).
- If robust, easy to compress information: just quantize the spectral phase!

Let's collect $m < n$ measurements about $\boldsymbol{x}$ from this linear model:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\epsilon} \in \mathbb{C}^m, \qquad \text{(CS)}$$

with:

- a low-complexity vector $\boldsymbol{x} \in \mathcal{L} \subset \mathbb{C}^n$ (e.g., a vectorized image) with $\mathcal{L}$ the set of sparse signals, low-rank matrices, ...
- a complex sensing matrix $\boldsymbol{A} \in \mathbb{C}^{m \times n}$,
- a given (additive) noise $\boldsymbol{\epsilon} \in \mathbb{C}^m$ and $\|\boldsymbol{\epsilon}\| \leqslant \varepsilon$.

Let's collect $m < n$ measurements about $\boldsymbol{x}$ from this linear model:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\epsilon} \in \mathbb{C}^m, \qquad \text{(CS)}$$

with:
- a low-complexity vector $\boldsymbol{x} \in \mathcal{L} \subset \mathbb{C}^n$ (e.g., a vectorized image) with $\mathcal{L}$ the set of sparse signals, low-rank matrices, ...
- a complex sensing matrix $\boldsymbol{A} \in \mathbb{C}^{m \times n}$,
- a given (additive) noise $\boldsymbol{\epsilon} \in \mathbb{C}^m$ and $\|\boldsymbol{\epsilon}\| \leqslant \varepsilon$.

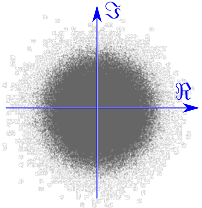
Compressive sensing:

Despite $m < n$, if m larger than $\mathcal{L}$'s "dimension", and $\boldsymbol{A}$ is "random", the vector $\boldsymbol{x}$ can be exactly recovered (or estimated if $\boldsymbol{\epsilon} \neq 0$).

[Candès and Tao, 2005; Foucart and Rauhut, 2013]

Let's be more specific ... let's focus on the Gaussian case.

Restricted isometry property



For some $0 < \delta < 1$ and $k < m < n$, if

$$m \geqslant C\,\delta^{-2} k\,\log(n/k),$$

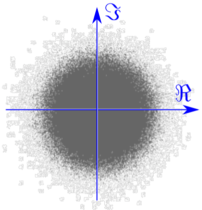
and $\sqrt{m}\,A_{ij} \sim_{\text{i.i.d.}} \mathbb{C}\mathcal{N}(0,2) \sim \mathcal{N}(0,1) + \mathrm{i}\,\mathcal{N}(0,1)$,

then, *with high probability* (w.h.p.),

$$(1-\delta)\|\boldsymbol{v}\|^2 \leqslant \|\boldsymbol{A}\boldsymbol{v}\|^2 \leqslant (1+\delta)\|\boldsymbol{v}\|^2, \quad \forall k\text{-sparse } \boldsymbol{v}. \qquad (\mathrm{RIP}(k,\delta))$$

Let's be more specific ... let's focus on the Gaussian case.

Restricted isometry property



For some $0 < \delta < 1$ and $k < m < n$, if

$$m \geqslant C\,\delta^{-2} k\, \log(n/k),$$

and $\sqrt{m}\, A_{ij} \sim_{\text{i.i.d.}} \mathbb{C}\mathcal{N}(0,2) \sim \mathcal{N}(0,1) + \mathrm{i}\,\mathcal{N}(0,1)$,

then, *with high probability* (w.h.p.),

$$(1-\delta)\|\boldsymbol{v}\|^2 \leqslant \|\boldsymbol{A}\boldsymbol{v}\|^2 \leqslant (1+\delta)\|\boldsymbol{v}\|^2, \quad \forall k\text{-sparse } \boldsymbol{v}. \qquad (\text{RIP}(k,\delta))$$

So, why does CS work? Given $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ with $\boldsymbol{x}$ a k-sparse signal, we have

- $\text{RIP}(2k,\delta) \Rightarrow \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{u}\|^2 = \|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{u})\|^2 \approx \|\boldsymbol{x}-\boldsymbol{u}\|^2$, for all k-sparse $\boldsymbol{u}$.

$\Rightarrow$ $\boldsymbol{A}$ is *essentially* invertible over the set of sparse vectors;
just estimate $\boldsymbol{x}$ by finding a sparse $\boldsymbol{u}$ zeroing or minimizing $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{u}\|^2$!

The RIP supports (one of) the "fundamental theorem(s) of CS"

Theorem: If $\boldsymbol{A}$ is RIP$(2k, \delta)$ with $0 < \delta < \delta_0$ (*e.g.*, $\delta_0 = 1/\sqrt{2}$), then the *basis pursuit denoise* estimate:

$$\hat{\boldsymbol{x}} = \mathop{\arg\min}_{\boldsymbol{u} \in \mathbb{C}^n} \underbrace{\|\boldsymbol{u}\|_1}_{\text{sparsity promoting}} \quad \text{s.t.} \quad \underbrace{\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{u}\| \leqslant \varepsilon}_{\text{data fidelity}}, \qquad \text{(BPDN)}$$

See, *e.g.*, Candès, 2008; Foucart and Rauhut, 2013.

The RIP supports (one of) the "fundamental theorem(s) of CS"

Theorem: If $\boldsymbol{A}$ is RIP$(2k, \delta)$ with $0 < \delta < \delta_0$ (*e.g.*, $\delta_0 = 1/\sqrt{2}$), then the *basis pursuit denoise* estimate:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u} \in \mathbb{C}^n} \underbrace{\|\boldsymbol{u}\|_1}_{\text{sparsity promoting}} \quad \text{s.t.} \quad \underbrace{\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{u}\| \leqslant \varepsilon}_{\text{data fidelity}}, \qquad \text{(BPDN)}$$

satisfies the *instance optimality*

$$\underbrace{\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|}_{\text{Rec. error; } \approx \text{ MSE}} \leqslant C \underbrace{\frac{\|\boldsymbol{x} - \boldsymbol{x}_k\|_1}{\sqrt{k}}}_{\text{deviation to sparsity}} + \underbrace{D\,\varepsilon}_{\text{noise}}.$$

See, *e.g.*, Candès, 2008; Foucart and Rauhut, 2013.

Phase-Only Sensing Model for CS

Inspired by Oppenheim and Lim, 1981; Boufounos, 2013,

let's consider the phase-only (non-linear) compressive sensing model:

$$\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) + \boldsymbol{\epsilon} \in \mathbb{C}^m, \qquad \text{(PO-CS)}$$

where:

- $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ is complex, but $\boldsymbol{x}$ is a real (⚠), k-sparse vector;
- $\mathrm{sign}_{\mathbb{C}}(re^{\mathrm{i}\theta}) := e^{\mathrm{i}\theta}$ (and 0 if $r = 0$), applied pointwise;
- and $\boldsymbol{\epsilon} \in \mathbb{C}^m$ a bounded noise with $\|\boldsymbol{\epsilon}\|_\infty \leqslant \tau$ for some $\tau \geqslant 0$.

Phase-Only Sensing Model for CS

Inspired by Oppenheim and Lim, 1981; Boufounos, 2013,

let's consider the phase-only (non-linear) compressive sensing model:

$$\boldsymbol{z} = \operatorname{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) + \boldsymbol{\epsilon} \in \mathbb{C}^m, \qquad \text{(PO-CS)}$$

where:

- $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ is complex, but $\boldsymbol{x}$ is a real (⚠), k-sparse vector;
- $\operatorname{sign}_{\mathbb{C}}(re^{i\theta}) := e^{i\theta}$ (and 0 if $r = 0$), applied pointwise;
- and $\boldsymbol{\epsilon} \in \mathbb{C}^m$ a bounded noise with $\|\boldsymbol{\epsilon}\|_\infty \leqslant \tau$ for some $\tau \geqslant 0$.

Key observations:

1. If $\boldsymbol{x} \to C\boldsymbol{x}$ with $C > 0$, $\boldsymbol{z}$ is unchanged (signal amplitude is lost)

Phase-Only Sensing Model for CS

Inspired by Oppenheim and Lim, 1981; Boufounos, 2013,

let's consider the phase-only (non-linear) compressive sensing model:

$$\boldsymbol{z} = \operatorname{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) + \boldsymbol{\epsilon} \in \mathbb{C}^m, \qquad \text{(PO-CS)}$$

where:

- $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ is complex, but $\boldsymbol{x}$ is a real (⚠), k-sparse vector;
- $\operatorname{sign}_{\mathbb{C}}(re^{\mathrm{i}\theta}) := e^{\mathrm{i}\theta}$ (and 0 if $r = 0$), applied pointwise;
- and $\boldsymbol{\epsilon} \in \mathbb{C}^m$ a bounded noise with $\|\boldsymbol{\epsilon}\|_\infty \leqslant \tau$ for some $\tau \geqslant 0$.

Key observations:

1. If $\boldsymbol{x} \to C\boldsymbol{x}$ with $C > 0$, $\boldsymbol{z}$ is unchanged (signal amplitude is lost)
2. If both $\boldsymbol{A}$ and $\boldsymbol{x}$ are real, then $\boldsymbol{z} \in \{\pm 1\}^m$ (real PO-CS $\to$ 1-bit CS)

Fact: In noiseless 1-bit CS, best estimate s.t. $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| = \Omega(1/m)$ if $m \uparrow$.
[Boufounos and Baraniuk, 2008; Jacques et al., 2013; Plan and Vershynin, 2012]

Principle: Turn the non-linear PO model into linear one.

Principle: Turn the non-linear PO model into linear one. Step by step . . .

A. Let's normalize x: Any renormalized signal $c\boldsymbol{x}$ ($c > 0$), and in particular

$$\boldsymbol{x}^\star := \frac{\kappa\sqrt{m}}{\|\boldsymbol{A}\boldsymbol{x}\|_1}\,\boldsymbol{x}, \quad \text{with } \kappa := \sqrt{\tfrac{\pi}{2}},$$

preserves PO measurements, *i.e.*, $\mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}^\star)$.

Therefore, we focus on the recovery of $\boldsymbol{x}^\star$ ($\to$ encodes signal direction)

that satisfies $\|\boldsymbol{A}\boldsymbol{x}^\star\|_1 = \kappa\sqrt{m}$.

Principle: Turn the non-linear PO model into linear one. Step by step ...

A. Let's normalize x: Any renormalized signal $c\boldsymbol{x}$ ($c > 0$), and in particular

$$\boldsymbol{x}^\star := \frac{\kappa\sqrt{m}}{\|\boldsymbol{A}\boldsymbol{x}\|_1}\,\boldsymbol{x}, \quad \text{with } \kappa := \sqrt{\tfrac{\pi}{2}},$$

preserves PO measurements, *i.e.*, $\operatorname{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) = \operatorname{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}^\star)$.

Therefore, we focus on the recovery of $\boldsymbol{x}^\star$ ($\rightarrow$ encodes signal direction)

that satisfies $\|\boldsymbol{A}\boldsymbol{x}^\star\|_1 = \kappa\sqrt{m}$.

Rationale:

- Well, it's useful for our proofs ☺
- For complex Gaussian $\sqrt{m}\,\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m\times n}(0,2)$ and $g \sim \mathcal{N}(0,1)$,

$$\mathbb{E}|g| = \kappa \quad \Rightarrow \quad \mathbb{E}\|\boldsymbol{A}\boldsymbol{x}\|_1 = \kappa\sqrt{m}\,\|\boldsymbol{x}\| \quad \Rightarrow \quad \|\boldsymbol{x}^\star\| \approx 1.$$

$\Rightarrow$ $\boldsymbol{x}^\star$ is (almost) a unit length vector, *i.e.*, a direction.

Principle: Turn the non-linear PO model into linear one. Step by step ...

B. Let's find linear constraints: From the noiseless model

$$\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}^\star),$$

we see that the vector $\boldsymbol{u} = \boldsymbol{x}^\star \in \mathbb{R}^n$ respects both:

$$\underbrace{\langle \boldsymbol{z}, \boldsymbol{A}\boldsymbol{u} \rangle}_{= \|\boldsymbol{A}\boldsymbol{x}^\star\|_1 \text{ if } \boldsymbol{u}=\boldsymbol{x}^\star} = \kappa\sqrt{m} \quad \Leftrightarrow \langle \underbrace{\tfrac{1}{\kappa\sqrt{m}}\boldsymbol{A}^*\boldsymbol{z}}_{:=\boldsymbol{\alpha}_{\boldsymbol{z}}}, \boldsymbol{u} \rangle = 1 \qquad \text{(normalization)}$$

Principle: Turn the non-linear PO model into linear one. Step by step ...

B. Let's find linear constraints: From the noiseless model

$$\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}^\star),$$

we see that the vector $\boldsymbol{u} = \boldsymbol{x}^\star \in \mathbb{R}^n$ respects both:

$$\begin{cases} \underbrace{\langle \boldsymbol{z}, \boldsymbol{A}\boldsymbol{u}\rangle}_{=\|\boldsymbol{A}\boldsymbol{x}^\star\|_1 \text{ if } \boldsymbol{u}=\boldsymbol{x}^\star} = \kappa\sqrt{m} \quad \Leftrightarrow \big\langle \underbrace{\tfrac{1}{\kappa\sqrt{m}}\boldsymbol{A}^*\boldsymbol{z}}_{:=\boldsymbol{\alpha}_{\boldsymbol{z}}}, \boldsymbol{u}\big\rangle = 1 & \text{(normalization)} \\ \mathrm{diag}(\boldsymbol{z})^*\boldsymbol{A}\boldsymbol{u} = \big(\underbrace{z_1^* \cdot (\boldsymbol{A}\boldsymbol{u})_1}_{=|(\boldsymbol{A}\boldsymbol{x}^\star)_1| \text{ if } \boldsymbol{u}=\boldsymbol{x}^\star}, \cdots, \underbrace{z_m^* \cdot (\boldsymbol{A}\boldsymbol{u})_m}_{=|(\boldsymbol{A}\boldsymbol{x}^\star)_m| \text{ if } \boldsymbol{u}=\boldsymbol{x}^\star}\big)^\top \in \mathbb{R}^m_+ & \text{(phase consistency)} \end{cases}$$

Principle: Turn the non-linear PO model into linear one. Step by step . . .

B. Let's find linear constraints: From the noiseless model

$$\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}^\star),$$

we see that the vector $\boldsymbol{u} = \boldsymbol{x}^\star \in \mathbb{R}^n$ respects both:

$$\begin{cases} \underbrace{\langle \boldsymbol{z}, \boldsymbol{A}\boldsymbol{u}\rangle}_{=\|\boldsymbol{A}\boldsymbol{x}^\star\|_1 \text{ if } \boldsymbol{u}=\boldsymbol{x}^\star} = \kappa\sqrt{m} \quad \Leftrightarrow \langle \underbrace{\tfrac{1}{\kappa\sqrt{m}}\boldsymbol{A}^*\boldsymbol{z}}_{:=\boldsymbol{\alpha}_{\boldsymbol{z}}}, \boldsymbol{u}\rangle = 1 & \text{(normalization)} \\ \mathrm{diag}(\boldsymbol{z})^* \boldsymbol{A}\boldsymbol{u} = \big(\underbrace{z_1^* \cdot (\boldsymbol{A}\boldsymbol{u})_1}_{=|(\boldsymbol{A}\boldsymbol{x}^\star)_1| \text{ if } \boldsymbol{u}=\boldsymbol{x}^\star}, \cdots, \underbrace{z_m^* \cdot (\boldsymbol{A}\boldsymbol{u})_m}_{=|(\boldsymbol{A}\boldsymbol{x}^\star)_m| \text{ if } \boldsymbol{u}=\boldsymbol{x}^\star} \big)^\top \in \mathbb{R}_+^m & \text{(phase consistency)} \end{cases}$$

Let's relax "phase consistency": we only impose $\mathrm{diag}(\boldsymbol{z})^* \boldsymbol{A}\boldsymbol{u} \in \mathbb{R}^m$, that is

$$0 = \Im(\mathrm{diag}(\boldsymbol{z})^* \boldsymbol{A}\boldsymbol{u}) = (\mathrm{diag}(z)^{\Re} \boldsymbol{A}^{\Im} - \mathrm{diag}(z)^{\Im} \boldsymbol{A}^{\Re})\boldsymbol{u} =: \boldsymbol{H}_z \boldsymbol{u}.$$

Moreover, "normalization" means

$$\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{u}\rangle = 1 \quad \Leftrightarrow \quad \langle \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{u}\rangle = 1,\ \langle \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{u}\rangle = 0.$$

In summary, $\boldsymbol{u} = \boldsymbol{x}^\star$ respects the relaxed, real $m+2$ constraints ...

$$\boldsymbol{A}_z \boldsymbol{u} = \boldsymbol{e}_1 := (1, 0, \cdots, 0)^\top \quad \Rightarrow$$

This is a linear sensing model! Like "$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$"

with

$$\boldsymbol{A}_z := (\boldsymbol{\alpha}_z^{\Re}, \boldsymbol{\alpha}_z^{\Im}, \boldsymbol{H}_z^\top)^\top \in \mathbb{R}^{(m+2)\times n}.$$

In other words,

- A good estimate $\hat{\boldsymbol{x}}$ of $\boldsymbol{x}^\star$ should respect the linear model $\boldsymbol{A}_z \hat{\boldsymbol{x}} = \boldsymbol{e}_1$ since $\boldsymbol{x}^\star \in \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{A}_z \hat{\boldsymbol{u}} = \boldsymbol{e}_1\}$.
- We know this estimate should be sparse (as for $\boldsymbol{x}^\star$)

In summary, $\boldsymbol{u} = \boldsymbol{x}^\star$ respects the relaxed, real $m+2$ constraints ...

$$\boldsymbol{A}_z \boldsymbol{u} = \boldsymbol{e}_1 := (1, 0, \cdots, 0)^\top$$

$\Rightarrow$ This is a linear sensing model! Like "$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$"

with

$$\boldsymbol{A}_z := (\boldsymbol{\alpha}_z^{\Re}, \boldsymbol{\alpha}_z^{\Im}, \boldsymbol{H}_z^\top)^\top \in \mathbb{R}^{(m+2)\times n}.$$

In other words,

- A good estimate $\hat{x}$ of $\boldsymbol{x}^\star$ should respect the linear model $\boldsymbol{A}_z \hat{x} = \boldsymbol{e}_1$ since $\boldsymbol{x}^\star \in \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{A}_z \hat{\boldsymbol{u}} = \boldsymbol{e}_1\}$.
- We know this estimate should be sparse (as for $\boldsymbol{x}^\star$)

$\Rightarrow$ As in linear CS, we can compute $\hat{x}$ from a *basis pursuit* program (BP)

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{u} \in \mathbb{C}^n} \|\boldsymbol{u}\|_1 \ \text{ s.t. } \ \boldsymbol{A}_z \boldsymbol{u} = \boldsymbol{e}_1, \qquad (\text{BP}(\boldsymbol{A}_z, \boldsymbol{e}_1))$$

Question: How far is $\hat{x}$ from $\boldsymbol{x}^\star$? Well, let's see if $\boldsymbol{A}_z$ respects the RIP!

Given $\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x})$, how could $\boldsymbol{A}_{\boldsymbol{z}} := (\boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{H}_{\boldsymbol{z}}^\top)^\top$ respect the RIP?

For a sparse $\boldsymbol{v}$,

$$\|\boldsymbol{A}_z \boldsymbol{v}\|^2 := |\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle|^2 + \|\boldsymbol{H}_z \boldsymbol{v}\|^2$$

Given $\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x})$, how could $\boldsymbol{A}_{\boldsymbol{z}} := (\boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{H}_{\boldsymbol{z}}^{\top})^{\top}$ respect the RIP?

For a sparse $\boldsymbol{v}$,

$$\|\boldsymbol{A}_z \boldsymbol{v}\|^2 := |\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle|^2 + \|\boldsymbol{H}_z \boldsymbol{v}\|^2$$

you can show that, for complex Gaussian $\boldsymbol{A}$:

- $\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle = \langle \frac{1}{\kappa\sqrt{m}} \boldsymbol{A}^* \boldsymbol{z}, \boldsymbol{v}\rangle \approx \langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \boldsymbol{v}\rangle \approx$ projection of $\boldsymbol{v}$ onto $\mathcal{X} := \mathbb{R}\,\boldsymbol{x}$

  Proof: $\frac{1}{m}\mathbb{E}\langle \mathrm{sign}_{\mathbb{C}}\, \boldsymbol{A}\boldsymbol{u}, \sqrt{m}\boldsymbol{A}\boldsymbol{v}\rangle = \kappa \langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \boldsymbol{v}\rangle$ if $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{n-1}$
  $\to$ *sign product embedding* (SPE) $\equiv$ extension $\forall k$-sparse $\boldsymbol{v}$ (*whp*).
  $\to$ pick $\boldsymbol{u} = \boldsymbol{x}$

Given $\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x})$, how could $\boldsymbol{A}_{\boldsymbol{z}} := (\boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{H}_{\boldsymbol{z}}^{\top})^{\top}$ respect the RIP?

For a sparse $\boldsymbol{v}$,

$$\|\boldsymbol{A}_z \boldsymbol{v}\|^2 := |\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle|^2 + \|\boldsymbol{H}_z \boldsymbol{v}\|^2$$

you can show that, for complex Gaussian $\boldsymbol{A}$:

- $\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle = \langle \frac{1}{\kappa\sqrt{m}} \boldsymbol{A}^* \boldsymbol{z}, \boldsymbol{v}\rangle \approx \langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \boldsymbol{v}\rangle \approx$ projection of $\boldsymbol{v}$ onto $\mathcal{X} := \mathbb{R}\,\boldsymbol{x}$

  Proof: $\frac{1}{m}\mathbb{E}\langle \mathrm{sign}_{\mathbb{C}}\, \boldsymbol{A}\boldsymbol{u}, \sqrt{m}\boldsymbol{A}\boldsymbol{v}\rangle = \kappa\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \boldsymbol{v}\rangle$ if $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{n-1}$
  $\rightarrow$ *sign product embedding* (SPE) $\equiv$ extension $\forall k$-sparse $\boldsymbol{v}$ (*whp*).
  $\rightarrow$ pick $\boldsymbol{u} = \boldsymbol{x}$

- $\boldsymbol{H}_z \boldsymbol{x} = 0$ (by construction)
  $\rightarrow$ $\boldsymbol{H}_z \boldsymbol{v} = \boldsymbol{H}_z \boldsymbol{v}^{\perp}$, with $\boldsymbol{v}^{\perp} := \boldsymbol{v} - \langle \boldsymbol{v}, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\rangle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in \mathcal{X}^{\perp}$

Given $\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x})$, how could $\boldsymbol{A}_{\boldsymbol{z}} := (\boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{H}_{\boldsymbol{z}}^{\top})^{\top}$ respect the RIP?

For a sparse $\boldsymbol{v}$,

$$\|\boldsymbol{A}_z\boldsymbol{v}\|^2 := |\langle\boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle|^2 + \|\boldsymbol{H}_z\boldsymbol{v}\|^2$$

you can show that, for complex Gaussian $\boldsymbol{A}$:

- $\langle\boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle = \langle\frac{1}{\kappa\sqrt{m}}\boldsymbol{A}^*\boldsymbol{z}, \boldsymbol{v}\rangle \approx \langle\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \boldsymbol{v}\rangle \approx$ projection of $\boldsymbol{v}$ onto $\mathcal{X} := \mathbb{R}\,\boldsymbol{x}$

  Proof: $\frac{1}{m}\mathbb{E}\langle\mathrm{sign}_{\mathbb{C}}\,\boldsymbol{A}\boldsymbol{u}, \sqrt{m}\boldsymbol{A}\boldsymbol{v}\rangle = \kappa\langle\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \boldsymbol{v}\rangle$ if $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{n-1}$
  $\to$ *sign product embedding* (SPE) $\equiv$ extension $\forall k$-sparse $\boldsymbol{v}$ (*whp*).
  $\to$ pick $\boldsymbol{u} = \boldsymbol{x}$
- $\boldsymbol{H}_z\boldsymbol{x} = 0$ (by construction)
  $\to \boldsymbol{H}_z\boldsymbol{v} = \boldsymbol{H}_z\boldsymbol{v}^{\perp}$, with $\boldsymbol{v}^{\perp} := \boldsymbol{v} - \langle\boldsymbol{v}, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\rangle\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in \mathcal{X}^{\perp}$
- and $\boldsymbol{H}_z$ RIP on $2k$-sparse signals $\cap\,\mathcal{X}^{\perp}$:

$$\left\|\boldsymbol{H}_z\boldsymbol{v}^{\perp}\right\|^2 \approx \|\boldsymbol{v}^{\perp}\|^2.$$

Given $\boldsymbol{z} = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x})$, how could $\boldsymbol{A}_{\boldsymbol{z}} := (\boldsymbol{\alpha}_{\boldsymbol{z}}^{\Re}, \boldsymbol{\alpha}_{\boldsymbol{z}}^{\Im}, \boldsymbol{H}_{\boldsymbol{z}}^\top)^\top$ respect the RIP?

For a sparse $\boldsymbol{v}$,

$$\|\boldsymbol{A}_z \boldsymbol{v}\|^2 := |\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle|^2 + \|\boldsymbol{H}_z \boldsymbol{v}\|^2 \approx \langle \tfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \boldsymbol{v}\rangle^2 + \|\boldsymbol{v}^\perp\|^2 = \|\boldsymbol{v}\|^2$$

you can show that, for complex Gaussian $\boldsymbol{A}$:

- $\langle \boldsymbol{\alpha}_{\boldsymbol{z}}, \boldsymbol{v}\rangle = \langle \frac{1}{\kappa\sqrt{m}} \boldsymbol{A}^* \boldsymbol{z}, \boldsymbol{v}\rangle \approx \langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \boldsymbol{v}\rangle \approx$ projection of $\boldsymbol{v}$ onto $\mathcal{X} := \mathbb{R}\,\boldsymbol{x}$

  Proof: $\frac{1}{m}\mathbb{E}\langle \mathrm{sign}_{\mathbb{C}}\, \boldsymbol{A}\boldsymbol{u}, \sqrt{m}\boldsymbol{A}\boldsymbol{v}\rangle = \kappa\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \boldsymbol{v}\rangle$ if $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{n-1}$
  $\to$ *sign product embedding* (SPE) $\equiv$ extension $\forall k$-sparse $\boldsymbol{v}$ (*whp*).
  $\to$ pick $\boldsymbol{u} = \boldsymbol{x}$
- $\boldsymbol{H}_{\boldsymbol{z}}\boldsymbol{x} = 0$ (by construction)
  $\to \boldsymbol{H}_{\boldsymbol{z}}\boldsymbol{v} = \boldsymbol{H}_{\boldsymbol{z}}\boldsymbol{v}^\perp$, with $\boldsymbol{v}^\perp := \boldsymbol{v} - \langle \boldsymbol{v}, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\rangle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in \mathcal{X}^\perp$
- and $\boldsymbol{H}_{\boldsymbol{z}}$ RIP on $2k$-sparse signals $\cap\, \mathcal{X}^\perp$:

$$\|\boldsymbol{H}_{\boldsymbol{z}}\boldsymbol{v}^\perp\|^2 \approx \|\boldsymbol{v}^\perp\|^2.$$

Final statement:

Theorem: Given $\boldsymbol{x}$ and $0 < \delta < 1$, $\sqrt{m}\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m \times n}(0, 2)$, if

$$m \geqslant C\delta^{-2} k \log(n/k),$$

then, w.h.p., $\boldsymbol{A}_z$ satisfies the RIP (k, δ).

Final statement:

Theorem: Given $\boldsymbol{x}$ and $0 < \delta < 1$, $\sqrt{m}\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m\times n}(0, 2)$, if

$$m \geqslant C\delta^{-2} k \log(n/k),$$

then, w.h.p., $\boldsymbol{A}_z$ satisfies the RIP (k, δ).

Consequences:

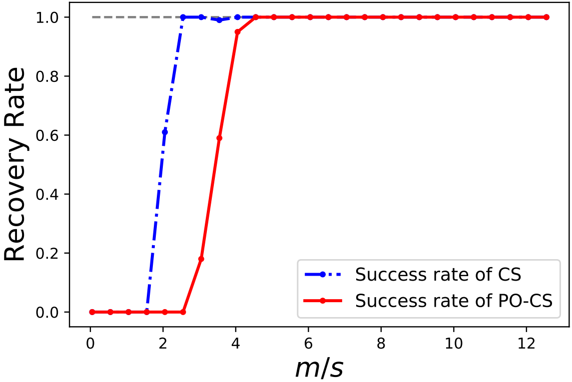
- For $\hat{\boldsymbol{x}} = \mathrm{BP}(\boldsymbol{A}_z, \boldsymbol{e}_1)$, if $\boldsymbol{A}_z$ is RIP$(\delta < \delta_0, 2k)$, we get exact reconstruction of signal direction, *i.e.*, $\hat{\boldsymbol{x}} = \boldsymbol{x}^\star$!
- + Stability & robustness (aka *instance optimality*) with BPDN (see paper)

Let's plot a *phase-transition curve*: we generate $\sqrt{m}\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m\times 256}(0,2)$ &

- 20-sparse vectors in $\mathbb{R}^{256}$;
- $m \in [1, 256]$ and average over 100 trials;
- Reconstruction successful if SNR $\geqslant$ 60 dB.

Let's plot a *phase-transition curve*: we generate $\sqrt{m}\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m\times 256}(0,2)$ &

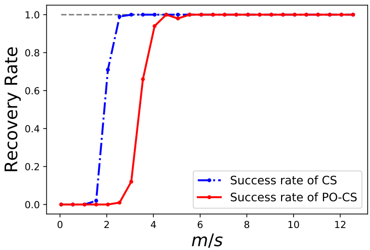
- 20-sparse vectors in $\mathbb{R}^{256}$;
- $m \in [1, 256]$ and average over 100 trials;
- Reconstruction successful if SNR $\geqslant$ 60 dB.



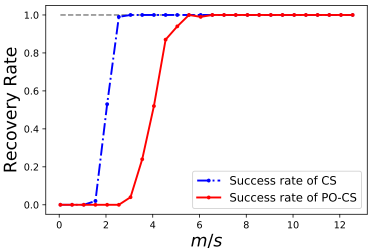
Let's be a little more daring . . . and forget Gauss

Let's be a little more daring . . . and forget Gauss

Bernoulli random matrix
$A_{ij} \sim_{\text{iid}} \{\pm 1 \pm \mathrm{i}\}$

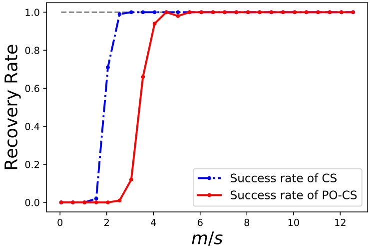


Random partial Fourier
(A = sub-sampled $\mathcal{F}(\boldsymbol{x})$)

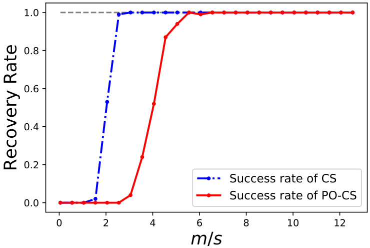


Let's be a little more daring . . . and forget Gauss

Bernoulli random matrix
$A_{ij} \sim_{\rm iid} \{\pm 1 \pm \mathrm{i}\}$



Random partial Fourier
(A = sub-sampled $\mathcal{F}(\boldsymbol{x})$)



Interestingly:

- These results are not covered by theory.
- Bernoulli random matrices do not work for 1-bit CS.
- Fourier sensing has PO-CS counter-examples (that cannot be recovered)!

$$e.g., \text{ for } \boldsymbol{x}' := \boldsymbol{h} * \boldsymbol{x} \text{ with } \hat{h}_k > 0, \forall k, \quad \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}') = \mathrm{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}).$$

Take-Away Messages

1. In Gauss' world, despite:
   - the non-linearity of its sensing model,
   - and the bad example of 1-bit CS (the "real" PO-CS),

   phase-only compressive sensing works "*as well as*" (linear) CS.

2. What is recovered/estimated is the signal direction (via $\boldsymbol{x}^\star$).

3. Applications: phase-quantization procedures with bounded distortion
   e.g., in radar, MRI, . . .

4. Open/Closed questions:
   - (minor) Extension to complex signals & uniform result
     $\rightarrow$ Chen and Ng, 2023
   - (major) Theoretical extension to other random sensing matrices.

. — . — .

Thank you!

LJ, T. Feuillen, "The importance of phase in complex compressive sensing", IEEE Transactions on Information Theory, 67(6), 4150-4161. arXiv:2001.02529

Boufounos, Petros T (2013). "Sparse signal reconstruction from phase-only measurements". In: *Proc. Int. Conf. Sampling Theory and Applications (SampTA)],(July 1-5 2013)*. Citeseer.

Boufounos, Petros T and Richard G Baraniuk (2008). "1-bit compressive sensing". In: *2008 42nd Annual Conference on Information Sciences and Systems*. IEEE, pp. 16–21.

Candès, EJ and T Tao (2005). "Decoding by linear programming". In: *IEEE Transactions on Information Theory* 51.12, pp. 4203–4215.

Candès, Emmanuel J. (May 2008). "The restricted isometry property and its implications for compressed sensing". In: *Comptes Rendus Mathematique* 346.9-10, pp. 589–592.

Chen, Junren and Michael K Ng (2023). "Uniform exact reconstruction of sparse signals and low-rank matrices from phase-only measurements". In: *IEEE Transactions on Information Theory* 69.10, pp. 6739–6764.

Foucart, Simon and Holger Rauhut (2013). *A Mathematical Introduction to Compressive Sensing*. Springer New York.

Jacques, Laurent et al. (2013). "Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors". In: *IEEE Transactions on Information Theory* 59.4, pp. 2082–2102.

Oppenheim, A.V. and J.S. Lim (1981). "The importance of phase in signals". In: *Proceedings of the IEEE* 69.5, pp. 529–541.

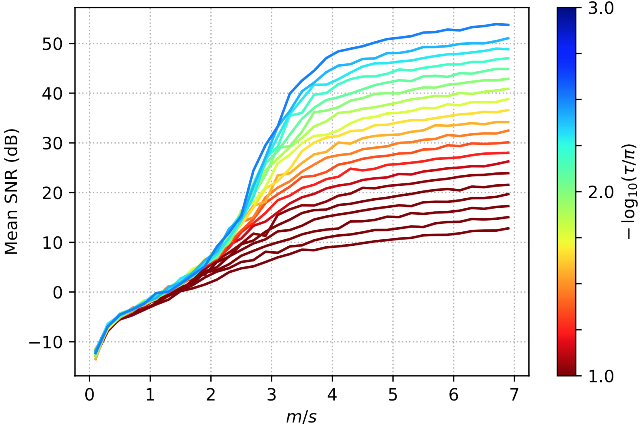
Plan, Yaniv and Roman Vershynin (2012). "Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach". In: *IEEE Transactions on Information Theory* 59.1, pp. 482–494.

— Extra slides —

Extra simulations: noisy case

We generate $\sqrt{m}\boldsymbol{A} \sim \mathbb{C}\mathcal{N}^{m\times 256}(0,2)$ &

- 20-sparse vectors in $\mathbb{R}^{256}$;
- $m \in [1, 256]$ and average over 100 trials;
- $\boldsymbol{z} = \text{sign}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{x}) + \boldsymbol{\xi}$, with $\boldsymbol{\xi} \in \mathbb{C}^m$ and $\|\boldsymbol{\xi}\|_\infty \leqslant \tau$.



Simplifying hypothesis

Phase-only observation in Compressive Sensing?

Simplifying hypothesis

Phase-only observation in Compressive Sensing?

Let's first simplify the context ...

1. We consider the sensing of real vectors $\boldsymbol{x} \in \mathbb{R}^n$.

Note: If complex signal $\boldsymbol{x}$, we can always rewrite

$$\boldsymbol{A}\,\boldsymbol{x} = (\boldsymbol{A}^{\Re} + \mathrm{i}\boldsymbol{A}^{\Im})(\boldsymbol{x}^{\Re} + \mathrm{i}\boldsymbol{x}^{\Im}) = \big(\boldsymbol{A},\, \mathrm{i}\boldsymbol{A}\big)\begin{pmatrix}\boldsymbol{x}^{\Re}\\ \boldsymbol{x}^{\Im}\end{pmatrix} = \overline{\boldsymbol{A}}\,\bar{\boldsymbol{x}},$$

with $\bar{\boldsymbol{x}} \in \mathbb{R}^{2n}$ and $\overline{\boldsymbol{A}} \in \mathbb{C}^{m\times 2n}$.

Simplifying hypothesis

Phase-only observation in Compressive Sensing?

Let's first simplify the context ...

1. We consider the sensing of real vectors $\boldsymbol{x} \in \mathbb{R}^n$.

Note: If complex signal $\boldsymbol{x}$, we can always rewrite

$$\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{A}^{\Re} + \mathrm{i}\boldsymbol{A}^{\Im})(\boldsymbol{x}^{\Re} + \mathrm{i}\boldsymbol{x}^{\Im}) = (\boldsymbol{A}, \mathrm{i}\boldsymbol{A}) \begin{pmatrix} \boldsymbol{x}^{\Re} \\ \boldsymbol{x}^{\Im} \end{pmatrix} = \overline{\boldsymbol{A}}\,\bar{\boldsymbol{x}},$$

with $\bar{\boldsymbol{x}} \in \mathbb{R}^{2n}$ and $\overline{\boldsymbol{A}} \in \mathbb{C}^{m \times 2n}$.

Caveat: This can impact the signal model
e.g., sparse in $\mathbb{C}^n \equiv$ group sparse in $\mathbb{R}^{2n}$.

Simplifying hypothesis

Phase-only observation in Compressive Sensing?

Let's first simplify the context ...

1. We consider the sensing of real vectors $\boldsymbol{x} \in \mathbb{R}^n$.

   Note: If complex signal $\boldsymbol{x}$, we can always rewrite

$$\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{A}^{\Re} + \mathrm{i}\boldsymbol{A}^{\Im})(\boldsymbol{x}^{\Re} + \mathrm{i}\boldsymbol{x}^{\Im}) = (\boldsymbol{A}, \mathrm{i}\boldsymbol{A}) \begin{pmatrix} \boldsymbol{x}^{\Re} \\ \boldsymbol{x}^{\Im} \end{pmatrix} = \overline{\boldsymbol{A}}\,\bar{\boldsymbol{x}},$$

   with $\bar{\boldsymbol{x}} \in \mathbb{R}^{2n}$ and $\overline{\boldsymbol{A}} \in \mathbb{C}^{m \times 2n}$.

   Caveat: This can impact the signal model
   e.g., sparse in $\mathbb{C}^n \equiv$ group sparse in $\mathbb{R}^{2n}$.

2. We focus here on the case of sparse vectors in $\mathbb{R}^n$.

   However, extension to any low-complexity signals is possible
   (with small "dimension", that is *Gaussian mean width*)