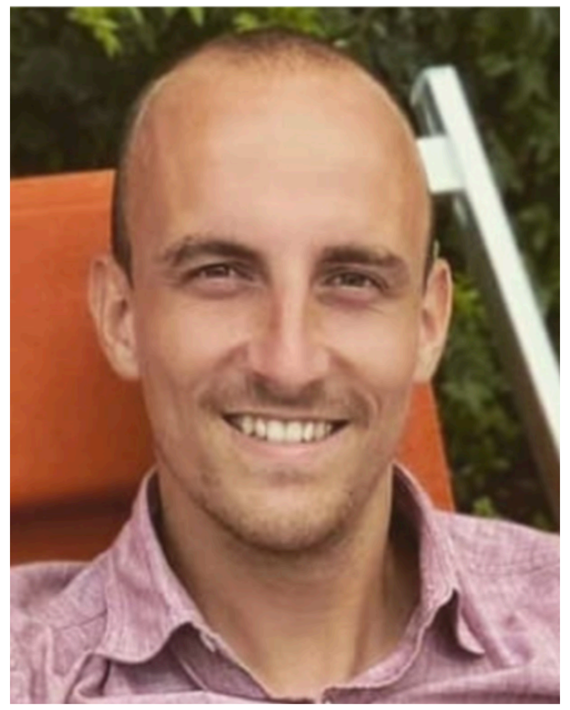
Rank-one projections for compressive radio interferometric imaging

Laurent Jacques

INMA, ICTEAM, UCLouvain, Belgium. IAS, Collegium, Lyon, France



O. Leblanc*

Y. Wiaux†

T. Chu†

L. Jacques*

*: ISPGroup, INMA, UCLouvain, Belgium. †: Heriot Watt, UK.

Introduction to random projections and compressive sensing

The multiple use of random projections in “data science”

Object

“$\boldsymbol{x}$”

signal
audio
volume
image
distribution
measure

$\xrightarrow{\boldsymbol{\Phi}}$

Random projection

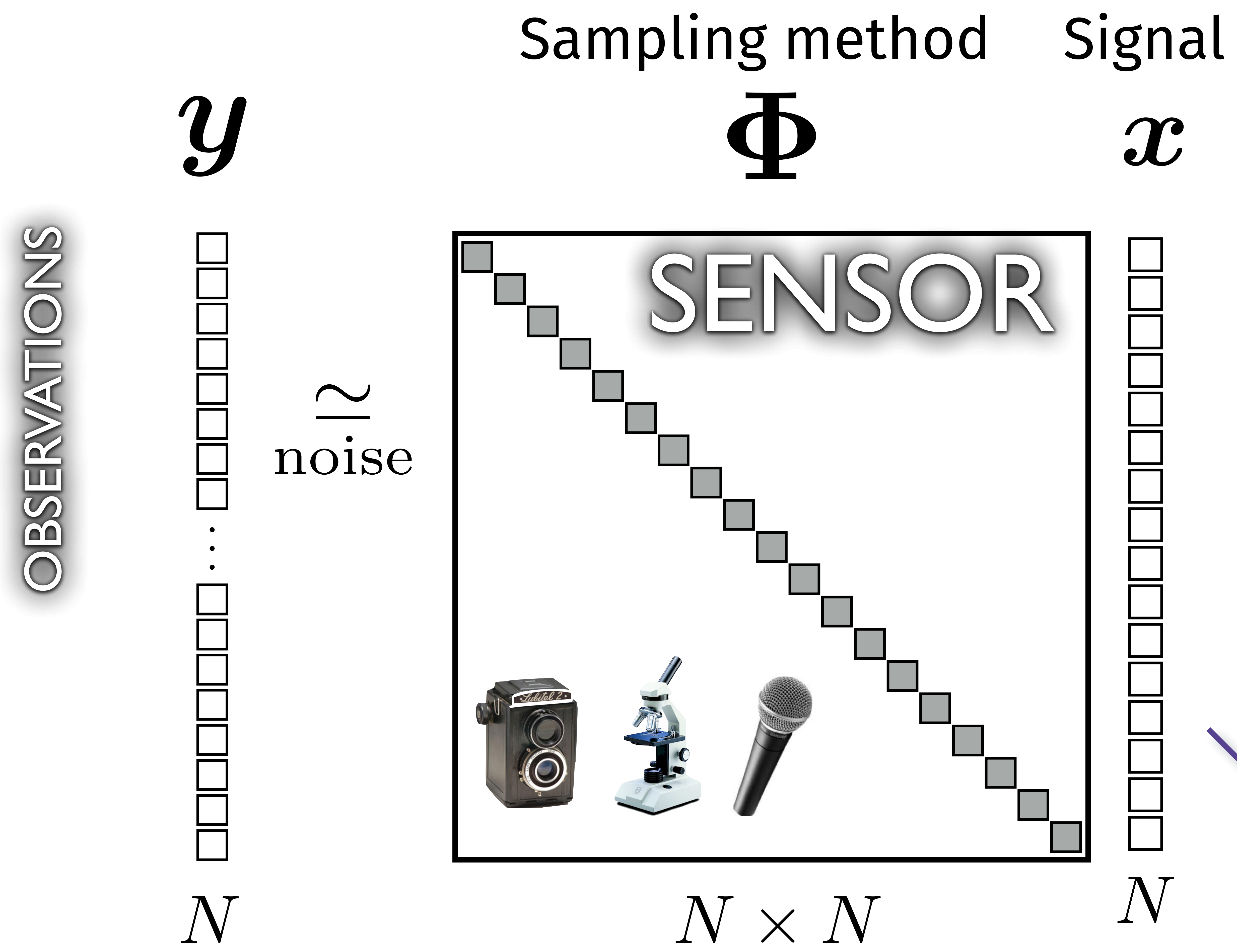
$$\boldsymbol{y} = \boldsymbol{\Phi}(\boldsymbol{x})$$

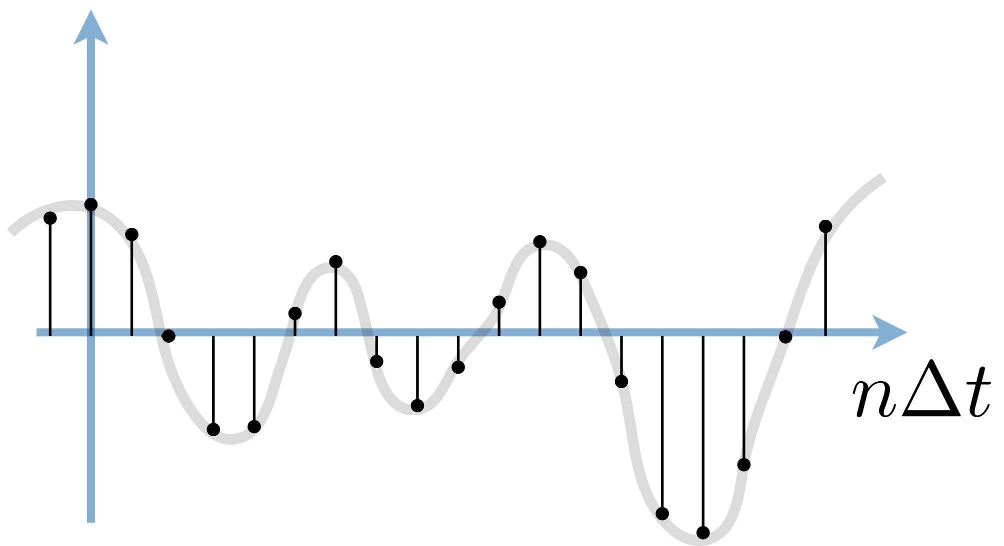
smaller dimension
physics-driven
distinctive
compressed
binarized

Random “projections" are ubiquitous in:

- Data mining & dimensionality reduction techniques
- Sensing and imaging methods (optics, astronomy, ...)
- Machine learning (sketching, explicit kernel, initialization, ...)
- Randomized numerical methods, ...

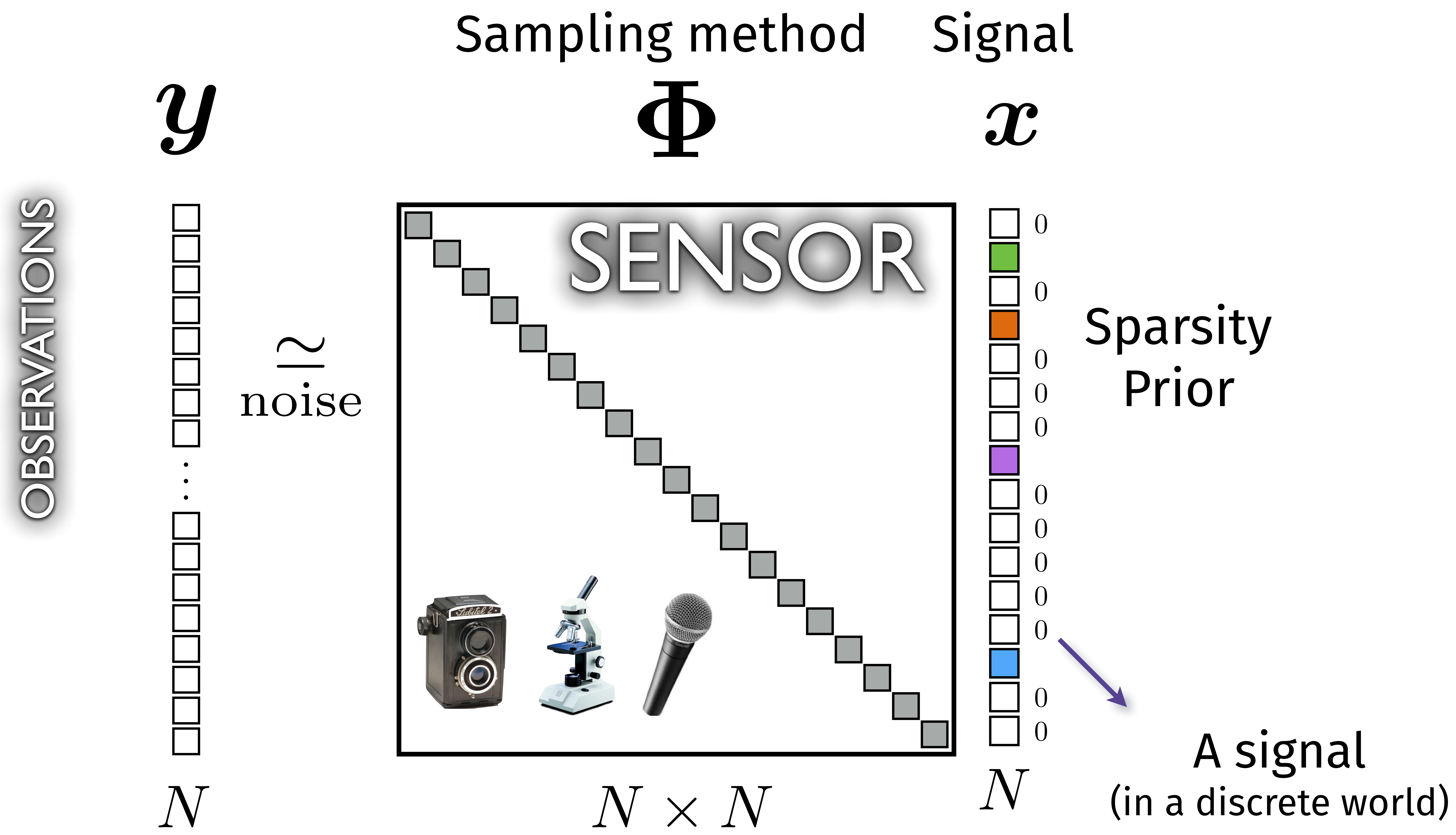
From “Nyquist” Sensing...





A signal
(in a discrete world)

From "Nyquist" Sensing...



If $\boldsymbol{x}$ is *sparse*, $\boldsymbol{\Phi}$ is a wasteful sensing process!

… to compressive sensing

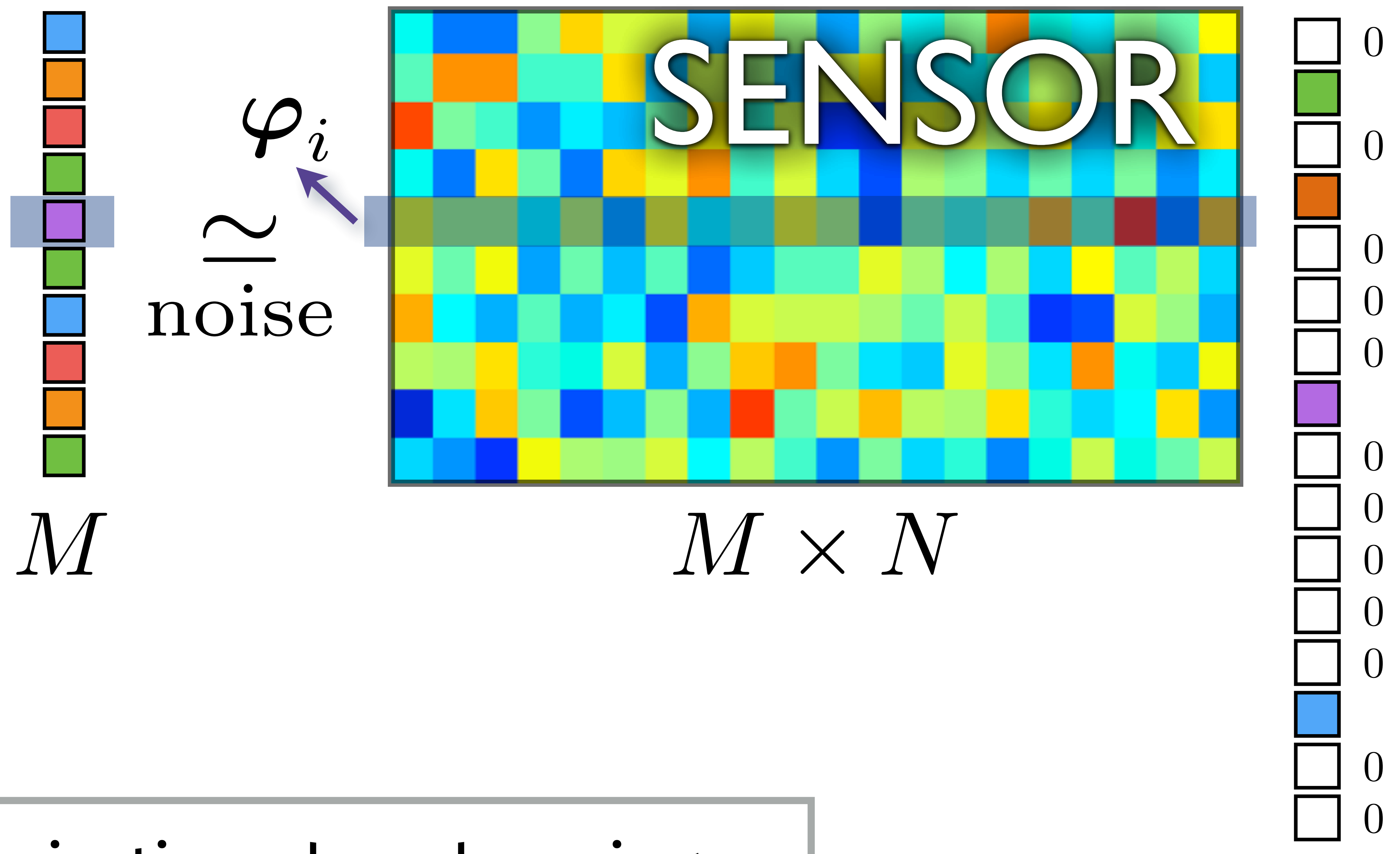
OBSERVATIONS

Sensing method   Signal

$\boldsymbol{y}$ $\boldsymbol{\Phi}$ $\boldsymbol{x}$

Generalized linear sensing

$$y_i \simeq \langle \boldsymbol{\varphi}_i, \boldsymbol{x} \rangle = \boldsymbol{\varphi}_i^T \boldsymbol{x}$$

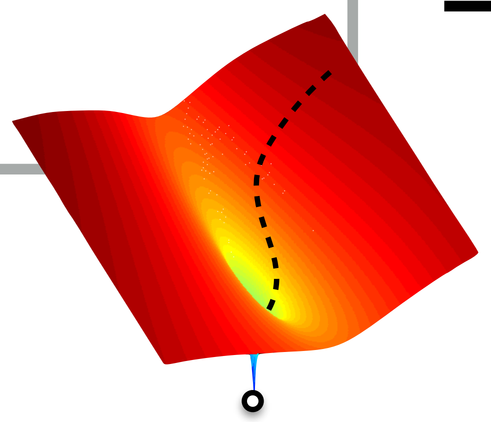


M $M \times N$ N

Sparsity Prior

$M \simeq$ d.o.f. $(\boldsymbol{x})$

Optimization, deep learning, greedy algorithms

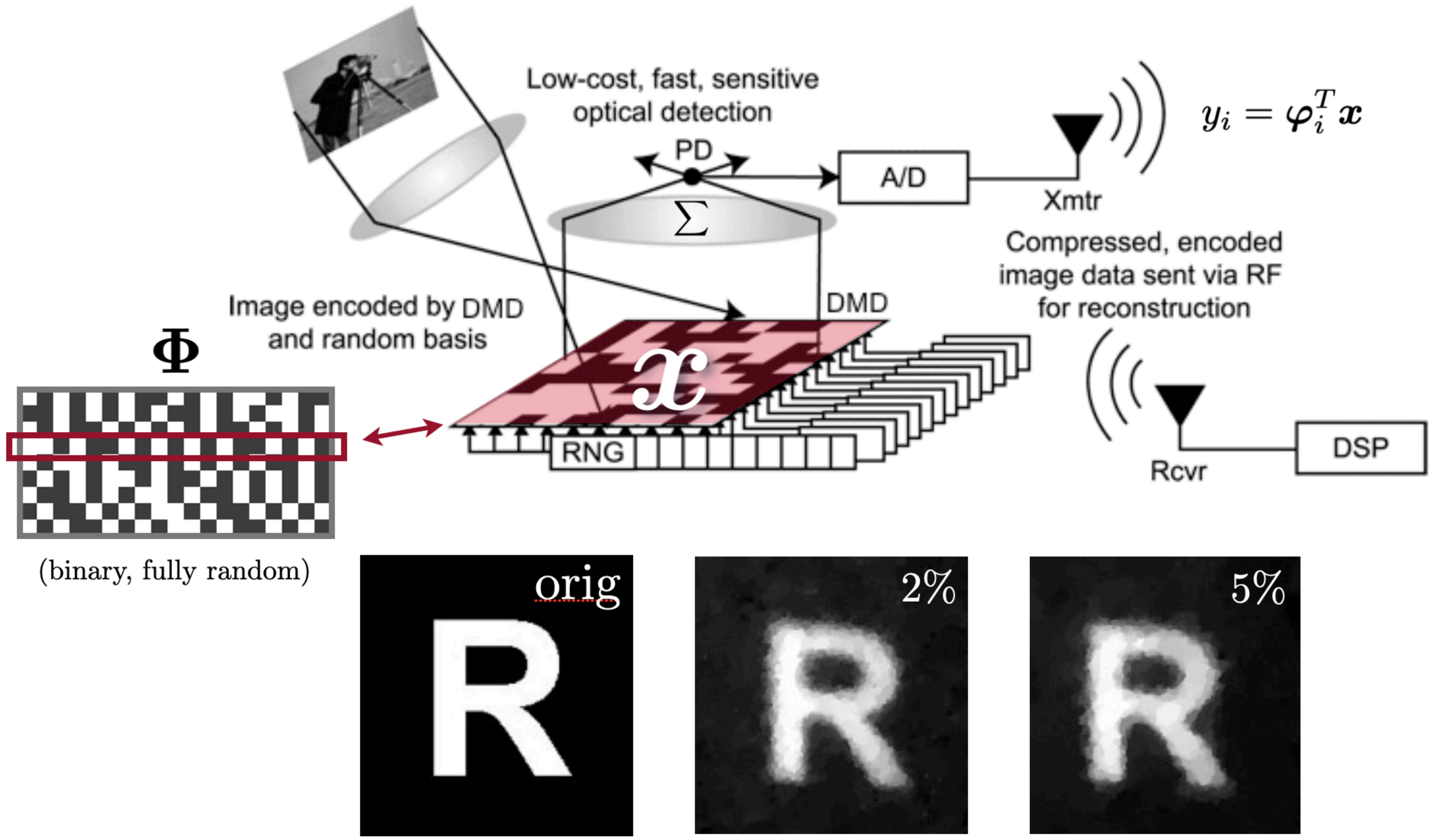


Prior (sparsity)

$\tilde{\boldsymbol{x}}$ (estimate)

Examples of compressive sensors

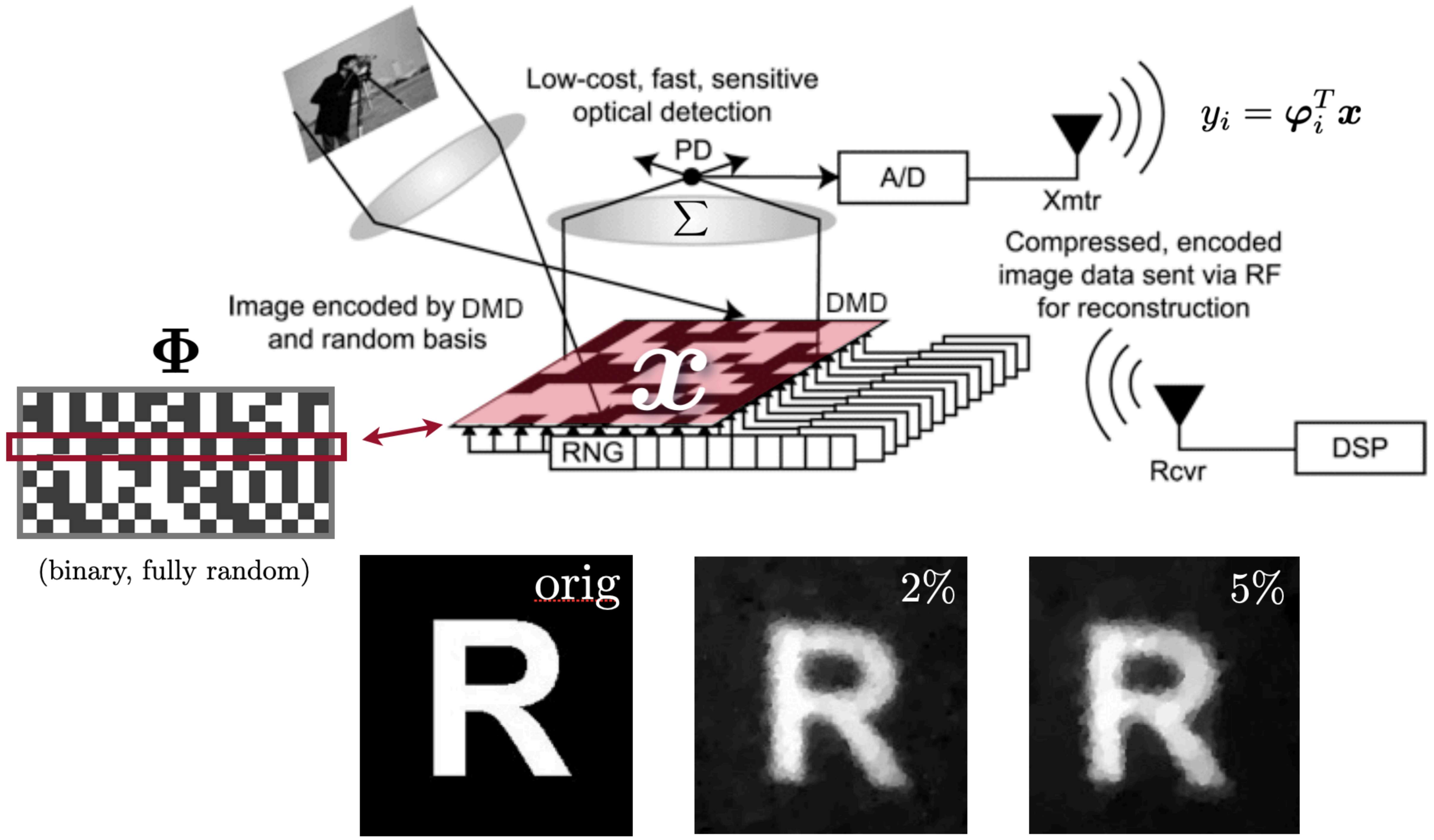
"Rice One-pixel Camera"



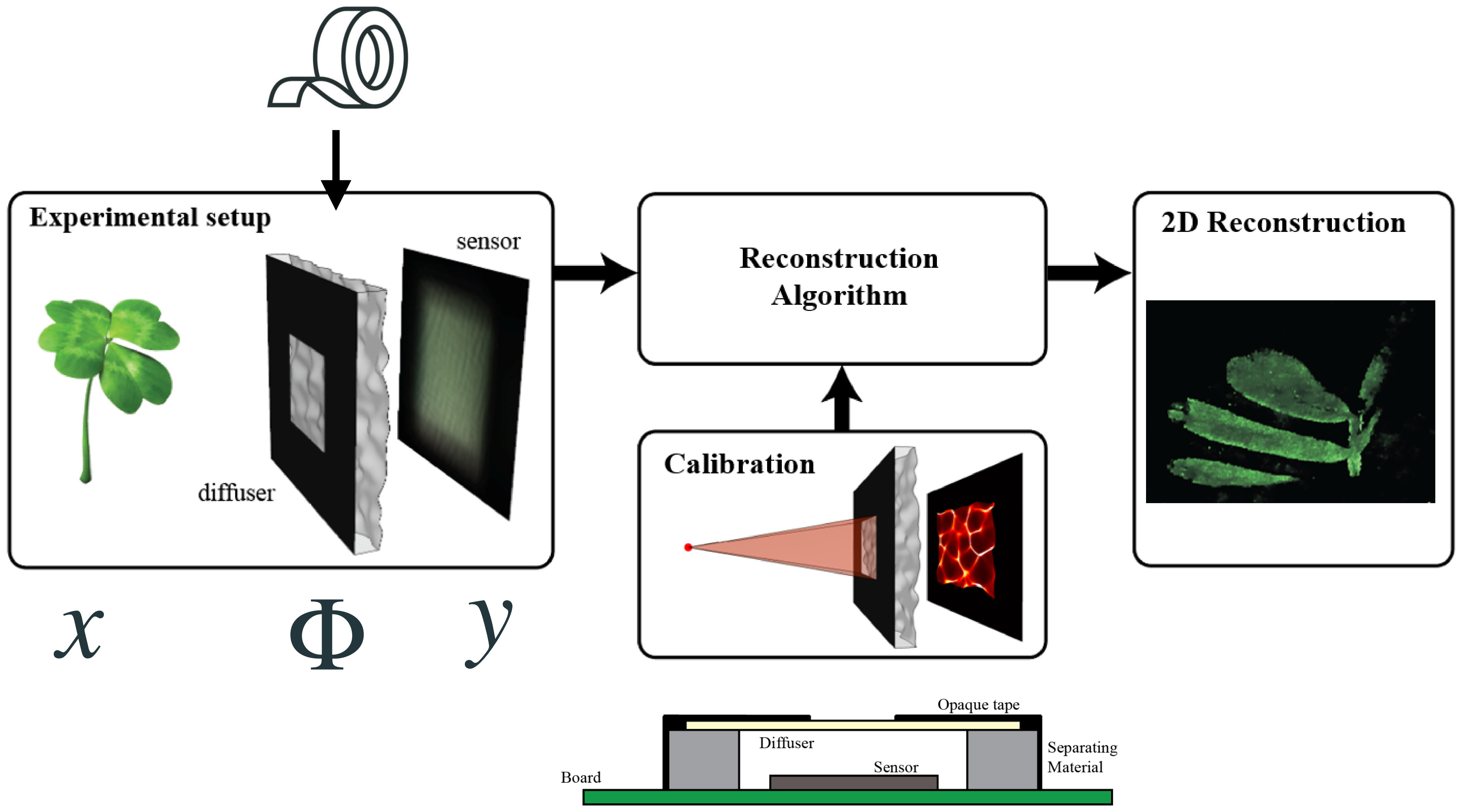
M. F. Duarte, M. A. Davenport, D. Takhar, J. N. Laska, T. Sun, K. F. Kelly and R. G. Baraniuk (2008) Single-pixel imaging via compressive sampling

Examples of compressive sensors

"Rice One-pixel Camera"



"Diffuser Cam"



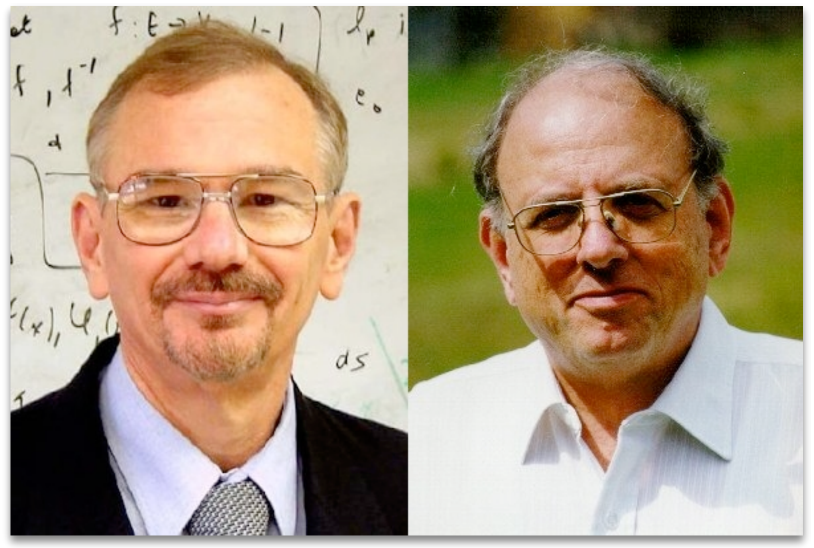
M. F. Duarte, M. A. Davenport, D. Takhar, J. N. Laska, T. Sun, K. F. Kelly and R. G. Baraniuk (2008) Single-pixel imaging via compressive sampling

N. Antipa, G. Kuo, R. Heckel, B. Mildenhall, E. Bostan, R. Ng, L. Waller, L. (2017). DiffuserCam: lensless single-exposure 3D imaging.

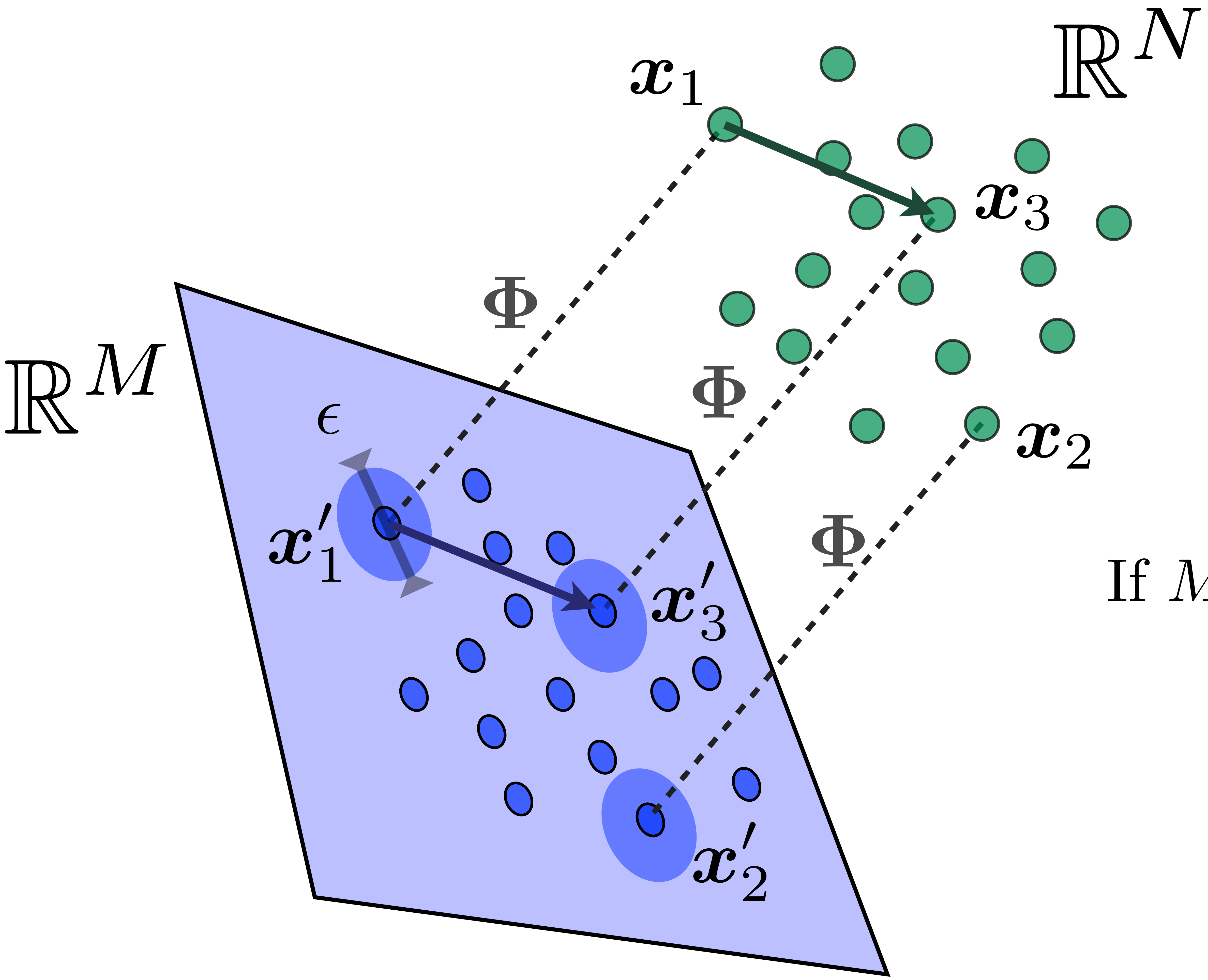
Why is it working? $\to$ embedding of low-complexity signals

Johnson-Lindenstrauss Lemma (1984)

For many random $M \times N$ matrices $\mathbf{\Phi}$ (e.g., Gaussian, Bernoulli, structured)



Johnson & Lindenstrauss



D points

$\mathcal{S} = \{\boldsymbol{x}_i : 1 \leqslant i \leqslant D\}$

Prior Information

If $M \geqslant C \log(D)$, then, with high probability,

$$\|\mathbf{\Phi}\boldsymbol{x}_i - \mathbf{\Phi}\boldsymbol{x}_j\| \approx_\epsilon \|\boldsymbol{x}_i - \boldsymbol{x}_j\|, \ \forall i, j$$

Why is it working? $\rightarrow$ embedding of low-complexity signals

For many random $M \times N$ matrices $\boldsymbol{\Phi}$ (e.g., Gaussian, Bernoulli, structured)

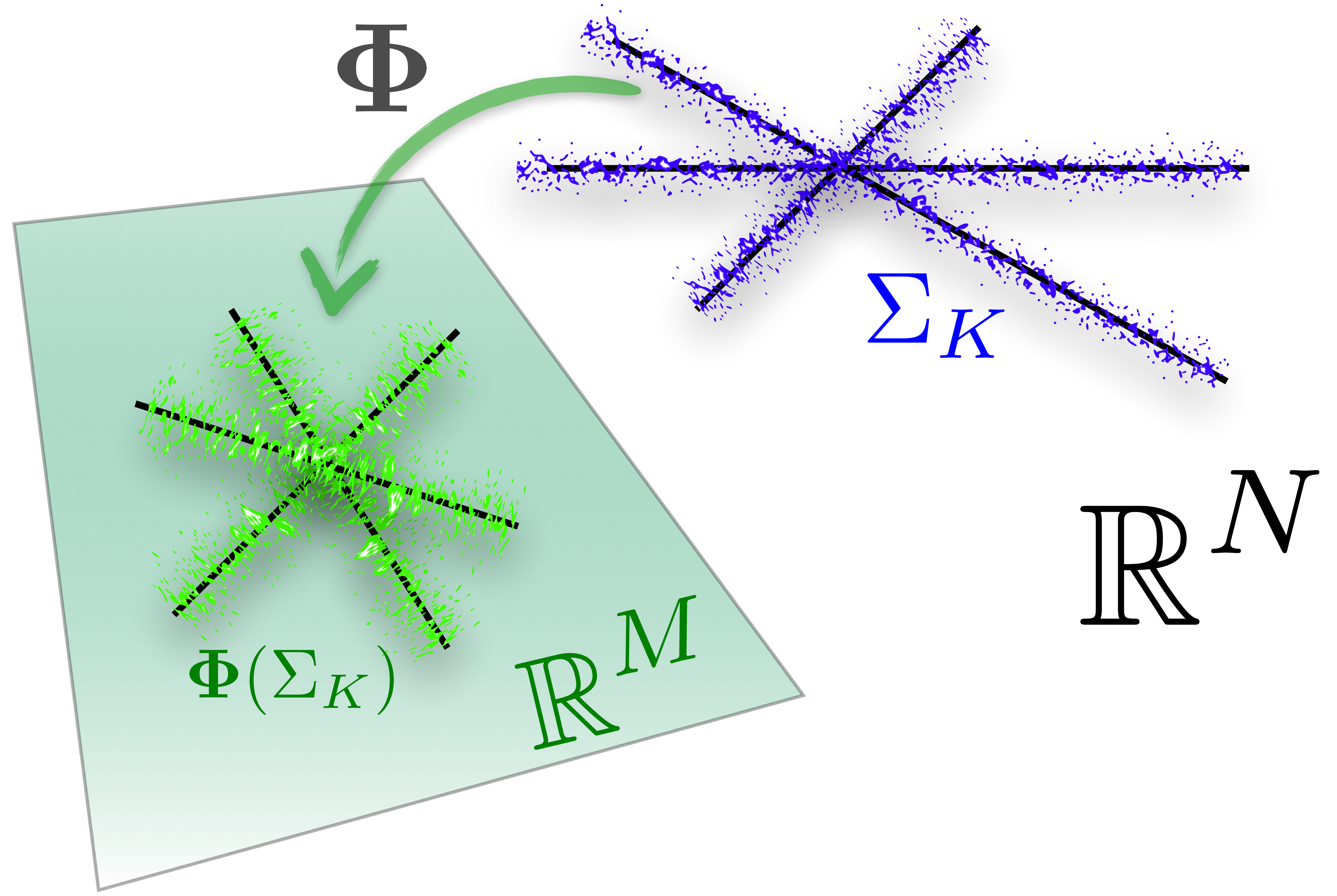
$$M \gtrsim \text{d.o.f.}(\Sigma_K) \asymp K \log(N/K)$$

$\Downarrow$ (with high probability) $\Downarrow$

$$\text{Geometry of } \boldsymbol{\Phi}(\Sigma_K) \approx \text{Geometry of } \Sigma_K$$

$$\underbrace{\boldsymbol{\Phi x} \approx \boldsymbol{\Phi x}'}_{\text{observations}} \;\Leftrightarrow\; \underbrace{\boldsymbol{x} \approx \boldsymbol{x}'}_{\text{true signals}}$$

with $\boldsymbol{x}, \boldsymbol{x}' \in \Sigma_K := \{\boldsymbol{u} : \|\boldsymbol{u}\|_0 = |\operatorname{supp}(\boldsymbol{u})| \leq K\}$

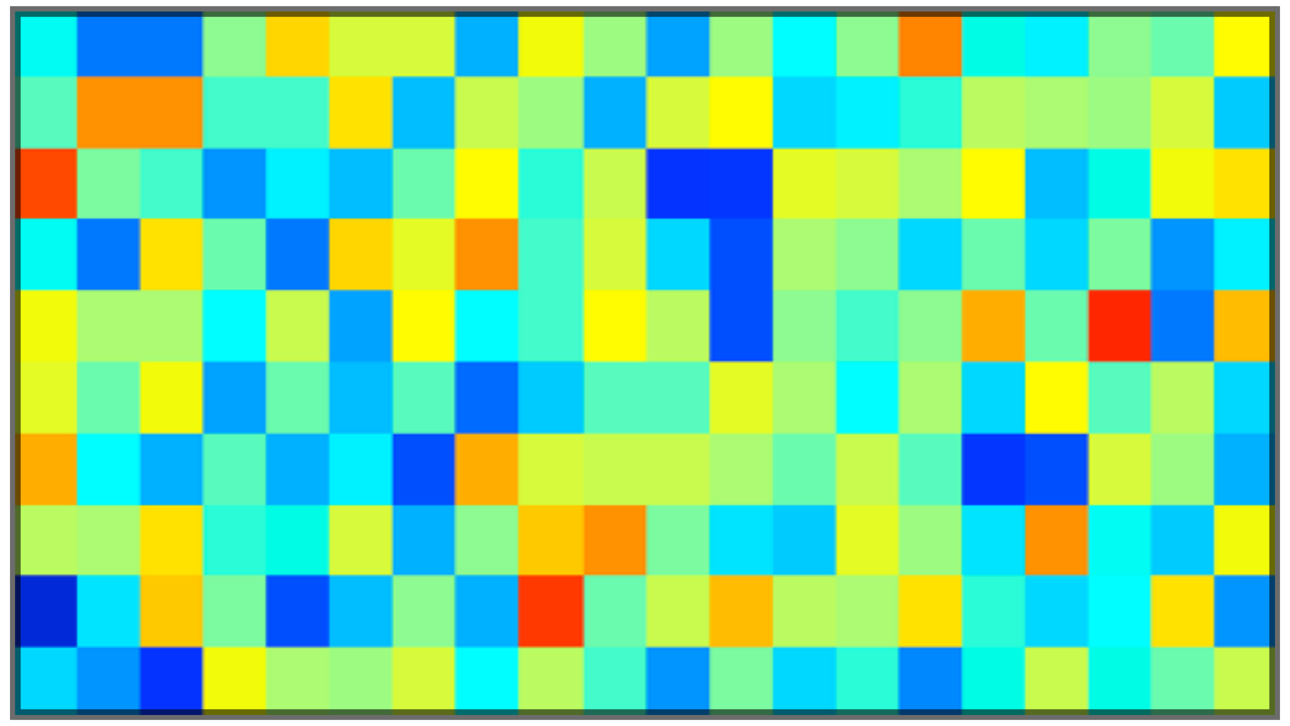


+ Many extensions to other sparsity models, low-rank matrices, ...

Structured random projections

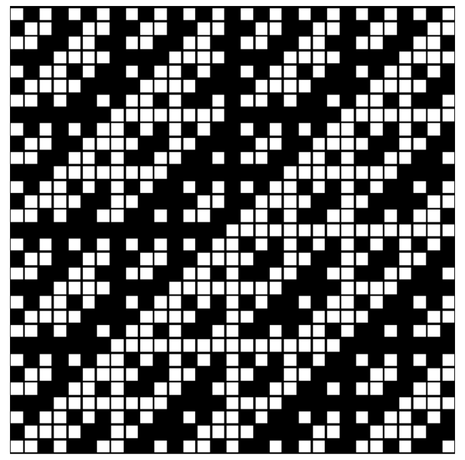
Challenge: dense matrices $\mathbf{\Phi}$ not optimal for:

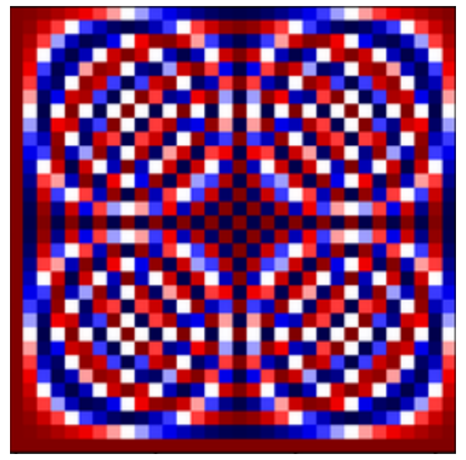
- memory and computational complexity
- physically friendly implementation
- sensing higher dimensional objects



Other solutions:

- Fourier (FFT) or Hadamard matrices (or derivatives)





$\longrightarrow$ random subsampling & modulation $\longrightarrow$ $\mathbf{\Phi}$

- Rank-one projections (ROP)

Focus on rank-one projections (ROP)

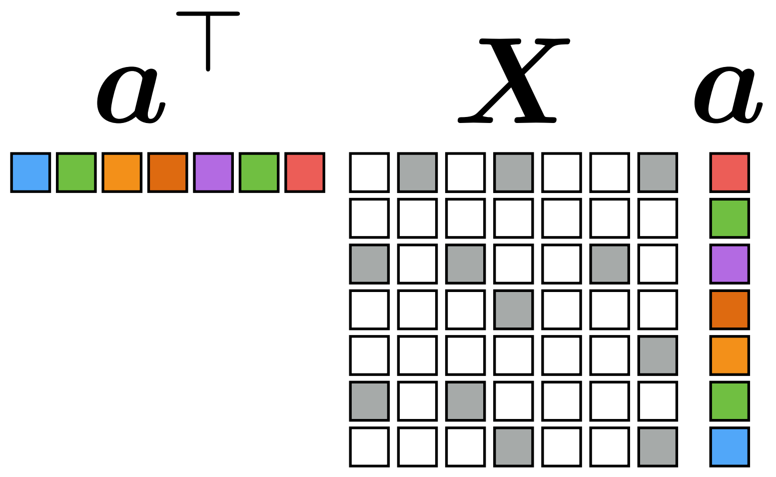
Object to project = symmetric $n \times n$ matrices $\boldsymbol{X} \in \mathbb{R}^{n\times n}$

e.g., image, volume, covariance matrices, ...

Projection with m random vectors $\{\boldsymbol{a}_j \sim_{\text{iid}} \boldsymbol{a}\}_{j=1}^m \subset \mathbb{R}^n$

(e.g., Gaussian)

$$\boldsymbol{y} := \boldsymbol{\Phi}(\boldsymbol{X}) := \Big(\underbrace{\boldsymbol{a}_j^\top \boldsymbol{X} \boldsymbol{a}_j}_{\langle \underbrace{\boldsymbol{a}_j \boldsymbol{a}_j^\top}_{\text{rank-one}}, \boldsymbol{X} \rangle_F} \Big)_{j=1}^m \in \mathbb{R}^m$$



Chen & Goldsmith (2015) Cai & Zhang (2015)

Focus on rank-one projections (ROP)

Object to project = symmetric $n \times n$ matrices $\boldsymbol{X} \in \mathbb{R}^{n \times n}$

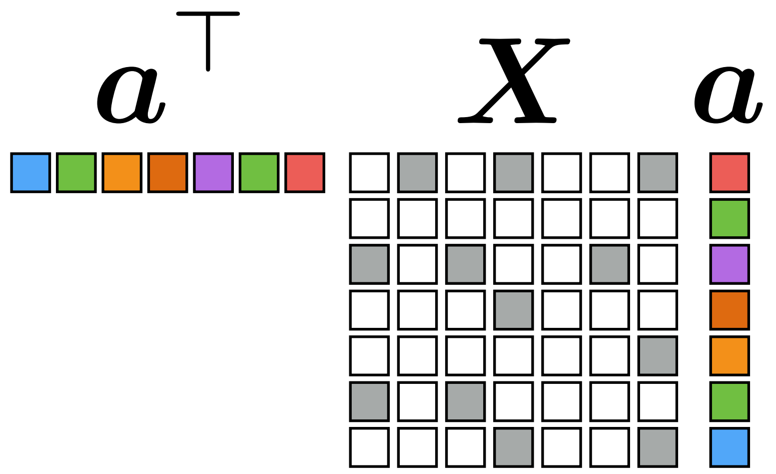
e.g., image, volume, covariance matrices, ...

Projection with m random vectors $\{\boldsymbol{a}_j \sim_{\text{iid}} \boldsymbol{a}\}_{j=1}^{m} \subset \mathbb{R}^n$

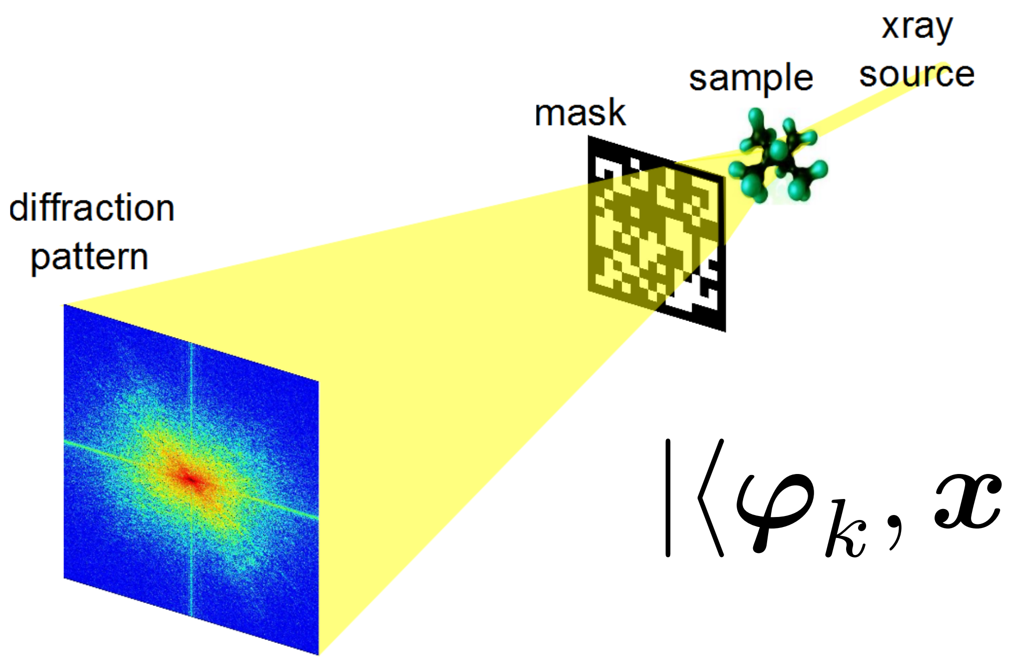
(e.g., Gaussian)

$$\boldsymbol{y} := \boldsymbol{\Phi}(\boldsymbol{X}) := \big(\underbrace{\boldsymbol{a}_j^\top \boldsymbol{X} \boldsymbol{a}_j}_{\langle \boldsymbol{a}_j \boldsymbol{a}_j^\top, \boldsymbol{X} \rangle_F} \big)_{j=1}^{m} \in \mathbb{R}^m$$

rank-one



Phase retrieval



$$|\langle \boldsymbol{\varphi}_k, \boldsymbol{x} \rangle|^2 = \boldsymbol{\varphi}_k^* (\boldsymbol{x}\boldsymbol{x}^*) \boldsymbol{\varphi}_k$$

Covariance matrix estimation

$$\begin{aligned} \mathcal{A}(\mathbb{E}\boldsymbol{x}\boldsymbol{x}^\top) &\approx \mathcal{A}(\tfrac{1}{N} \textstyle\sum_k \boldsymbol{x}_k \boldsymbol{x}_k) \\ &= \tfrac{1}{N} \textstyle\sum_k [(\boldsymbol{a}_j^\top \boldsymbol{x}_k)^2]_{j=1}^{m} \end{aligned}$$

for $\boldsymbol{x}_k \sim_{\text{iid}} \boldsymbol{x}$

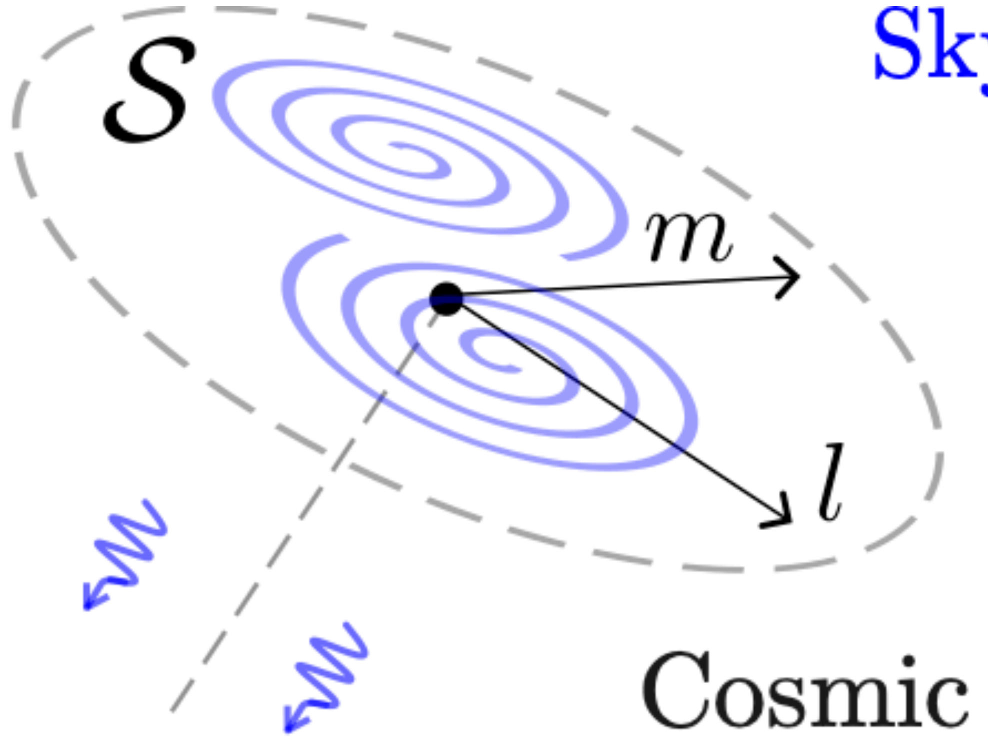
Chen & Goldsmith (2015) Cai & Zhang (2015)

Acquisition and imaging models in radio astronomy

Radio interferometric sensing model

$\mathcal{Q}$ antennas focused on

a (small) region $\mathscr{S}$ of the sky



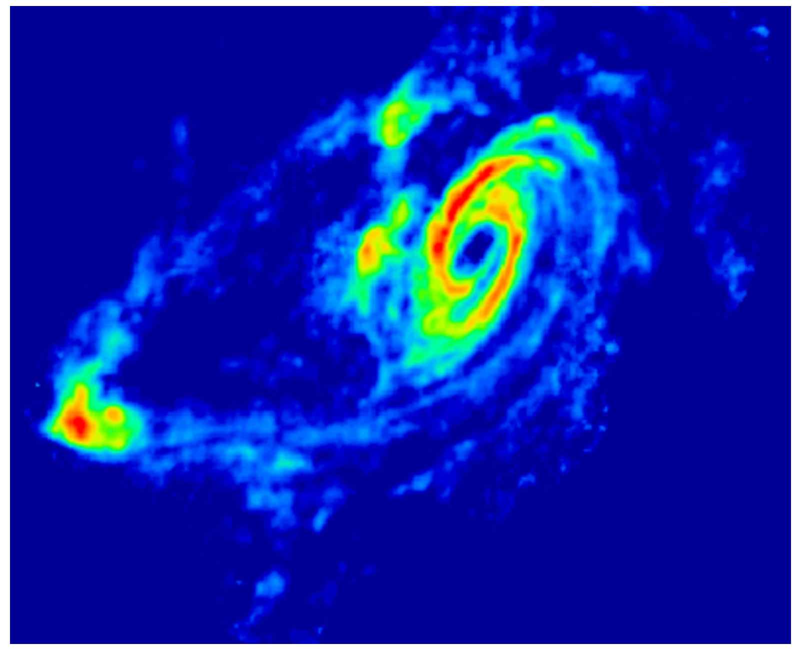
Sky intensity distribution

$$\sigma^2(\boldsymbol{l}) = \sigma^2(l, m)$$

Cosmic signals

$$s(\boldsymbol{l}, t) \underset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, \sigma^2(\boldsymbol{l}))$$

Stochastic



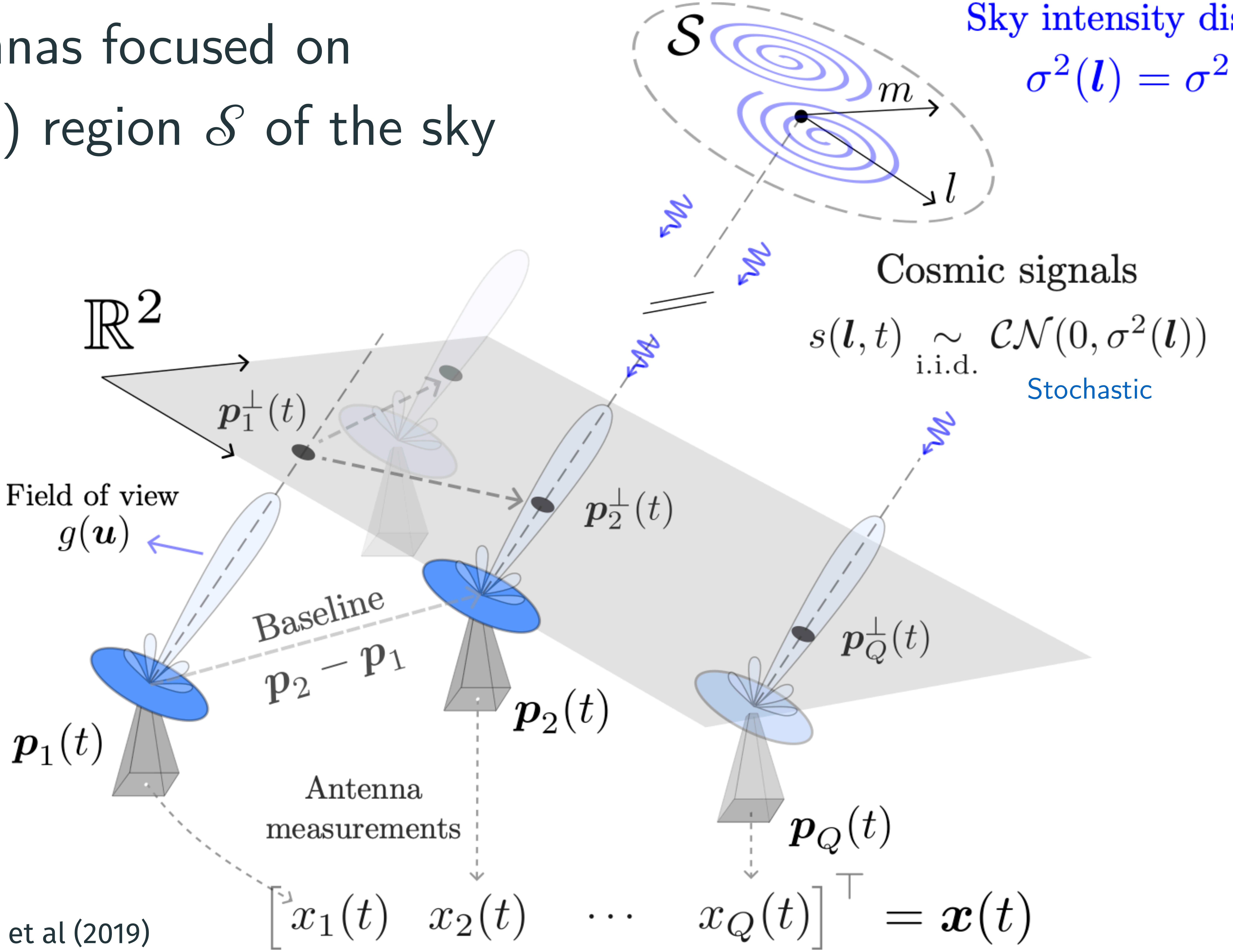
$$\sigma^2(\boldsymbol{l})$$

Arcminute Microkelvin Imager (AMI)

Van Der Veen et al (2019)

Radio interferometric sensing model

$\mathcal{Q}$ antennas focused on
a (small) region $\mathcal{S}$ of the sky



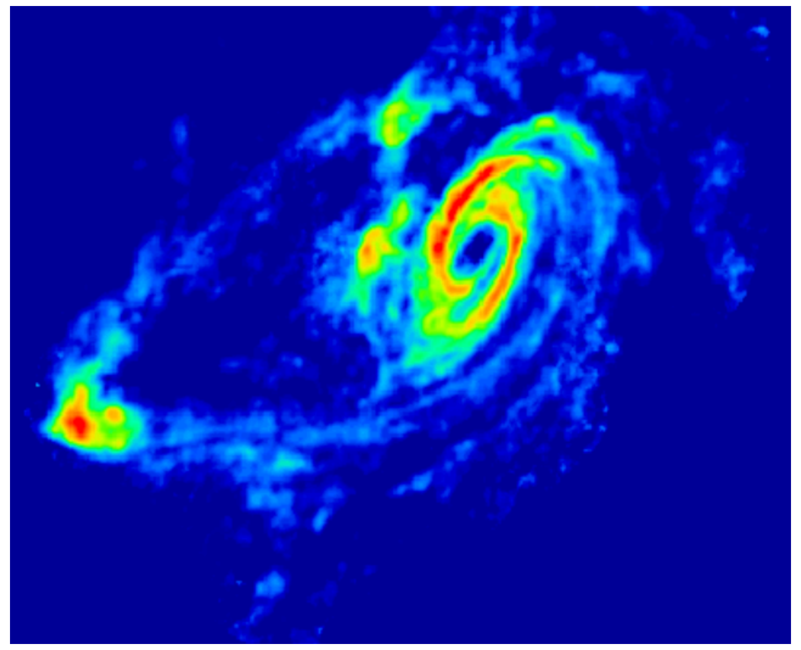
Sky intensity distribution

$$\sigma^2(\boldsymbol{l}) = \sigma^2(l, m)$$

Cosmic signals

$$s(\boldsymbol{l}, t) \underset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, \sigma^2(\boldsymbol{l}))$$

Stochastic



$$\sigma^2(\boldsymbol{l})$$

Arcminute Microkelvin Imager (AMI)

Van Der Veen et al (2019)

Radio interferometric sensing model

Sensing at q-th antenna signal: given the cosmic signal $s(\boldsymbol{l}, t) \underset{\text{i.i.d.}}{\sim} \mathcal{CN}\left(0, \sigma^2(\boldsymbol{l})\right)$

Stochastic $$\underbrace{x_q(t)}_{\text{signal}} = \int_{\mathbb{R}^2} \underbrace{s(\boldsymbol{l}, t)}_{\text{cosmic signal}} \underbrace{g(\boldsymbol{l})}_{\text{FOV}} \exp\left(\frac{\mathrm{i}2\pi}{\lambda} \underbrace{\boldsymbol{p}_q^{\perp}(t)^{\top}\boldsymbol{l}}_{\text{geometric delay}}\right) \mathrm{d}\boldsymbol{l}$$

$\Rightarrow$ *Fourier transform* of a stochastic signal on frequencies $\Omega(t) = \{\boldsymbol{p}_q^{\perp}(t)\}_{q=1}^{Q}$

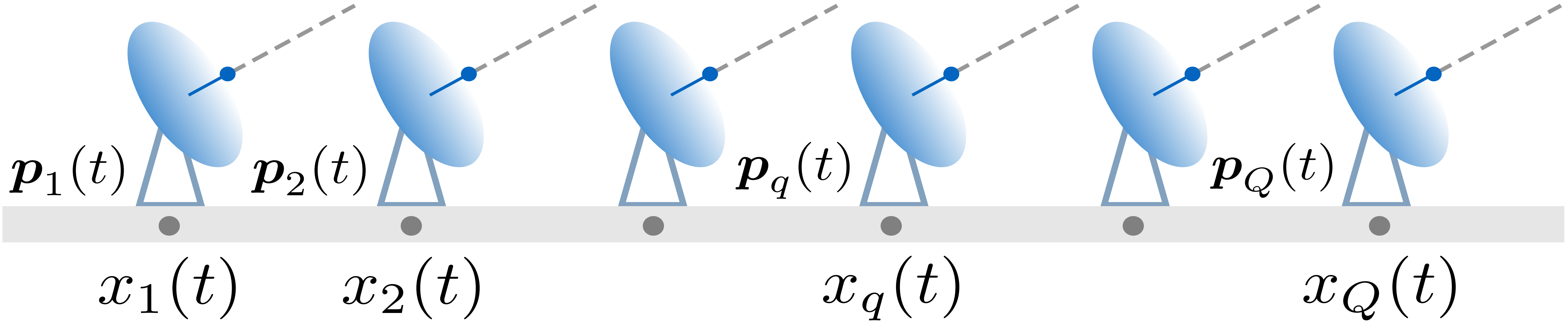
Radio interferometric sensing model

Sensing at q-th antenna signal: given the cosmic signal $s(\boldsymbol{l}, t) \underset{\text{i.i.d.}}{\sim} \mathcal{CN}\left(0, \sigma^2(\boldsymbol{l})\right)$

Stochastic
$$\underbrace{x_q(t)}_{\text{signal}} = \int_{\mathbb{R}^2} \underbrace{s(\boldsymbol{l}, t)}_{\text{cosmic signal}} \, \underbrace{g(\boldsymbol{l})}_{\text{FOV}} \exp\left(\frac{\mathrm{i}2\pi}{\lambda} \, \underbrace{\boldsymbol{p}_q^{\perp}(t)^{\top}\boldsymbol{l}}_{\text{geometric delay}}\right) \mathrm{d}\boldsymbol{l}$$

$\Rightarrow$ *Fourier transform* of a stochastic signal on frequencies $\Omega(t) = \{\boldsymbol{p}_q^{\perp}(t)\}_{q=1}^{Q}$

Radio-interferometry principle: let's make correlation/interference of antenna signals



$$\boldsymbol{x}(t) = (x_1(t), \ldots, x_Q(t))^{\top}$$

$\longrightarrow$

$$C_{ij}(t) := \mathbb{E}[x_i(t) x_j^*(t)]$$

$$\boldsymbol{C}(t) := \mathbb{E}[\boldsymbol{x}(t)\boldsymbol{x}^*(t)]$$

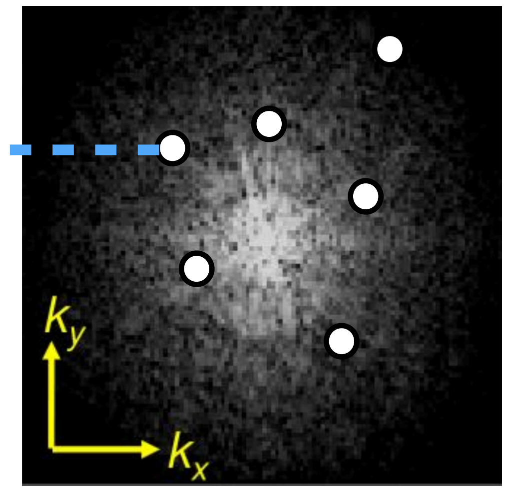
$Q \times Q$ covariance

Radio interferometric sensing model

By the Van Cittert-Zernike theorem (VCZ): $\boldsymbol{x}(t) = (x_1(t), \ldots, x_Q(t))^\top$

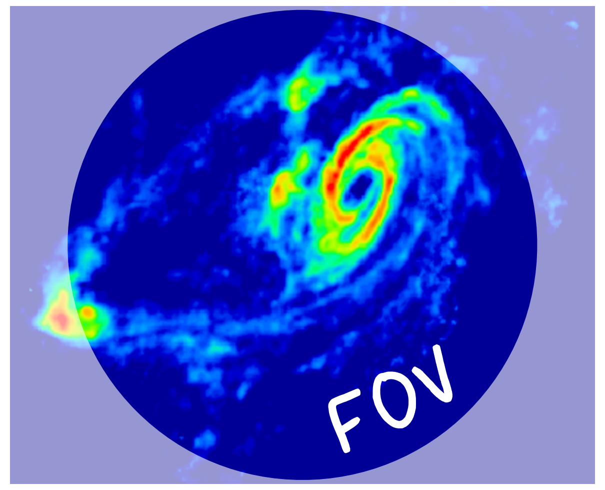
$\sigma^\circ(\boldsymbol{l})$

$$\boldsymbol{C}(t) := \mathbb{E}[\boldsymbol{x}(t)\boldsymbol{x}^*(t)] = \mathcal{I}_{\Omega(t)}[\sigma^\circ]$$



$\mathcal{F}[\sigma^\circ]$

FT $\leftarrow$



$g^2(\boldsymbol{l})\sigma^2(\boldsymbol{l})$

visibilities $\boldsymbol{\nu}_{kj}$

2D Fourier

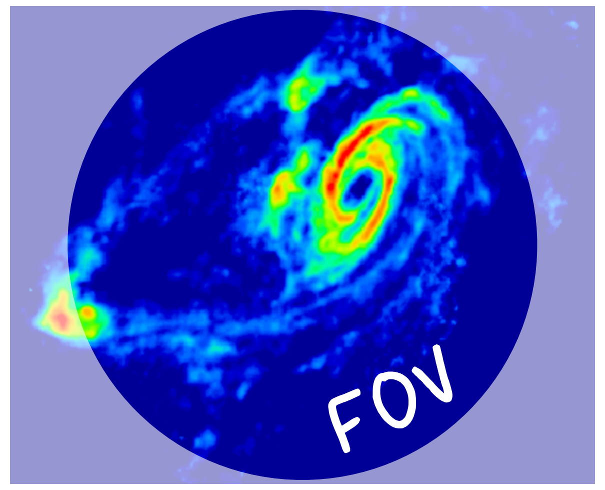
$$\text{with} \quad \big(\mathcal{I}_{\Omega(t)}(\sigma^\circ)\big)_{jk} := \mathcal{F}[\sigma^\circ]\Bigg(\underbrace{\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}}_{\in \mathcal{V} := \lambda^{-1}(\Omega - \Omega)}\Bigg)$$

$\Omega(t) = \{\boldsymbol{p}_q^\perp(t)\}_{q=1}^{Q}$

Radio interferometric sensing model

By the Van Cittert-Zernike theorem (VCZ): $\boldsymbol{x}(t) = (x_1(t), \dots, x_Q(t))^\top$

$\sigma^\circ(\boldsymbol{l})$



$g^2(\boldsymbol{l})\sigma^2(\boldsymbol{l})$

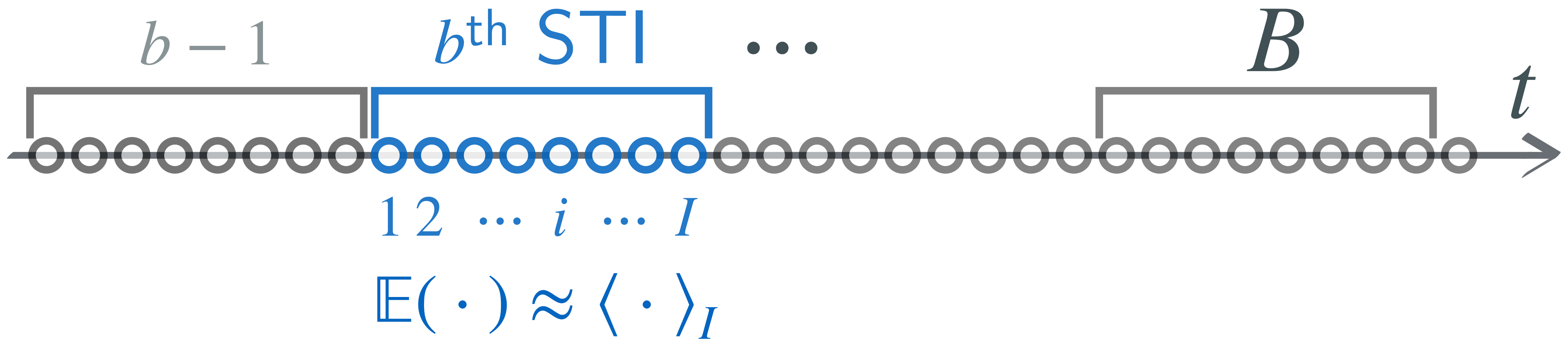
$$\boldsymbol{C}_b := \underbrace{\tfrac{1}{I}\sum_{i=1}^{I}\boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]}_{\text{STI} \to \langle \boldsymbol{x}_b[\cdot]\boldsymbol{x}_b^*[\cdot]\rangle_I} = \mathcal{I}_{\Omega_b}[\sigma^\circ]$$

with $$(\mathcal{I}_b(\sigma^\circ))_{jk} := \overset{\text{2D Fourier}}{\mathcal{F}[\sigma^\circ]}\Bigg(\underbrace{\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}}_{\in \mathcal{V}_b := \lambda^{-1}(\Omega_b - \Omega_b)}\Bigg)$$

In practice $\to$ time averaging

- B short-time integration intervals (STI) with I discrete time instants :

  $$\boldsymbol{x}(t) \to \boldsymbol{x}_b[i] \in \mathbb{R}^Q, \ \ \mathbb{E}(\,\cdot\,) \approx \langle\,\cdot\,\rangle_I$$



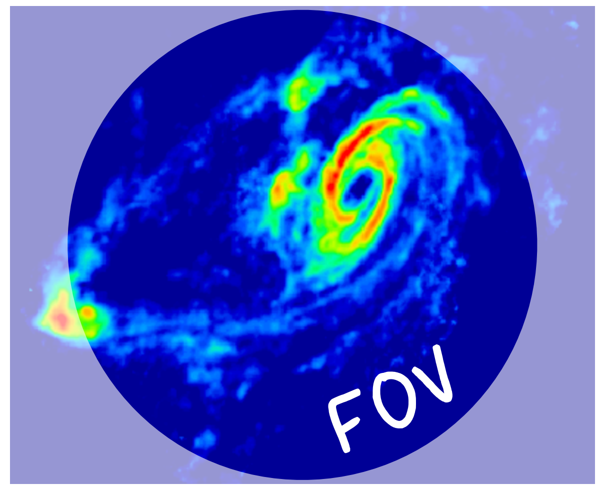
- Approx: over each STI, Earth & visibilities are fixed

Radio interferometric sensing model

By the Van Cittert-Zernike theorem (VCZ): $\boldsymbol{x}(t) = (x_1(t), \ldots, x_Q(t))^\top$

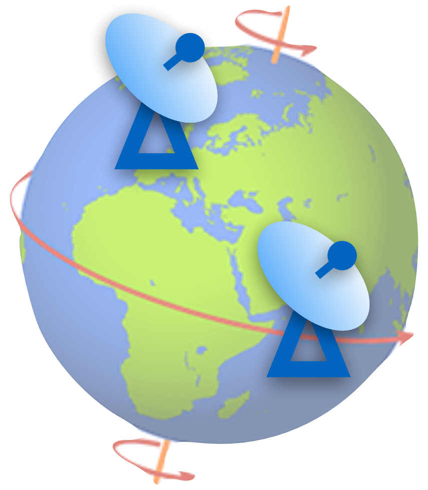
$$\boldsymbol{C}_b := \underbrace{\tfrac{1}{I}\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]}_{\text{STI } \to \ \langle \boldsymbol{x}_b[\cdot]\boldsymbol{x}_b^*[\cdot]\rangle_I} = \quad \mathcal{I}_{\Omega_b}[\sigma^\circ]$$

$\sigma^\circ(\boldsymbol{l})$



$g^2(\boldsymbol{l})\sigma^2(\boldsymbol{l})$

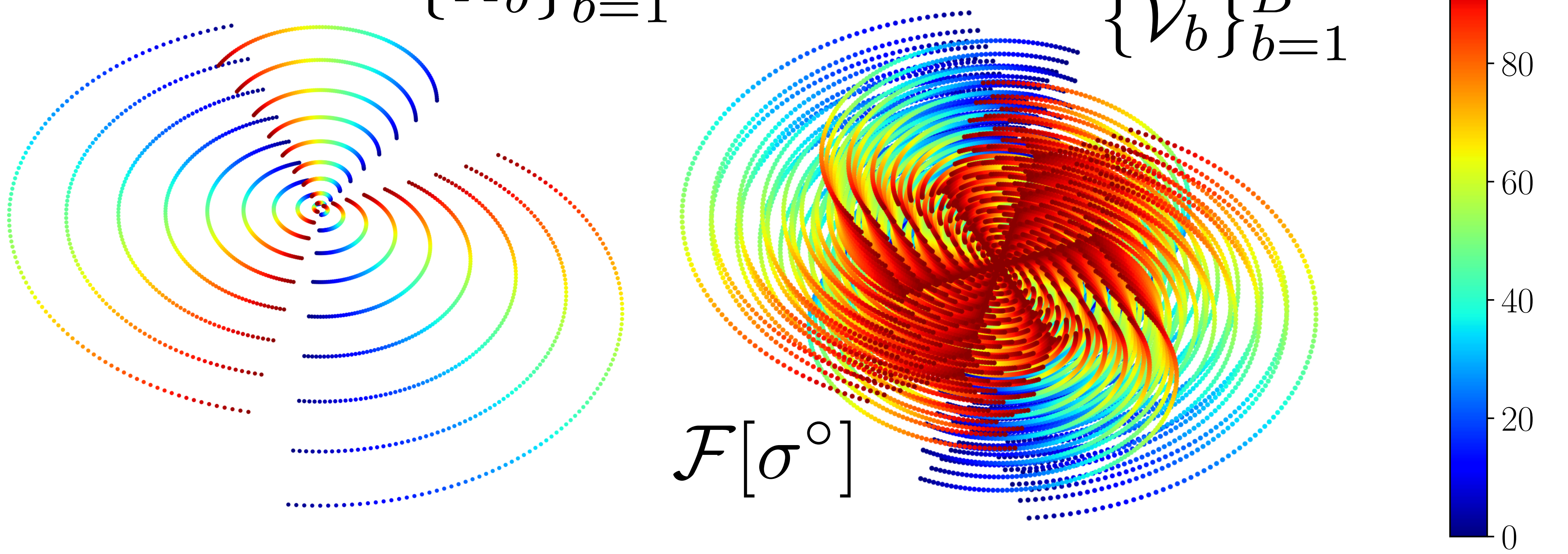
Eppur si muove



with $$(\mathcal{I}_b(\sigma^\circ))_{jk} := \overset{\text{2D Fourier}}{\mathcal{F}[\sigma^\circ]}\Big(\underbrace{\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}}_{\in \mathcal{V}_b := \lambda^{-1}(\Omega_b - \Omega_b)}\Big)$$

$\Downarrow$

$\{\Omega_b\}_{b=1}^B$ Fourier plane $\{\mathcal{V}_b\}_{b=1}^B$



$\mathcal{F}[\sigma^\circ]$

$O(Q^2)$ visibilities per STI
$\to O(Q^2 B)$ in total

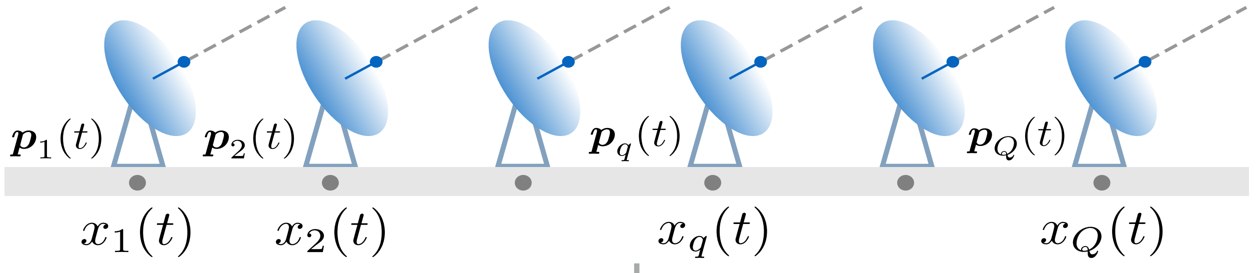
In practice $\to$ time averaging

- B short-time integration intervals (STI) with I discrete time instants :
  $\boldsymbol{x}(t) \to \boldsymbol{x}_b[i] \in \mathbb{R}^Q, \ \ \mathbb{E}(\,\cdot\,) \approx \langle\,\cdot\,\rangle_I$
- Approx: over each STI, Earth & visibilities are fixed

Radio interferometric sensing model

Two sensing operators :

Acquisition model



(antennas level) $\boldsymbol{x}(t)$

(QBI)

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

(correlator) $(QBI \to Q^2B)$

Covariances estimations

$$\boldsymbol{C}_b(\mathcal{X}_b) := \langle \boldsymbol{x}_b[\cdot]\boldsymbol{x}_b^*[\cdot]\rangle_I := \frac{1}{I}\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]$$

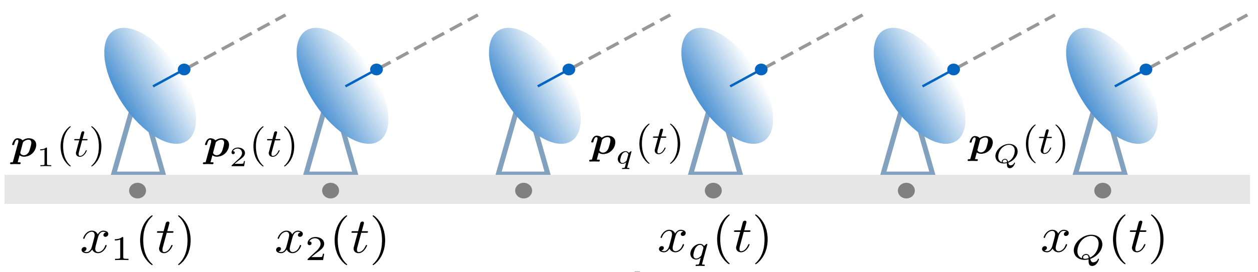
Acquisition operator

$$\mathcal{X} := \cup_b \mathcal{X}_b \to \boldsymbol{\Psi}(\mathcal{X}) := \underbrace{\bigcup_b \boldsymbol{C}_b(\mathcal{X}_b)}_{Q^2B \text{ values}}$$

Radio interferometric sensing model

Two sensing operators :

Acquisition model | **Imaging model** (for image reconstruction)



(antennas level) $\boldsymbol{x}(t)$

(*QBI*)

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

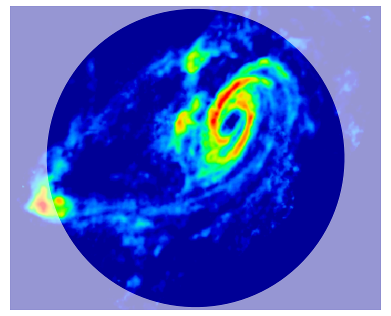
(correlator) $(QBI \to Q^2B)$

Covariances estimations

$$\boldsymbol{C}_b(\mathcal{X}_b) := \langle \boldsymbol{x}_b[\cdot]\boldsymbol{x}_b^*[\cdot]\rangle_I := \frac{1}{I}\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}_b^*[i]$$

Acquisition operator

$$\mathcal{X} := \cup_b \mathcal{X}_b \to \boldsymbol{\Psi}(\mathcal{X}) := \underbrace{\bigcup_b \boldsymbol{C}_b(\mathcal{X}_b)}_{Q^2B \text{ values}}$$

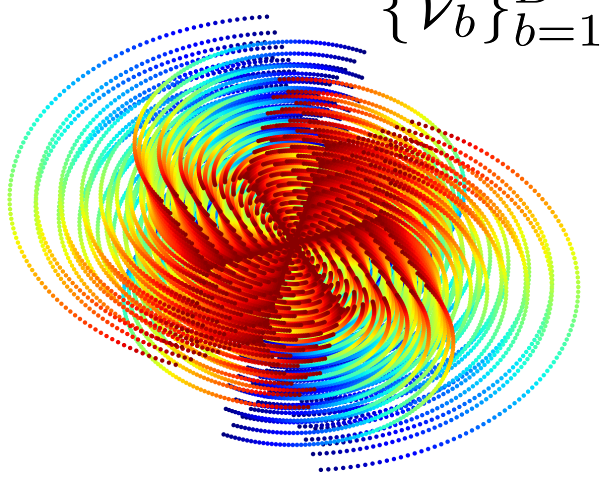


(image level) $\sigma^\circ(\boldsymbol{l})$

Sampling over a N-pixel grid

$$\boldsymbol{\sigma} \in \mathbb{R}^N$$

Given $\{\mathcal{V}_b\}_{b=1}^B =: \mathcal{V}$



Q^2B visibilities

(reconstruction level)

Imaging operator

$$\boldsymbol{\sigma} \xrightarrow[\text{NFFT}]{} \boldsymbol{\Phi}[\boldsymbol{\sigma}] = \mathcal{F}[\boldsymbol{\sigma}](\mathcal{V}) = \underbrace{\{\mathcal{I}_b(\boldsymbol{\sigma})\}_{b=1}^B}_{Q^2B \text{ values}}$$

VCZ

equal in expectation

$(N \to Q^2B)$

Imaging with radio-interferometry

Inverse problem solving:

(processed)
Interfero data

$$\text{Find a valid } \tilde{\boldsymbol{\sigma}} \text{ such that } \underbrace{\boldsymbol{\Phi}}_{\text{Imaging model}}[\tilde{\boldsymbol{\sigma}}] \approx \underbrace{\boldsymbol{\Psi}}_{\text{Acquisition model}}(\mathcal{X})$$

ill-posed problem
without priors

Prior (sparsity)

Imaging model

Acquisition model

Inverse problem solving:

(processed)
Interfero data
↓

$$\text{Find a valid } \tilde{\boldsymbol{\sigma}} \text{ such that } \boldsymbol{\Phi}[\tilde{\boldsymbol{\sigma}}] \approx \boldsymbol{\Psi}(\mathcal{X})$$

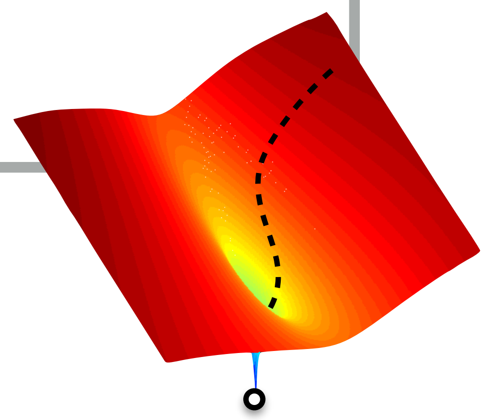
ill-posed problem
without priors

↑
Prior (sparsity)

Imaging model **Acquisition model**

$\downarrow \boldsymbol{\Phi}$

Interfero Data $\boldsymbol{\Psi}(\mathcal{X}) \longrightarrow$ Optimization, deep learning (DL), greedy algorithms $\tilde{\boldsymbol{\sigma}}$ Estimated Image



↑
Prior (sparsity)

Example of algorithms:

CLEAN (greedy, aka Matching Pursuit), Basis Pursuit (optim), uSARA (optim), R2D2 (DL)

J. A. Högbom (1974) Y. Wiaux et al (2009) R. Carrillo et al (2023) A. Aghabiglou et al (2023) ...

Challenges in radio-interferometry

Massive data stream 🐘

- $\#$visibilities $\mathscr{V} = \cup_{b=1}^{B} \mathscr{V}_b \to O(Q^2 B)$

  e.g., for the square-kilometer array (SKA)

  $Q = O(10^5)$, $B = O(100)$ → Storing $O(10^7)$ visibilities 😱
- Computing $\mathscr{F}[\sigma^\circ](\mathscr{V})$ via all $\{\boldsymbol{C}_b\}_{b=1}^{B}$ (with $I = O(10^9)$)

  → (Roughly) $O(IB\,Q^2) = O(10^9 \cdot 10^2 \cdot 10^{10}) = O(10^{21})$

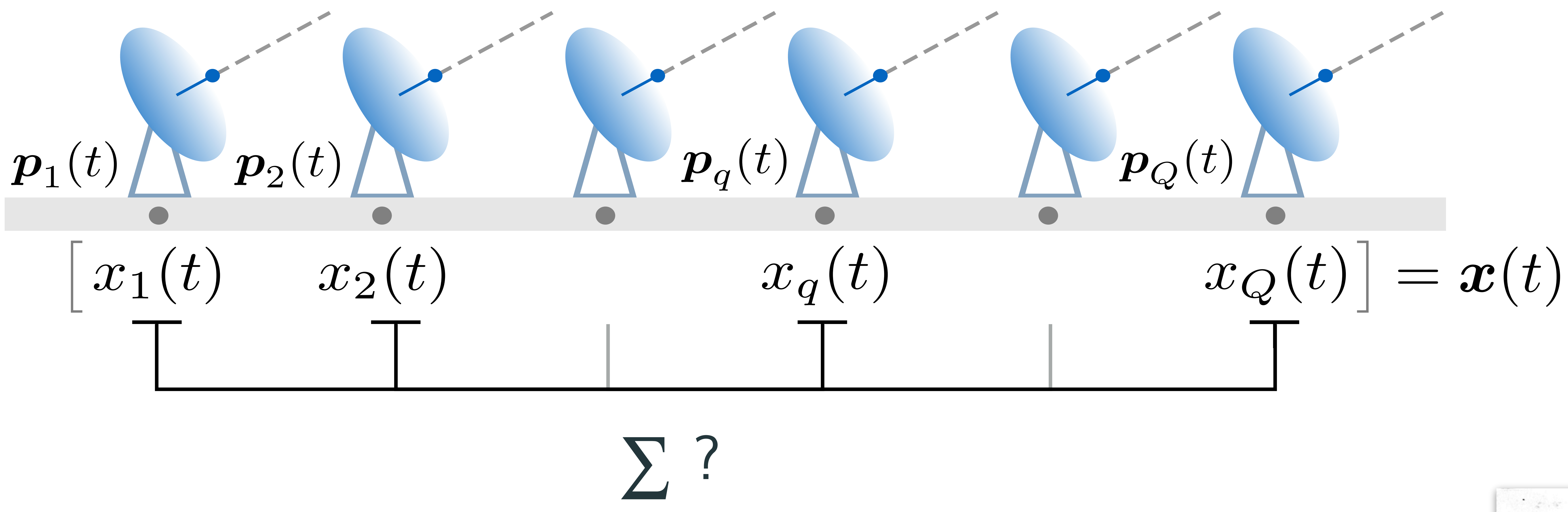
Solution: compressive radio-interferometric (RI) sensing scheme

- Dress up an old scheme, *beamforming*, in new clothes
- Compress measurements at antenna & reconstruction levels
- Supported by theoretical guarantees (under a few simplifications)

Compressive radio astronomy with rank-one projections and an old trick

Beamforming ≡ rank-one projections of covariance matrix

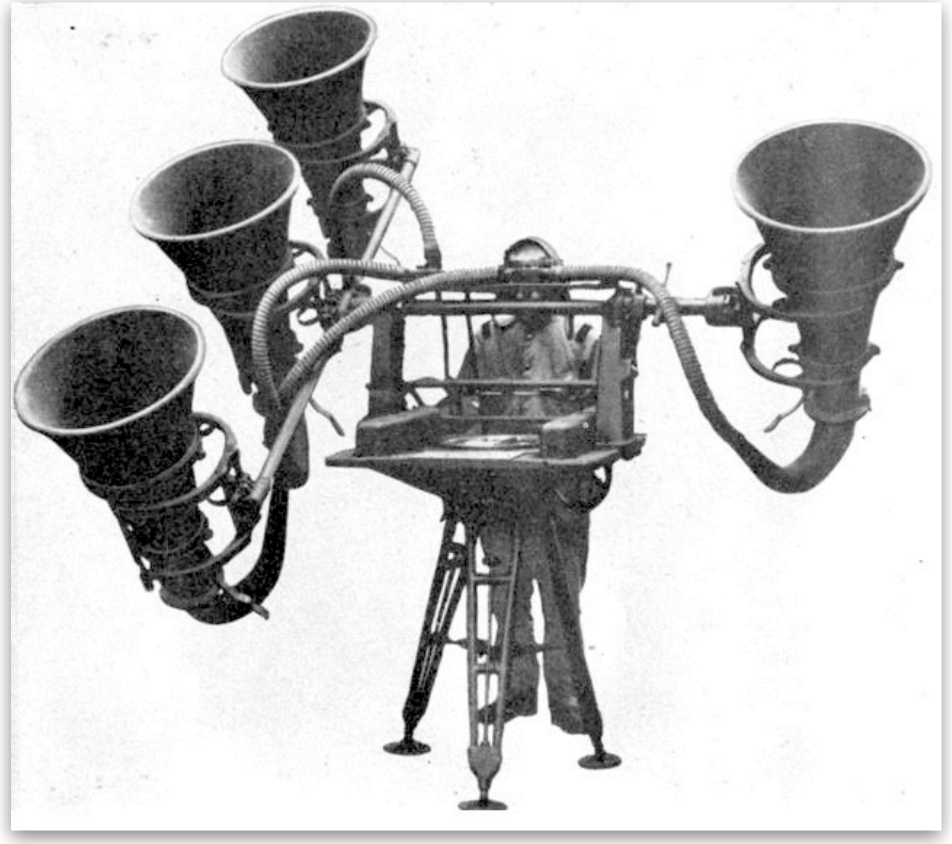
What if we create ***virtual*** antennas? Let's do beamforming



Beamforming ≡

creating virtual antenna for

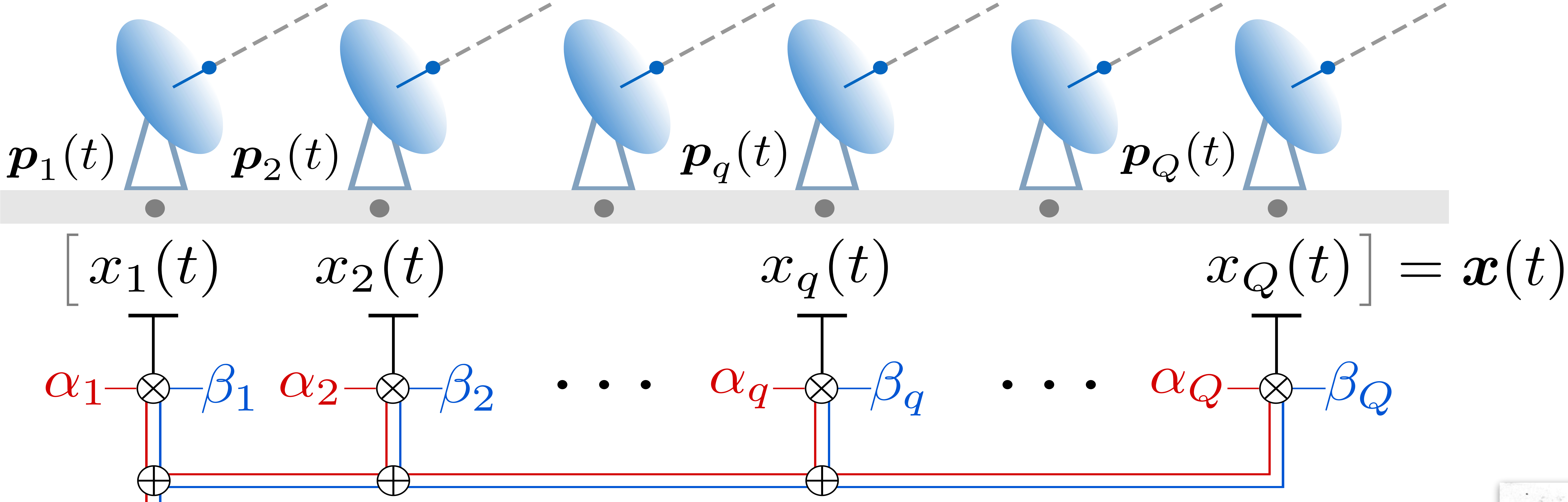
- spatial filtering,
- directionality, ...



Acoustic radar, Japan (1930)

Beamforming ≡ rank-one projections of covariance matrix

What if we create ***virtual*** antennas? Let's do beamforming



2 virtual antennas $\mu(t)$, $\nu(t)$

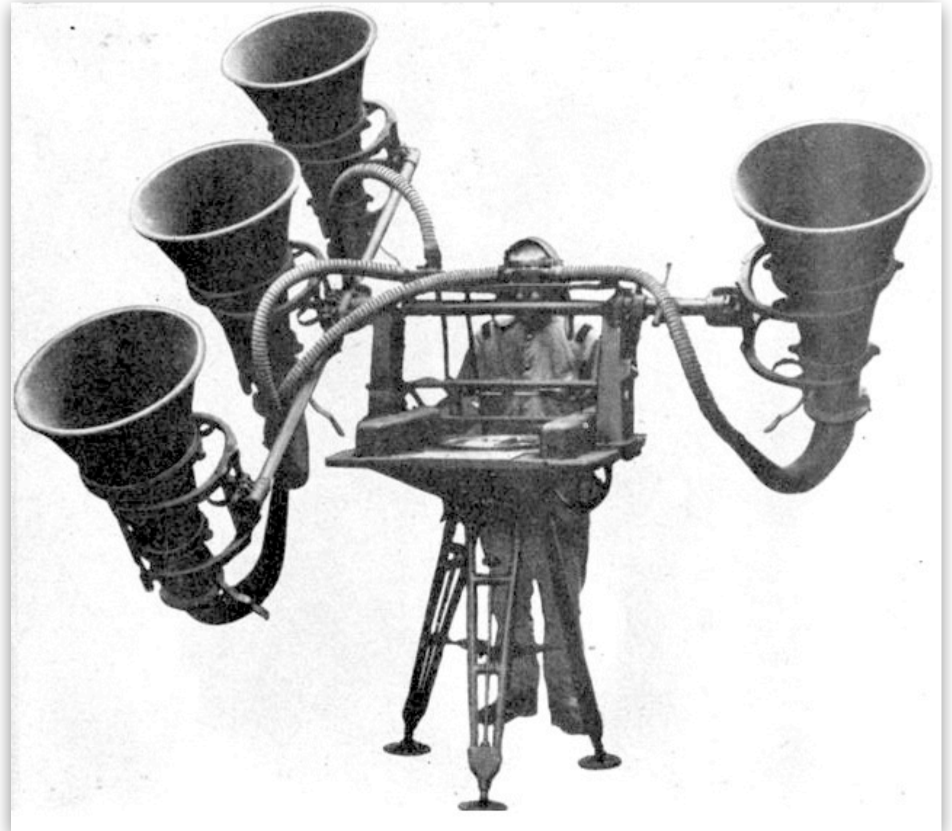
$$\mu(t) = \sum_{q=1}^{Q} \alpha_q^* x_q(t) = \langle \boldsymbol{\alpha}, \boldsymbol{x}(t) \rangle$$

$$\nu(t) = \sum_{q=1}^{Q} \beta_q^* x_q(t) = \langle \boldsymbol{\beta}, \boldsymbol{x}(t) \rangle$$

Beamforming ≡

optimising α_q, β_q for, e.g.,

- spatial filtering,
- directionality, ...

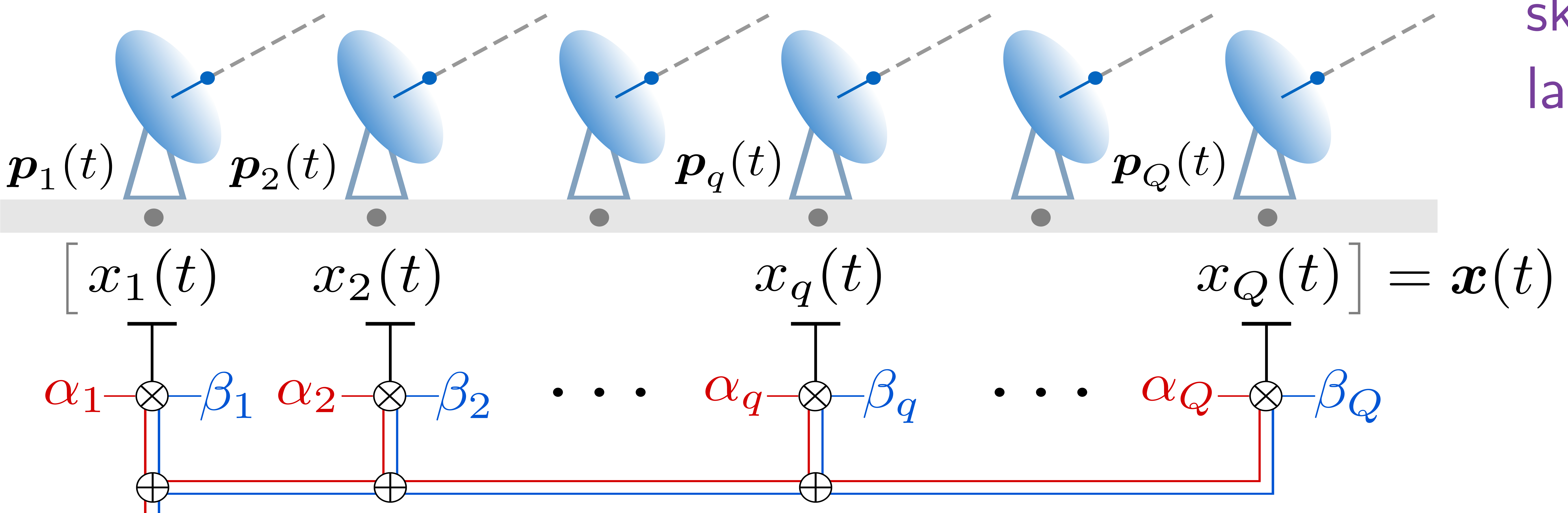


Acoustic radar, Japan (1930)

Given Q complex weights α_q, β_q

Beamforming ≡ rank-one projections of covariance matrix

What if we create ***virtual*** antennas? Let's do ~~beamforming~~ sketch-forming, lazy-forming



$$\mu(t) = \textstyle\sum_{q=1}^{Q} \alpha_q^* x_q(t) = \langle \boldsymbol{\alpha}, \boldsymbol{x}(t) \rangle$$

$$\nu(t) = \textstyle\sum_{q=1}^{Q} \beta_q^* x_q(t) = \langle \boldsymbol{\beta}, \boldsymbol{x}(t) \rangle$$

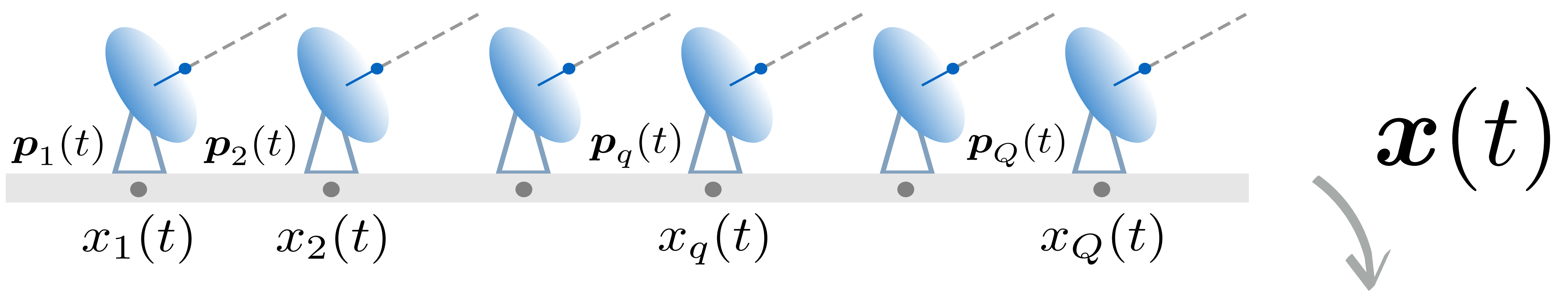
$$\xrightarrow[\text{Zernike}]{\text{Van Cittert}} \quad \mathbb{E}\mu\nu^* = \boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}_\Omega[\sigma^\circ] \boldsymbol{\beta}$$ ROP!!

with $(\boldsymbol{\mathcal{I}}_\Omega[\sigma^\circ])_{jk} = \mathcal{F}[\sigma^\circ]\big(\frac{\boldsymbol{p}_k^\perp - \boldsymbol{p}_j^\perp}{\lambda}\big)$

→ Let's take α & β *randomly* and compute P ROPs per *time/STI*

We need new sensing operators

Acquisition operator Given $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}_{p=1,b=1}^{P,B} \subset \mathbb{C}^Q$, $\{\gamma_{mb}\}_{m=1,b=1}^{M,B} \subset \mathbb{C}^Q$ (Not specified yet)

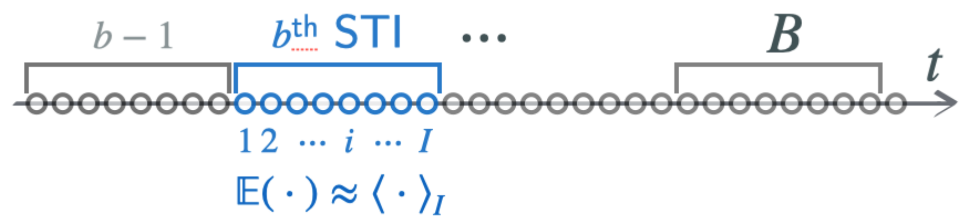


(sampled antenna signals) (QBI) (as before)

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

(B STI, I time samples per batch)



(1st compression @antennas level) $(QBI \to PB)$

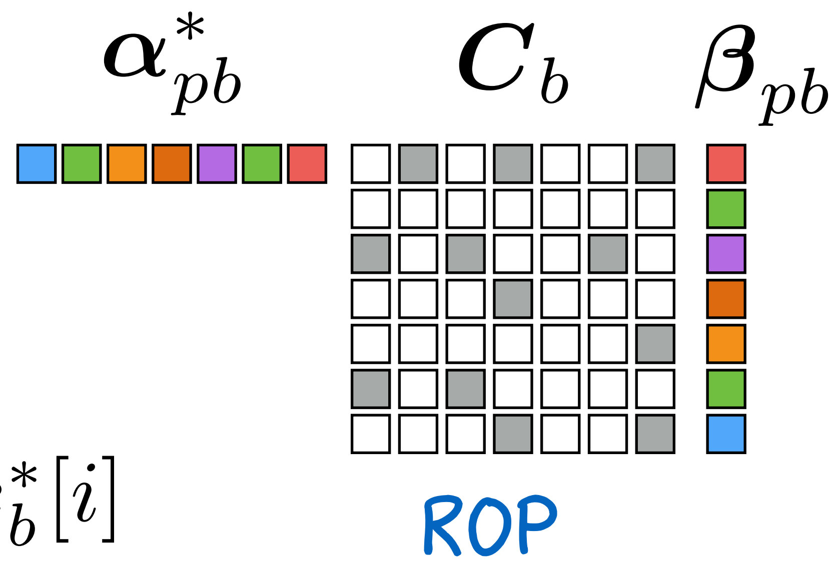
Random beamforming: for $p \in [P]$ ROPs per b

$$\mu_{pb}[i] := \langle \boldsymbol{\alpha}_{pb}, \boldsymbol{x}_b[i] \rangle,\ \ \nu_{pb}[i] := \langle \boldsymbol{\beta}_{pb}, \boldsymbol{x}_b[i] \rangle$$

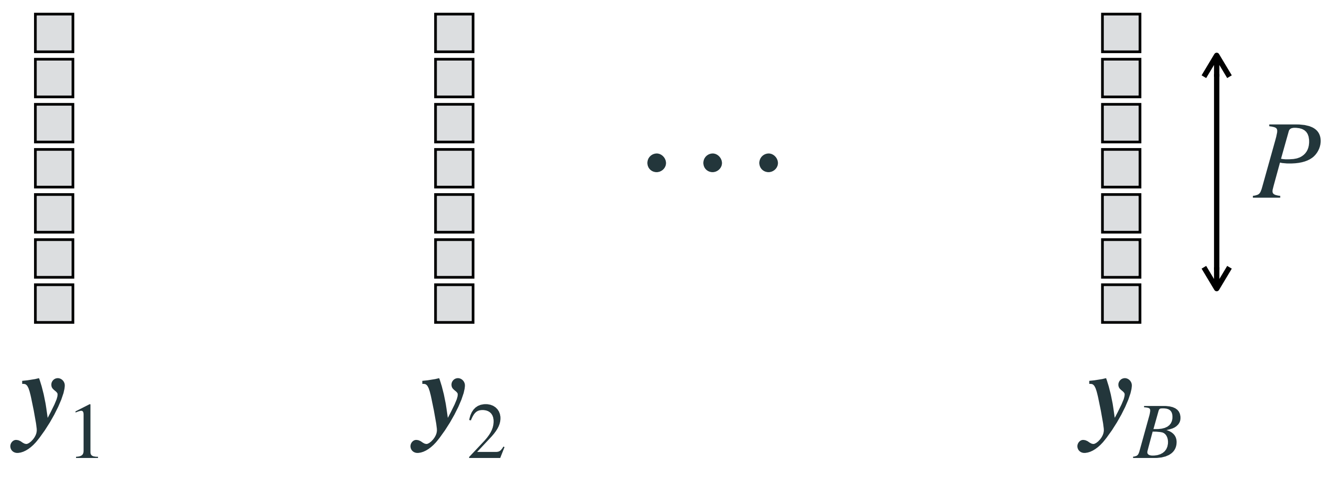
$$y_{pb} = \underbrace{\tfrac{1}{I} \textstyle\sum_{i=1}^{I} \mu_{pb}[i] \nu_{pb}[i]}_{\langle \mu_{pb}[\cdot] \nu_{pb}^*[\cdot] \rangle_I} = \boldsymbol{\alpha}_{pb}^* \boldsymbol{C}_b \boldsymbol{\beta}_{pb}$$

with $\boldsymbol{C}_b := \langle \boldsymbol{x}_b[\cdot] \boldsymbol{x}_b^*[\cdot] \rangle_I$

$:= \frac{1}{I} \sum_{i=1}^{I} \boldsymbol{x}_b[i] \boldsymbol{x}_b^*[i]$

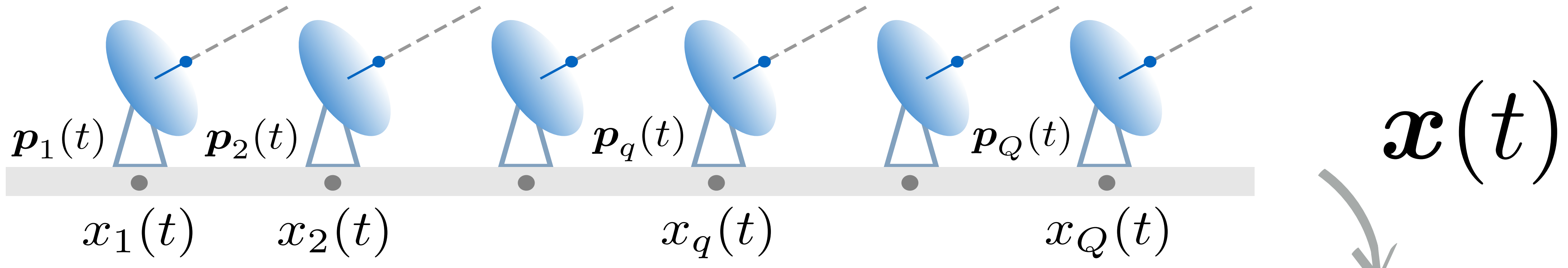


Still PB observations!



We need new sensing operators

Acquisition operator Given $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}_{p=1,b=1}^{P,B} \subset \mathbb{C}^Q$, $\{\gamma_{mb}\}_{m=1,b=1}^{M,B} \subset \mathbb{C}^Q$ (Not specified yet)

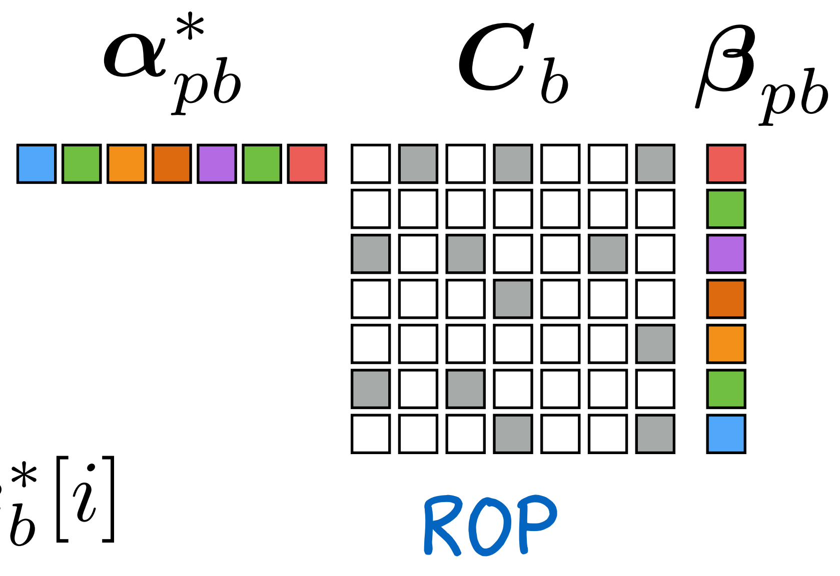


(1st compression @antennas level) $(QBI \to PB)$

Random beamforming: for $p \in [P]$ ROPs per b

$$\mu_{pb}[i] := \langle \boldsymbol{\alpha}_{pb}, \boldsymbol{x}_b[i] \rangle, \ \nu_{pb}[i] := \langle \boldsymbol{\beta}_{pb}, \boldsymbol{x}_b[i] \rangle$$

$$y_{pb} = \underbrace{\frac{1}{I} \sum_{i=1}^{I} \mu_{pb}[i] \nu_{pb}[i]}_{\langle \mu_{pb}[\cdot] \nu_{pb}^*[\cdot] \rangle_I} = \boldsymbol{\alpha}_{pb}^* \boldsymbol{C}_b \boldsymbol{\beta}_{pb}$$



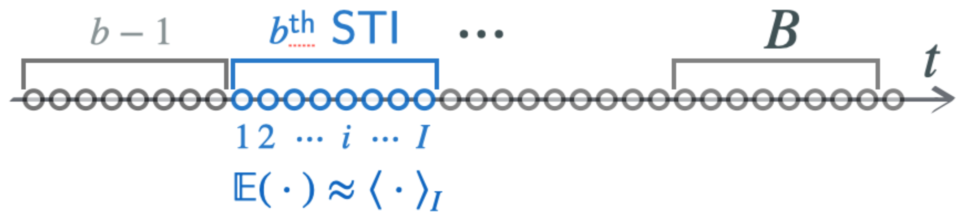
$$\text{with } \boldsymbol{C}_b := \langle \boldsymbol{x}_b[\cdot] \boldsymbol{x}_b^*[\cdot] \rangle_I := \frac{1}{I} \sum_{i=1}^{I} \boldsymbol{x}_b[i] \boldsymbol{x}_b^*[i]$$

(sampled antenna signals) (QBI) (as before)

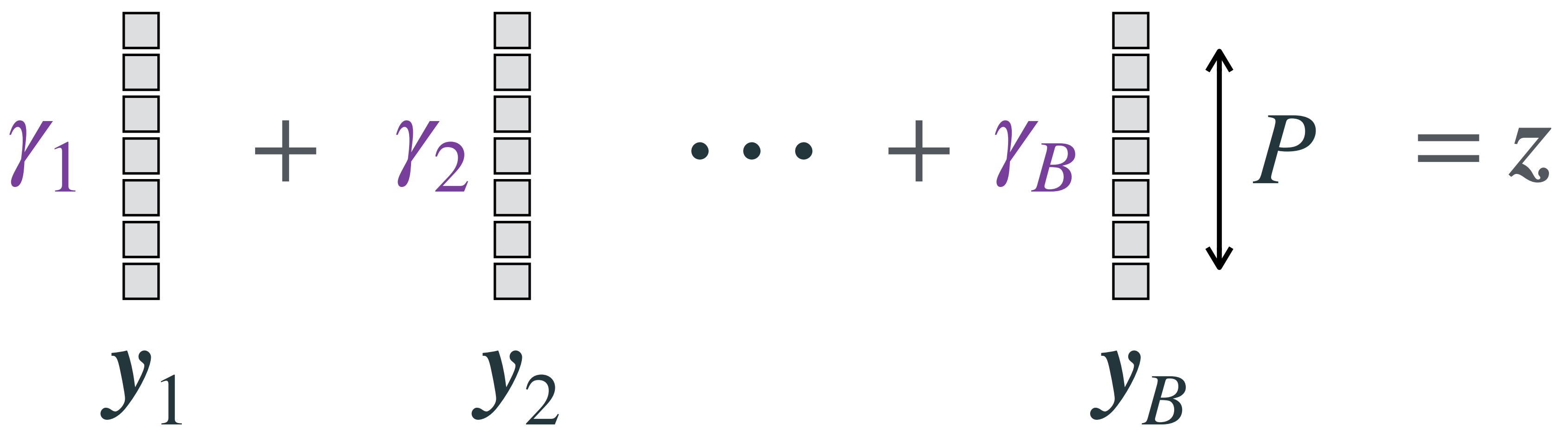
Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q, \ i \in [I]\}$$

(B STI, I time samples per batch)



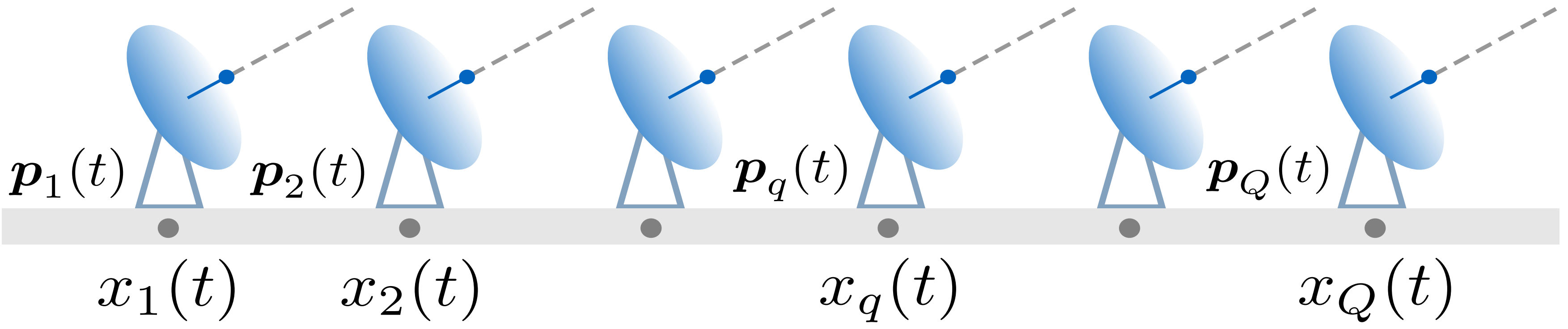
Still PB observations!



Let's modulate and sum M times

We need new sensing operators

Acquisition operator Given $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}_{p=1,b=1}^{P,B} \subset \mathbb{C}^Q$, $\{\gamma_{mb}\}_{m=1,b=1}^{M,B} \subset \mathbb{C}^Q$ (Not specified yet)



$\boldsymbol{x}(t)$

(sampled antenna signals) (QBI) (as before)

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

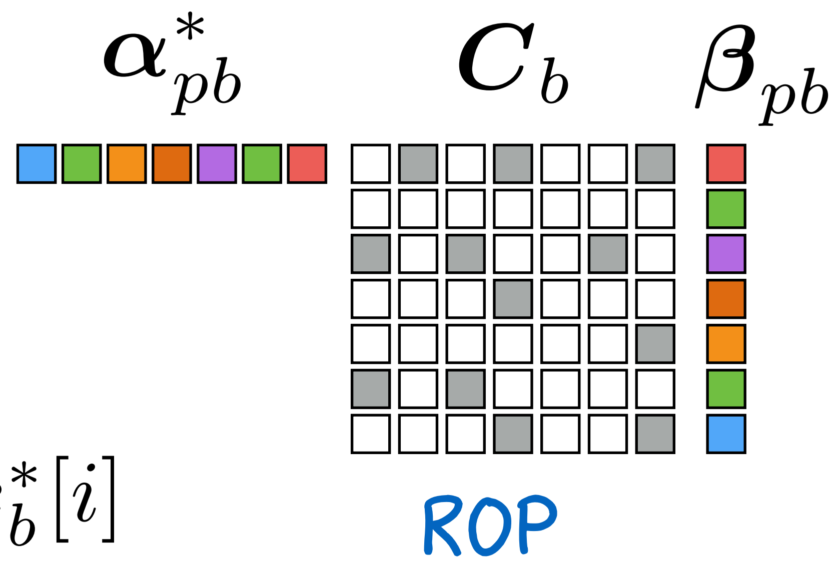
(B STI, I time samples per batch)

(1st compression @antennas level) ($QBI \to PB$)

Random beamforming: for $p \in [P]$ ROPs per b

$$\mu_{pb}[i] := \langle \boldsymbol{\alpha}_{pb}, \boldsymbol{x}_b[i] \rangle,\ \nu_{pb}[i] := \langle \boldsymbol{\beta}_{pb}, \boldsymbol{x}_b[i] \rangle$$

$$y_{pb} = \underbrace{\tfrac{1}{I} \textstyle\sum_{i=1}^{I} \mu_{pb}[i]\nu_{pb}[i]}_{\langle \mu_{pb}[\cdot]\nu^*_{pb}[\cdot]\rangle_I} = \boldsymbol{\alpha}^*_{pb}\boldsymbol{C}_b\boldsymbol{\beta}_{pb}$$



with $\boldsymbol{C}_b := \langle \boldsymbol{x}_b[\cdot]\boldsymbol{x}^*_b[\cdot]\rangle_I$

$$:= \tfrac{1}{I} \textstyle\sum_{i=1}^{I} \boldsymbol{x}_b[i]\boldsymbol{x}^*_b[i]$$

(2nd compression) ($PB \to PM$)

Bernoulli modulations: for $m \in [M]$ modulations

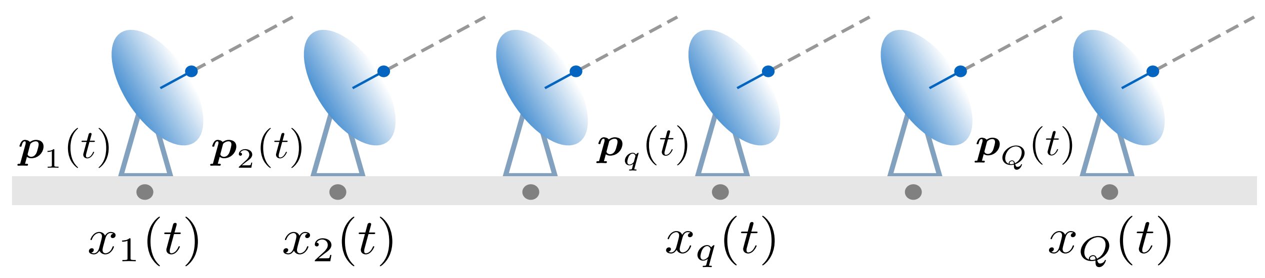
$$\mathcal{X} \to \widetilde{\boldsymbol{\Psi}}(\mathcal{X}) = \underbrace{\Big\{\boldsymbol{z}_m := \textstyle\sum_{b=1}^{B} \underbrace{\gamma_{mb}}_{\in\{\pm 1\}} \boldsymbol{y}_b\Big\}_{m=1}^{M}}_{PM \text{ values}}$$

$$\equiv \text{ROP of } \boldsymbol{C} := \mathrm{bdiag}(\boldsymbol{C}_1, \ldots, \boldsymbol{C}_B)$$

We need new sensing operators

Acquisition model

$\boldsymbol{x}(t)$



(1st compression @antennas level) (QBI)

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

Random beamforming: for $p \in [P]$ ROPs per b

$$\mu_{pb}[i] := \langle \boldsymbol{\alpha}_{pb}, \boldsymbol{x}_b[i] \rangle,\ \nu_{pb}[i] := \langle \boldsymbol{\beta}_{pb}, \boldsymbol{x}_b[i] \rangle$$

$$y_{pb} = \frac{1}{I} \sum_{i=1}^{I} \mu_{pb}[i] \nu_{pb}[i] = \underbrace{\boldsymbol{\alpha}_{pb}^* \boldsymbol{C}_b \boldsymbol{\beta}_{pb}}_{\text{ROP}}$$

$(QBI \to PB)$

Modulated ROPs

(2nd compression) $(PB \to PM)$

Bernoulli modulations: for $m \in [M]$ modulations

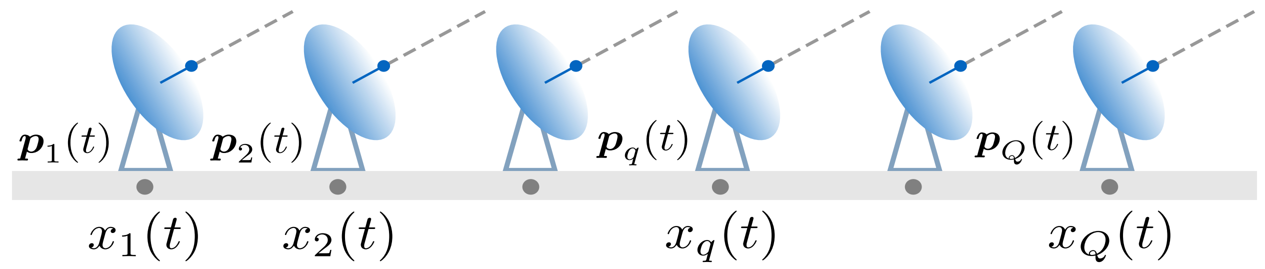
$$\mathcal{X} \to \widetilde{\boldsymbol{\Psi}}(\mathcal{X}) = \Big\{\boldsymbol{z}_m := \sum_{b=1}^{B} \underbrace{\gamma_{mb}}_{\in\{\pm 1\}} \boldsymbol{y}_b\Big\}_{m=1}^{M}$$

PM values

We need new sensing operators

Acquisition model | **Imaging model**

$\boldsymbol{x}(t)$



(1st compression @antennas level) (QBI)

Sampling B STIs, $b \in [B]$

$$\mathcal{X}_b := \{\boldsymbol{x}_b[i] \in \mathbb{C}^Q,\ i \in [I]\}$$

Random beamforming: for $p \in [P]$ ROPs per b

$$\mu_{pb}[i] := \langle \boldsymbol{\alpha}_{pb}, \boldsymbol{x}_b[i] \rangle,\ \nu_{pb}[i] := \langle \boldsymbol{\beta}_{pb}, \boldsymbol{x}_b[i] \rangle$$

$$y_{pb} = \tfrac{1}{I} \sum_{i=1}^{I} \mu_{pb}[i]\nu_{pb}[i] = \underbrace{\boldsymbol{\alpha}_{pb}^* \boldsymbol{C}_b \boldsymbol{\beta}_{pb}}_{\text{ROP}}$$

$(QBI \to PB)$

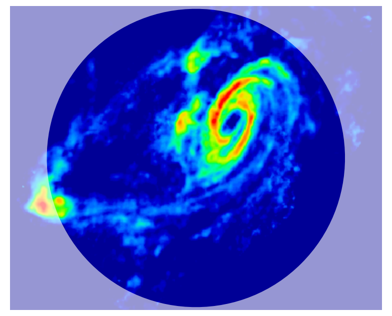
(2nd compression) $(PB \to PM)$

Bernoulli modulations: for $m \in [M]$ modulations

$$\mathcal{X} \to \widetilde{\boldsymbol{\Psi}}(\mathcal{X}) = \underbrace{\Big\{ \boldsymbol{z}_m := \textstyle\sum_{b=1}^{B} \underbrace{\gamma_{mb}}_{\in\{\pm 1\}} \boldsymbol{y}_b \Big\}_{m=1}^{M}}_{PM \text{ values}}$$

Modulated ROPs

$\sigma^\circ(\boldsymbol{l})$

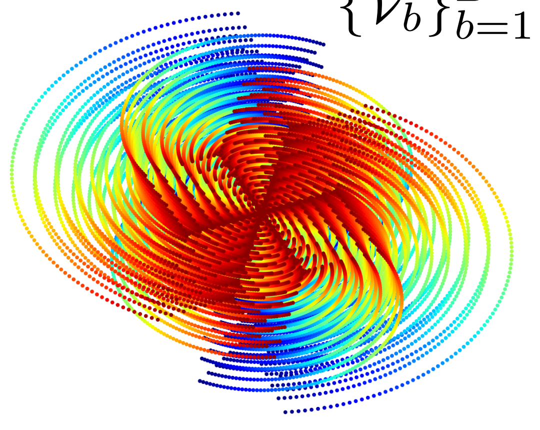


(image level)

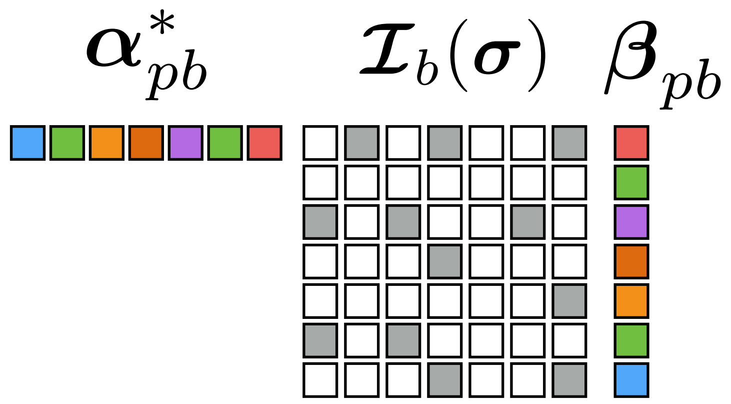
Sampling over a N-pixel grid

$$\boldsymbol{\sigma} \in \mathbb{R}^N$$

Given $\{\mathcal{V}_b\}_{b=1}^{B} =: \mathcal{V}$



Q^2B visibilities



(compression for the reconstruction level)

$$\boldsymbol{\sigma} \xrightarrow{\text{ROP}} \Big\{ \boldsymbol{y}'_b = \big[\boldsymbol{\alpha}_{pb}^* \overbrace{\mathcal{I}_b(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V}_b)} \boldsymbol{\beta}_{pb}\big)\big]_{p=1}^{P} \Big\}_{b=1}^{B}$$

$$\xrightarrow{\text{Mod.}} \underbrace{\widetilde{\boldsymbol{\Phi}}[\boldsymbol{\sigma}] = \big\{ \textstyle\sum_{b=1}^{B} \gamma_{mb} \boldsymbol{y}'_b \big\}_{m=1}^{M}}_{PM \text{ values}}$$

VCZ

equal in expectation

$(N \to PM)$

Reconstruction guarantees?

Questions:

- For which (distribution on) $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}, \gamma_{mb}\}$ can we estimate the image $\boldsymbol{\sigma}$?
- Which compression ratios can we reach?

Our answers:

1. **Theory**: ok if $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}$ are random and (sub)Gaussian

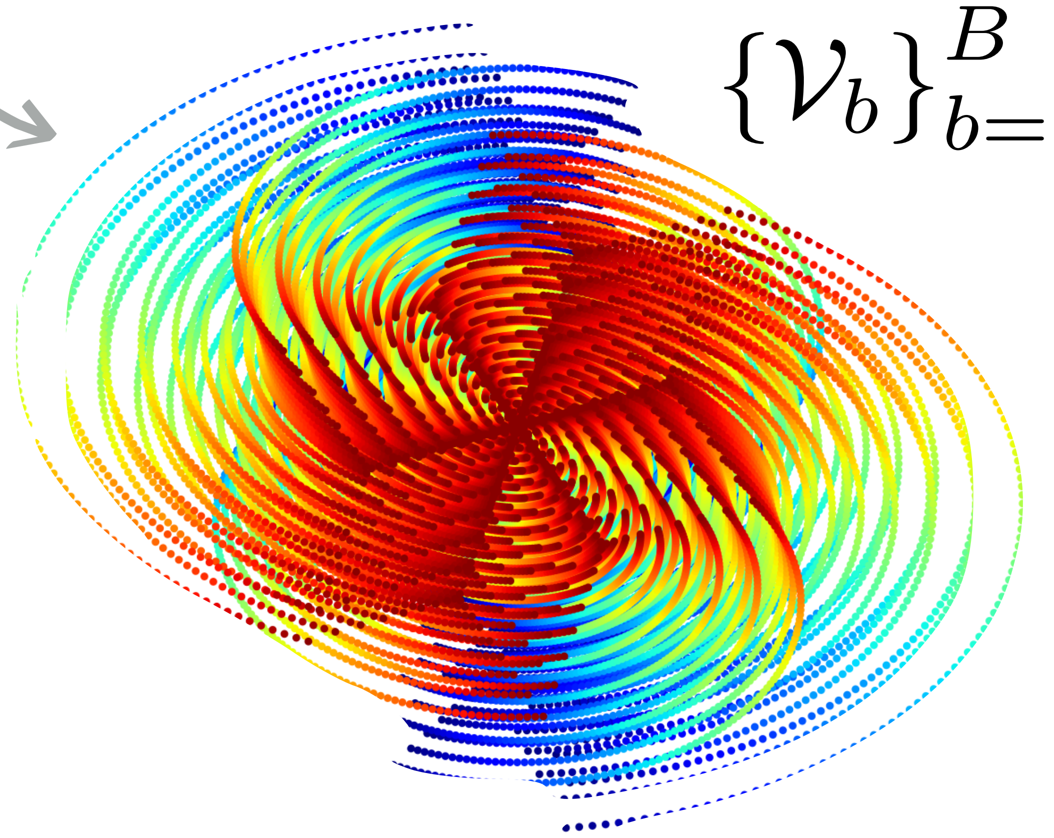
   but without modulations ($\gamma_{mb} = 1$, $M = 1$) and P large enough
2. **Experiments**: ok if $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}, \gamma_{mb}\}$ are random and (sub)Gaussian

Reconstruction guarantees? Theory for a simplified scenario

Batched ROP model: sum without modulation ($\gamma_{mb} = 1,\, M = 1$)

$$\widetilde{\boldsymbol{\Phi}}[\boldsymbol{\sigma}] = \textstyle\sum_{b=1}^{B}\big[\boldsymbol{\alpha}_{pb}^{*}\ \overbrace{\boldsymbol{\mathcal{I}}_b(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V}_b)}\ \boldsymbol{\beta}_{pb}\big]_{p=1}^{P} = \big[\boldsymbol{\alpha}_{p}^{*}\ \overbrace{\boldsymbol{\mathcal{I}}(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V})}\ \boldsymbol{\beta}_{p}\big]_{p=1}^{P}$$

with $\boldsymbol{\alpha}_p = [\boldsymbol{\alpha}_{pb}]_{b=1}^{B}$, $\boldsymbol{\beta}_p = [\boldsymbol{\beta}_{pb}]_{b=1}^{B}$, $\boldsymbol{\mathcal{I}} = \mathrm{bdiag}(\boldsymbol{\mathcal{I}}_1, \ldots, \boldsymbol{\mathcal{I}}_B)$.



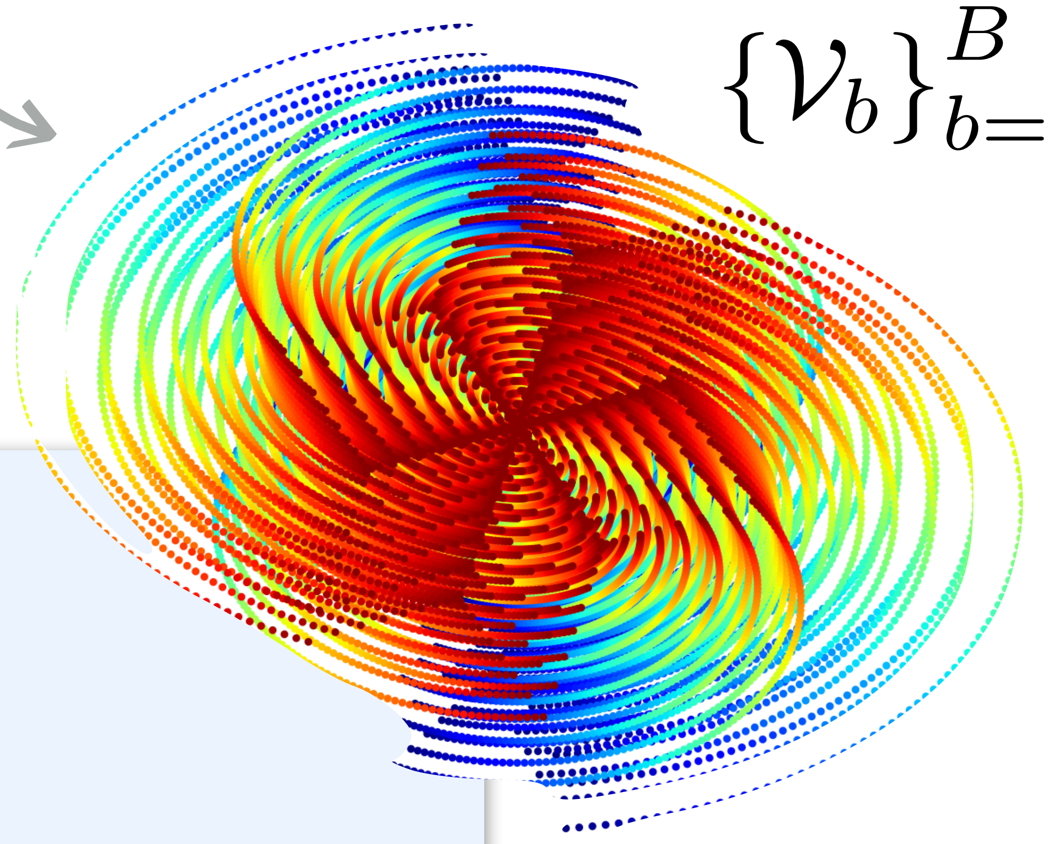
$\{\mathcal{V}_b\}_{b=1}^{B} =: \mathcal{V}$

Reconstruction guarantees? Theory for a simplified scenario

Batched ROP model: sum without modulation ($\gamma_{mb} = 1,\ M = 1$)

$$\widetilde{\boldsymbol{\Phi}}[\boldsymbol{\sigma}] = \sum_{b=1}^{B} \Big[\boldsymbol{\alpha}_{pb}^{*}\ \overbrace{\boldsymbol{\mathcal{I}}_b(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V}_b)}\ \boldsymbol{\beta}_{pb}\Big]_{p=1}^{P} = \Big[\boldsymbol{\alpha}_{p}^{*}\ \overbrace{\boldsymbol{\mathcal{I}}(\boldsymbol{\sigma})}^{\mathcal{F}[\boldsymbol{\sigma}](\mathcal{V})}\ \boldsymbol{\beta}_{p}\Big]_{p=1}^{P}$$

with $\boldsymbol{\alpha}_p = [\boldsymbol{\alpha}_{pb}]_{b=1}^{B}$, $\boldsymbol{\beta}_p = [\boldsymbol{\beta}_{pb}]_{b=1}^{B}$, $\boldsymbol{\mathcal{I}} = \mathrm{bdiag}(\boldsymbol{\mathcal{I}}_1, \ldots, \boldsymbol{\mathcal{I}}_B)$.



$\{\mathcal{V}_b\}_{b=1}^{B} =: \mathcal{V}$

(under specific simplifying assumptions)

If $\{\boldsymbol{\alpha}_{pb}, \boldsymbol{\beta}_{pb}\}$ are (sub)Gaussian, given a sparsity level K
and provided $P = O(K)$ and $Q^2B = O(K)$ (up to logs),
then, with high probability, given the observations $\boldsymbol{z} = \tilde{\boldsymbol{\Phi}}[\boldsymbol{\sigma}] + \underbrace{\text{noise}}_{\|\cdot\|_1 \leqslant \epsilon}$,
an ℓ_1-minimization gives an estimate $\boldsymbol{\sigma}'$ with

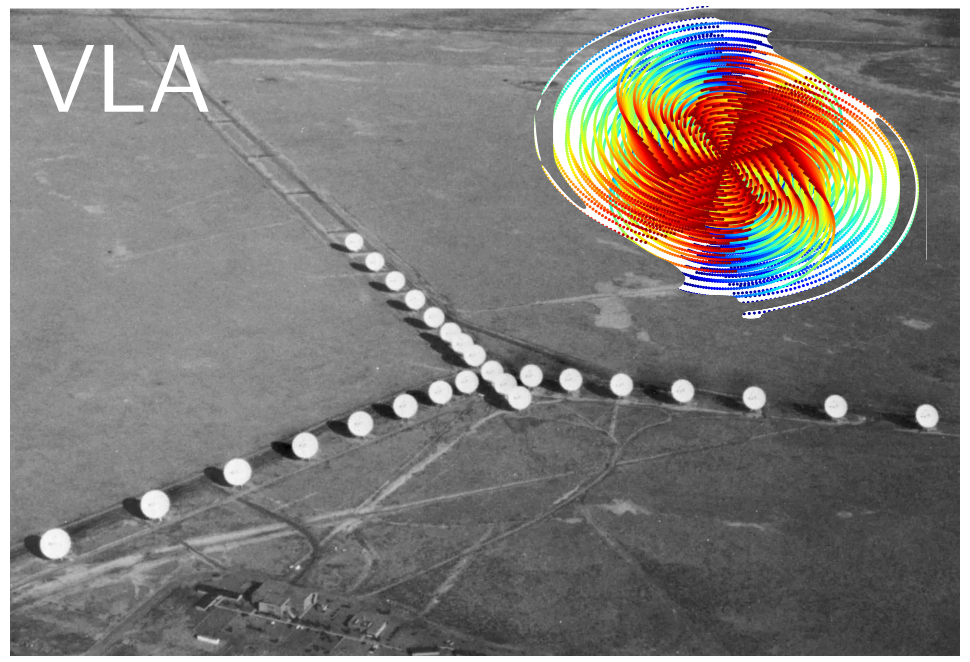
$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}'\|_2 \leqslant C\frac{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{N_{\mathrm{p}}}$$

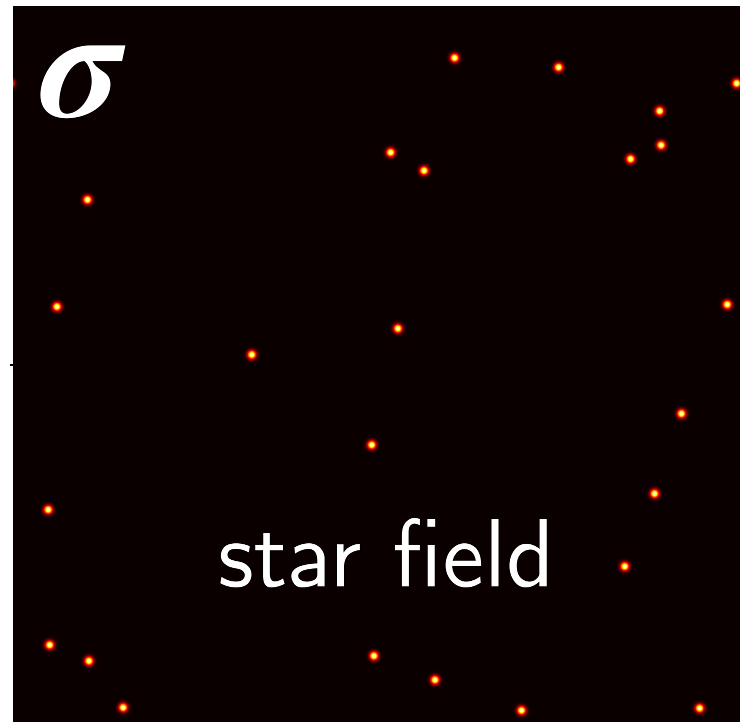
for some $C, D > 0$.

Reconstruction guarantees? Simulations

Modulated ROP model (MROP):

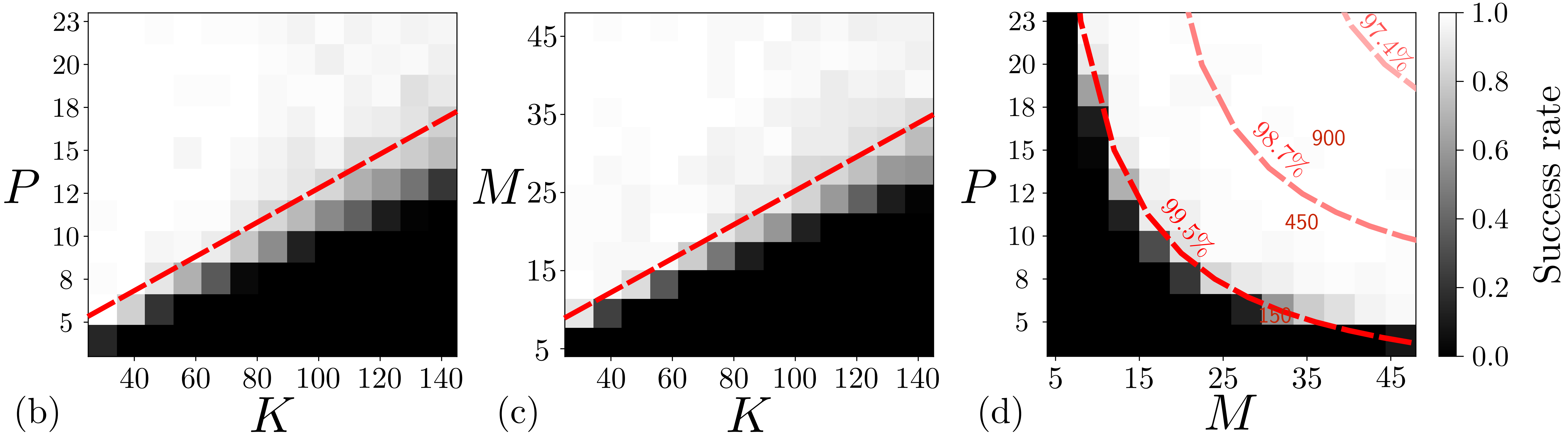
- Monte Carlo simulations
- $N = 10^4$, $B = 100$, $Q = 27$
- Various K, P, M
- Very Large Array (VLA) visibility/frequency coverage





$N = 100 \times 100$

Phase transition diagrams (success if SNR > 40 dB)



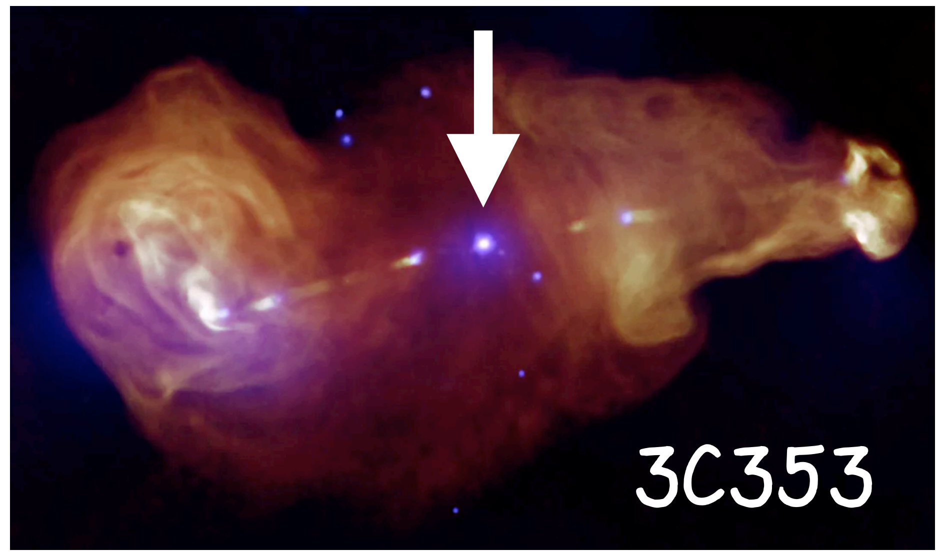
High reconstruction success as soon as $PM \geqslant CK$, with $C \simeq 5$.

$\rightarrow \quad PM \ll \; Q^2 B = 70\,200$

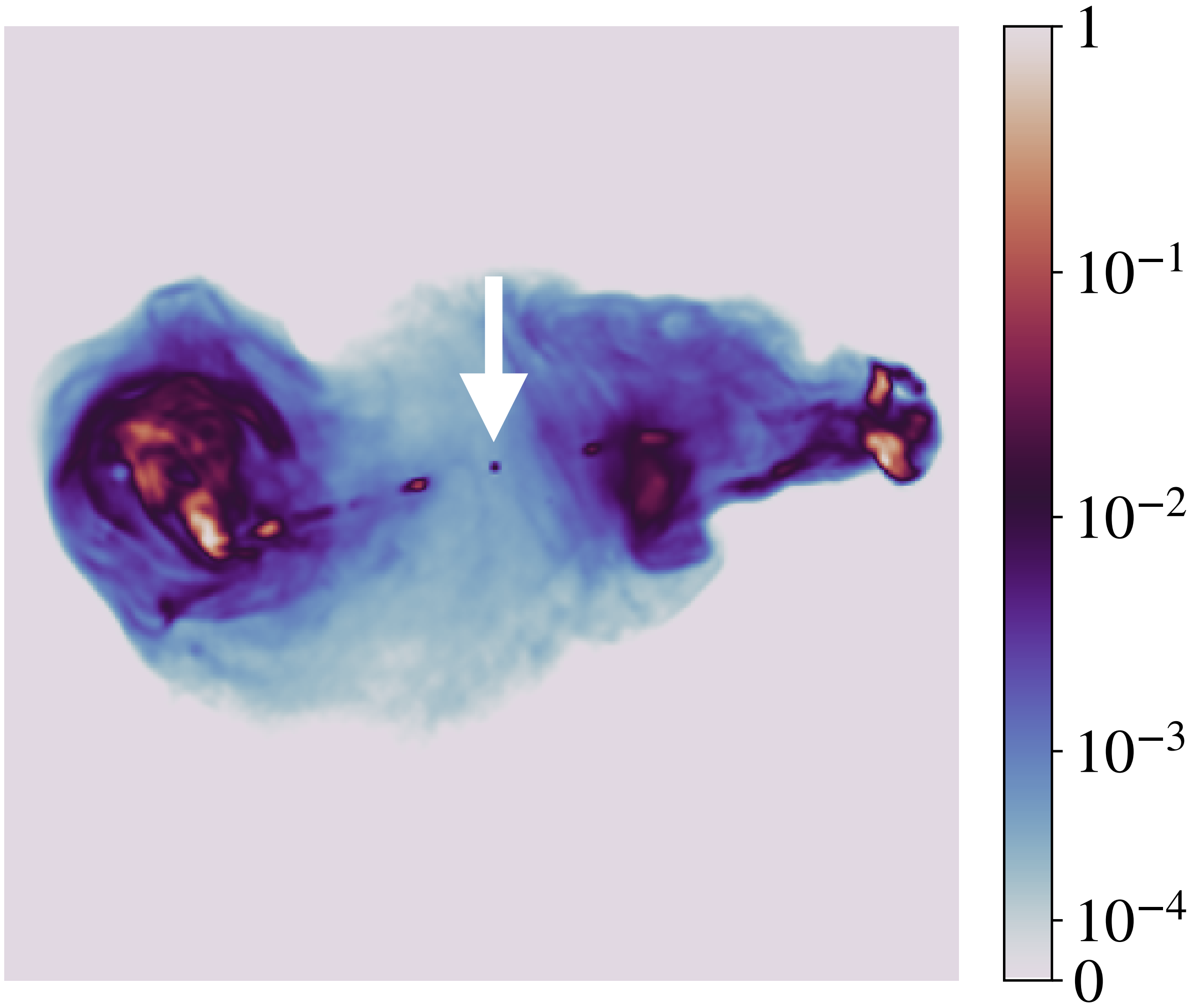
Radio Galaxy 3C353, uSARA reconstruction

Reconstruction for the image of the Radio Galaxy 3C353

- Classical reconstruction ≡ all visibilities
- Basic subsampled visibilities
- Baseline dependent averaging (BDA*)
- and, ours, the MROP sensing models

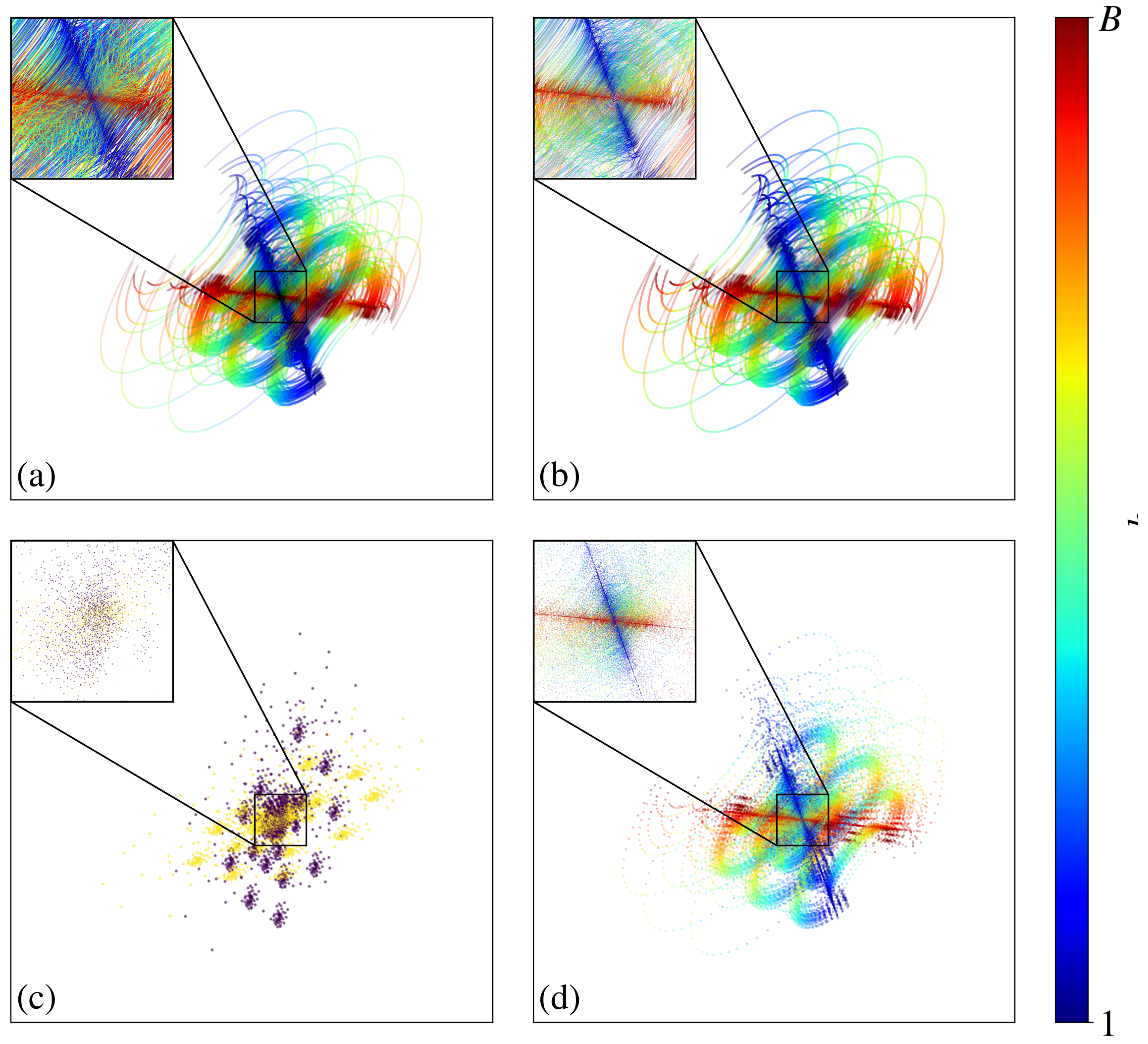


Chandra & VLA



Parameters:

- #Visibilities ($V = Q^2 B$) / #Pixels (N) = 14.5
- Total number of measurements: $D = PM$
- ROP compressions levels: D/N
  $\in \{9.8\ 10^{-1}, 3.8\ 10^{-1}, 6.1\ 10^{-2}\}$
- Simulated MeerKat visibilities (Q=64)
- uSARA reconstruction (Terris et al, 2022)
  (inverse problem solving with "average sparsity" prior)



MeerKat visibilities

* S.J. Wijnholds et al (2018)

Radio Galaxy 3C353, uSARA reconstruction

Two compression metrics

D/VB and D/N

Total # measurements:

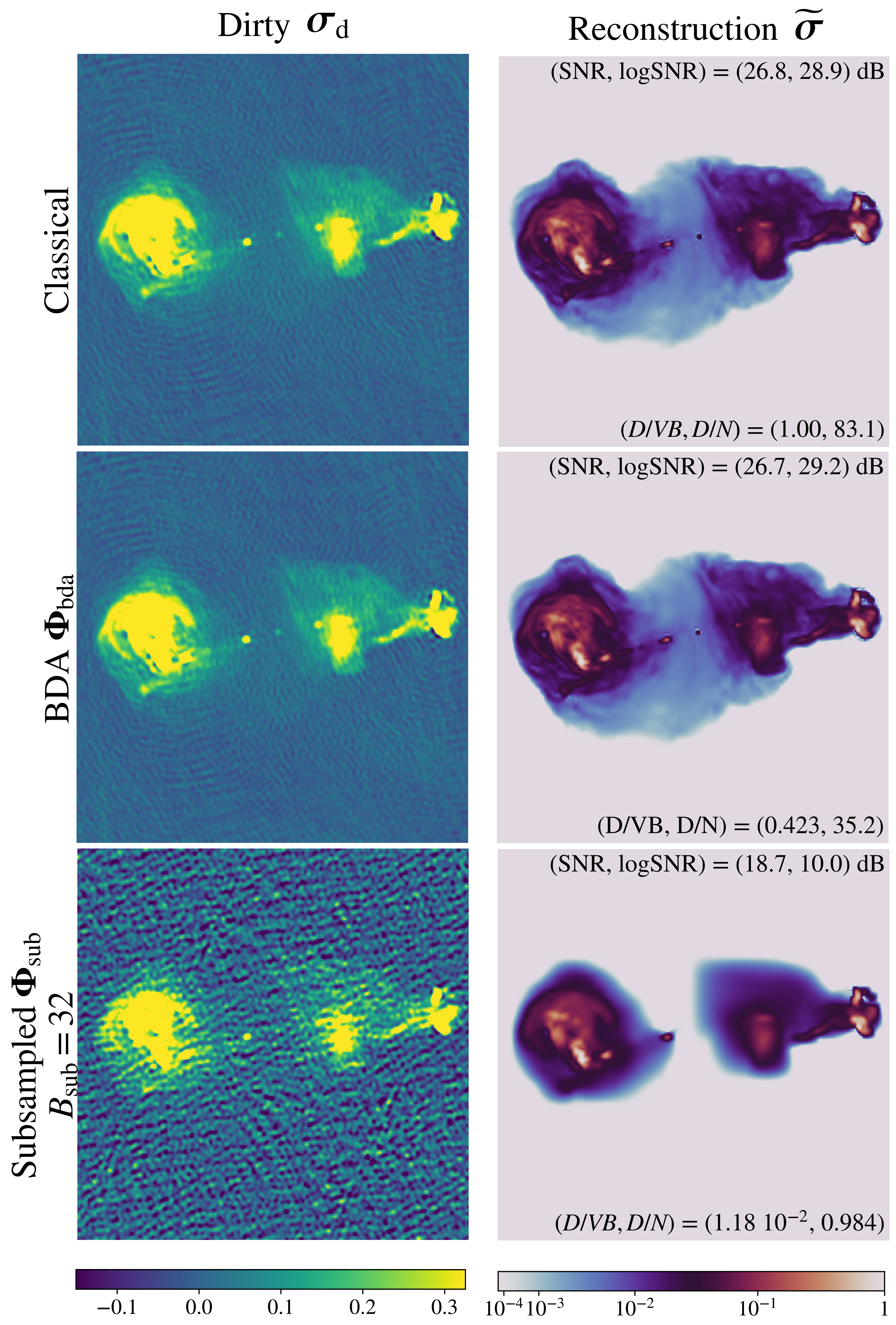
$D = PM$

Total # visibilities:

$V = Q^2B$

Total # pixels:

N



Higher Compression Rates

Radio Galaxy 3C353, uSARA reconstruction

Two compression metrics

D/VB and D/N

Total # measurements:

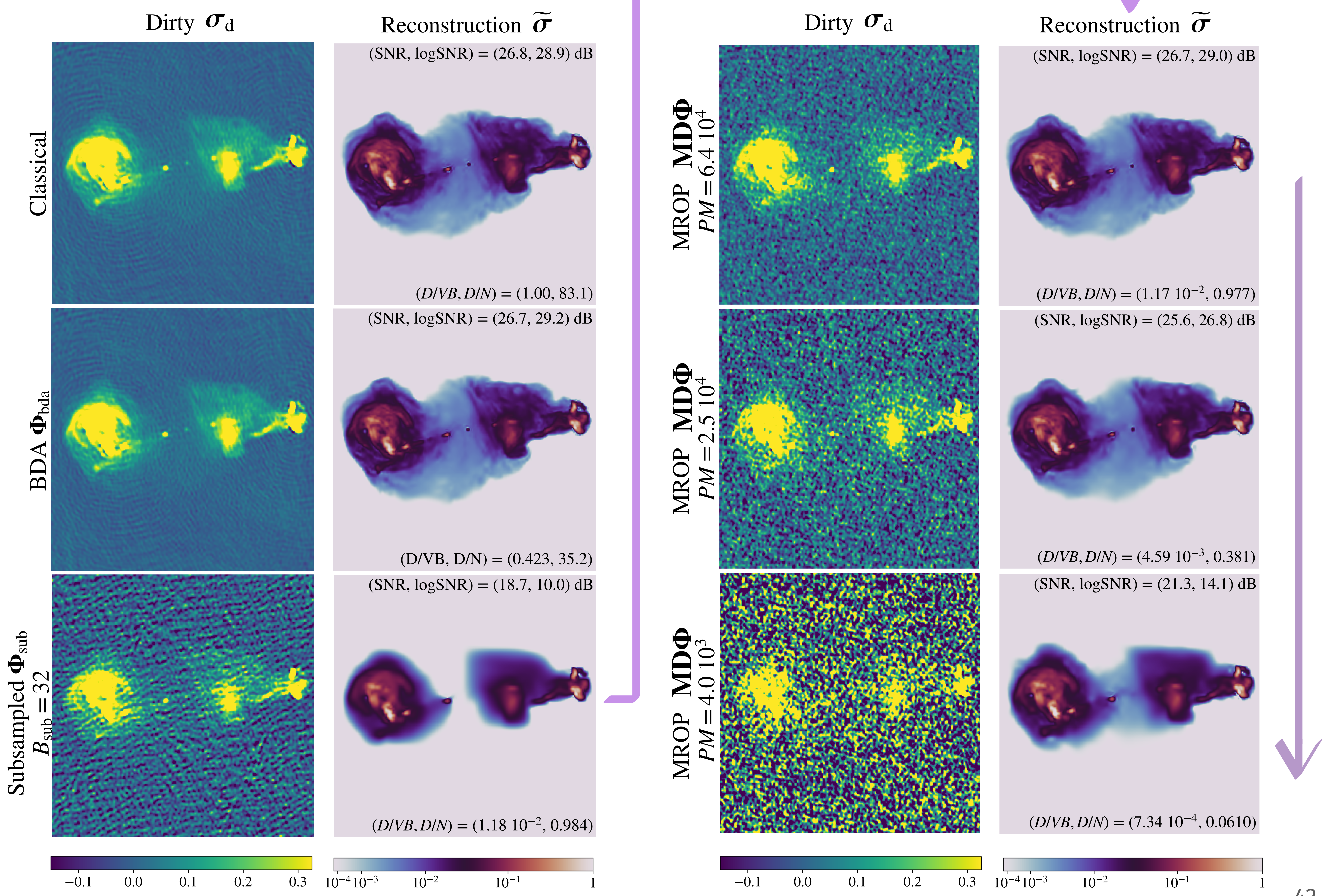
$D = PM$

Total # visibilities:

$V = Q^2 B$

Total # pixels:

N



Conclusions and perspectives

Summary:

- Interferometry and “beamforming” $\rightarrow$ ROP + Fourier
  (also used in Optics: random illumination with multicore fibers)
- Additional compression for multiple STI $\rightarrow$ modulated ROP
- Theory + simulations and experimental data confirmation

Conclusions and perspectives

Summary:

- Interferometry and “beamforming” $\rightarrow$ ROP + Fourier
  (also used in Optics: random illumination with multicore fibers)
- Additional compression for multiple STI $\rightarrow$ modulated ROP
- Theory + simulations and experimental data confirmation

Open questions:

- Integrating frequency weighting?
- Faster ROP models?
- Calibration through beamforming sensing?

Thank you for your attention!

- **O. Leblanc, Y. Wiaux, L. Jacques, "Compressive radio-interferometric sensing with random beamforming as rank-one signal covariance projections", IEEE TCI https://arxiv.org/abs/2409.15031**
- **O. Leblanc, C. S. Chu, L. Jacques, Y. Wiaux, "MROP: Modulated Rank-One Projections for compressive radio interferometric imaging", MNRAS, 2025**
- **O. Leblanc, M. Hofer, S. Sivankutty, H. Rigneault, L. Jacques (2023). "Interferometric lensless imaging: rank-one projections of image frequencies with speckle illuminations". Submitted to IEEE TCI, arXiv:2306.12698.**

- Chen, Y., Chi, Y., & Goldsmith, A. J. (2015). Exact and stable covariance estimation from quadratic sampling via convex programming. *IEEE Transactions on Information Theory*, *61*(7), 4034-4059.
- Cai, T. T., & Zhang, A. (2015). ROP: Matrix recovery via rank-one projections. The Annals of Statistics, 43(1), 102-138.
- A.-J. Van Der Veen, S. J. Wijnholds, A. M. Sardarabadi. "Signal Processing for Radio Astronomy." In Handbook of Signal Processing Systems, by Springer International Publishing, 2019.
- Terris M., Dabbech A., Tang C., Wiaux Y., 2022, Monthly Notices of the Royal Astronomical Society, 518, 604
- "Acoustic Location and Sound Mirrors", http://www.douglas-self.com/MUSEUM/COMMS/ear/ear.htm

Extra slides

Beyond astronomy... a more general distributed or dynamical context

Linked covariance model:

Given B datasets $\mathcal{X}_b = \{x_i^{(b)}\}_{i=1}^N \subset \mathbb{R}^d$, all of the same size N, with $1 \leq b \leq B$.

with $x_i^{(b)} \sim_{\text{iid}} \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_b})$, with $\Sigma_b = \mathcal{S}_b(\theta) \in \mathbb{R}^{d \times d}$, for some **common** $\theta \in \mathbb{R}^K$

Johannes Maly
(LMU)

Beyond astronomy... a more general distributed or dynamical context

Linked covariance model:

Given B datasets $\mathscr{X}_b = \{x_i^{(b)}\}_{i=1}^N \subset \mathbb{R}^d$, all of the same size N, with $1 \le b \le B$.

with $x_i^{(b)} \sim_{\text{iid}} \mathscr{N}(\mathbf{0}, \mathbf{\Sigma_b})$, with $\Sigma_b = \mathscr{S}_b(\theta) \in \mathbb{R}^{d \times d}$, for some **common** $\theta \in \mathbb{R}^K$

1st measurement process: for $\alpha_i^{(b)}, \beta_i^{(b)} \sim \mathscr{N}(\mathbf{0}, \boldsymbol{I}_\mathbf{d})$

$$\mathcal{X}_b \to \mathcal{A}_b(\mathcal{X}_b) = \big\{\boldsymbol{\alpha}_i^{(b)\top}\big[\tfrac{1}{N}\textstyle\sum_{k=1}^N \boldsymbol{x}_k^{(b)}\boldsymbol{x}_k^{(b)\top}\big]\boldsymbol{\beta}_i^{(b)}\big\}_{i=1}^m, \quad \mathcal{A}(\mathcal{X}) := \{\mathcal{A}_b(\mathcal{X}_b)\}_{b=1}^B \in \mathbb{R}^{mB}$$

with $\frac{1}{m}\mathbb{E}\mathcal{A}_b^*\mathcal{A}_b(\mathcal{X}_b) = \hat{\Sigma}_b \approx \Sigma_b$

Measurement compression: for $\gamma^{(g)} \sim \mathscr{U}\{\pm 1\}^B$, $1 \le g \le G$,

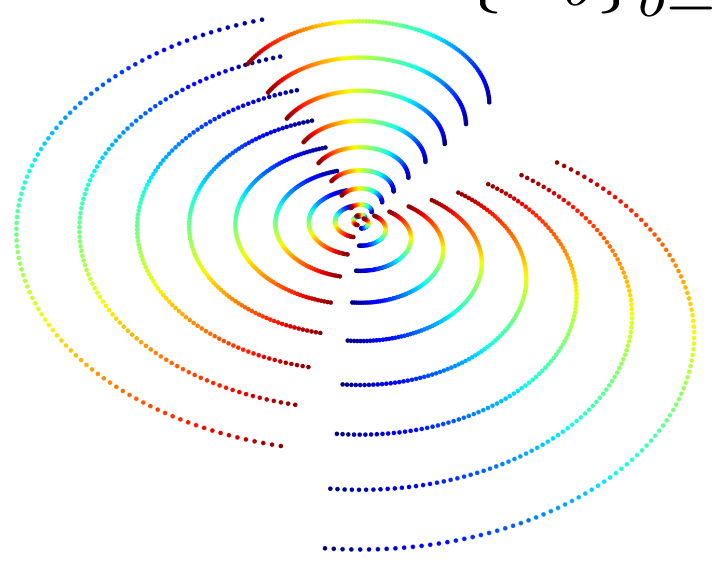
$$\boldsymbol{z} = (\boldsymbol{z}_1, \dots, z_G)^\top \text{ with } \boldsymbol{z}_g := \textstyle\sum_{b=1}^B \gamma_b^{(g)} \mathcal{A}_b(\mathcal{X}_b)$$

Research question: Can we estimate θ from z ?

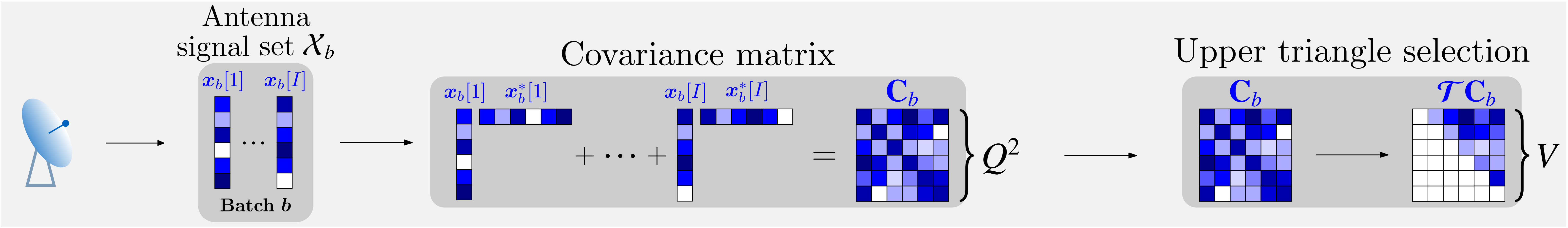
Johannes Maly
(LMU)

Visibility-based acquisition of MROP

$\{\Omega_b\}_{b=1}^{B}$

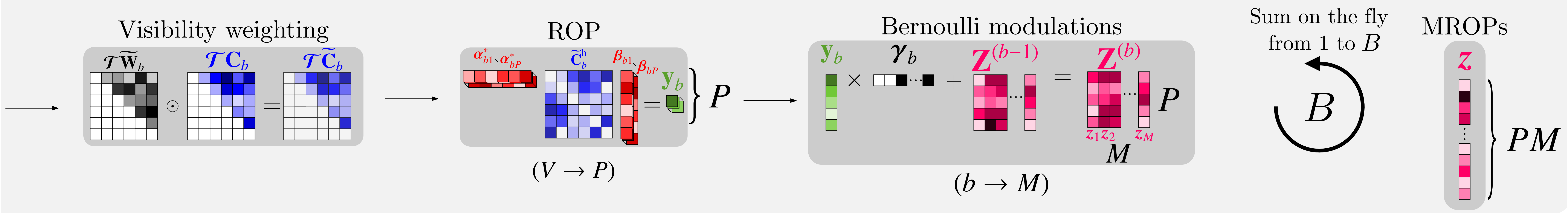


at each STI (b), compute the covariance matrix (classical)



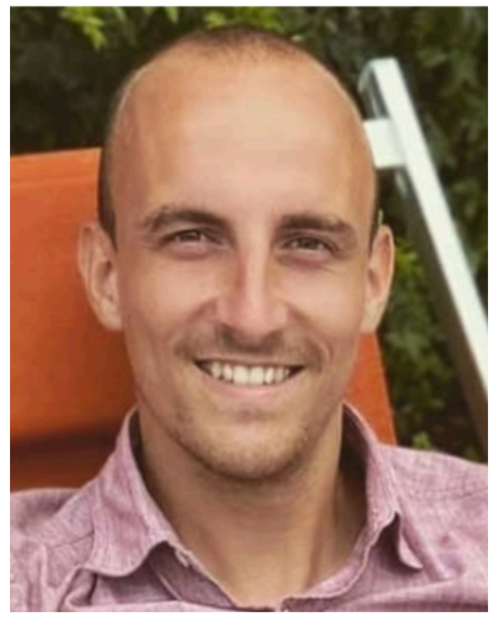
(visibility domain)

then do ROP across STIs after visibility weighting



(compressive interferometry #1)

Lensless interferometry & rank-one projections


O. Leblanc*


L. Jacques*

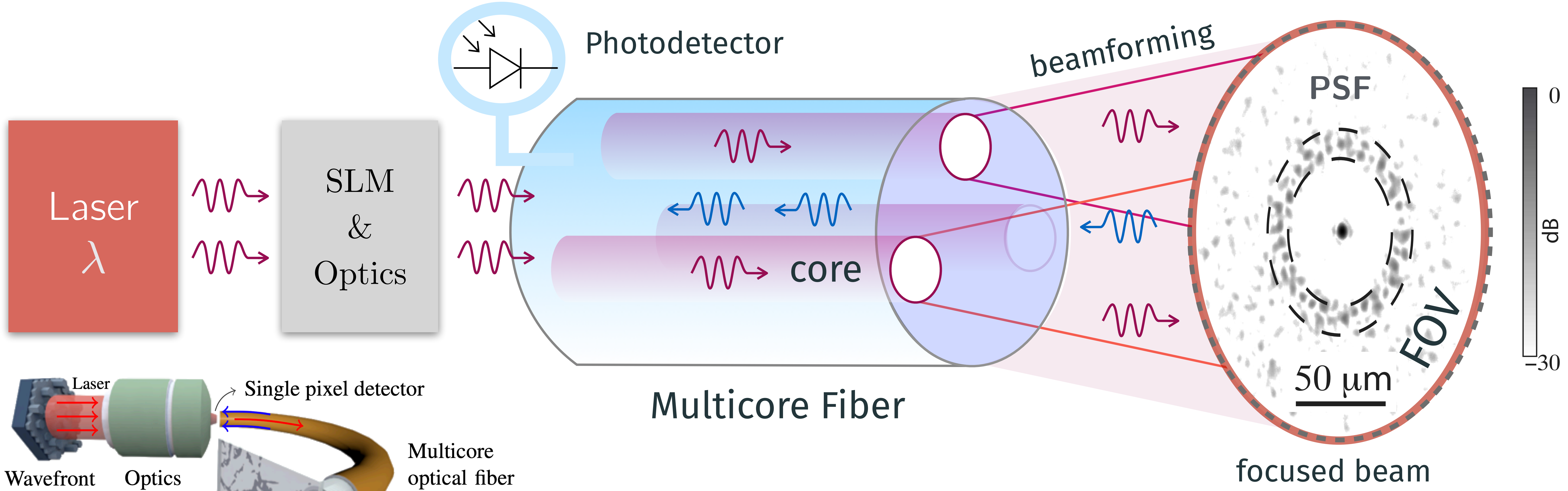

M. Hofer†

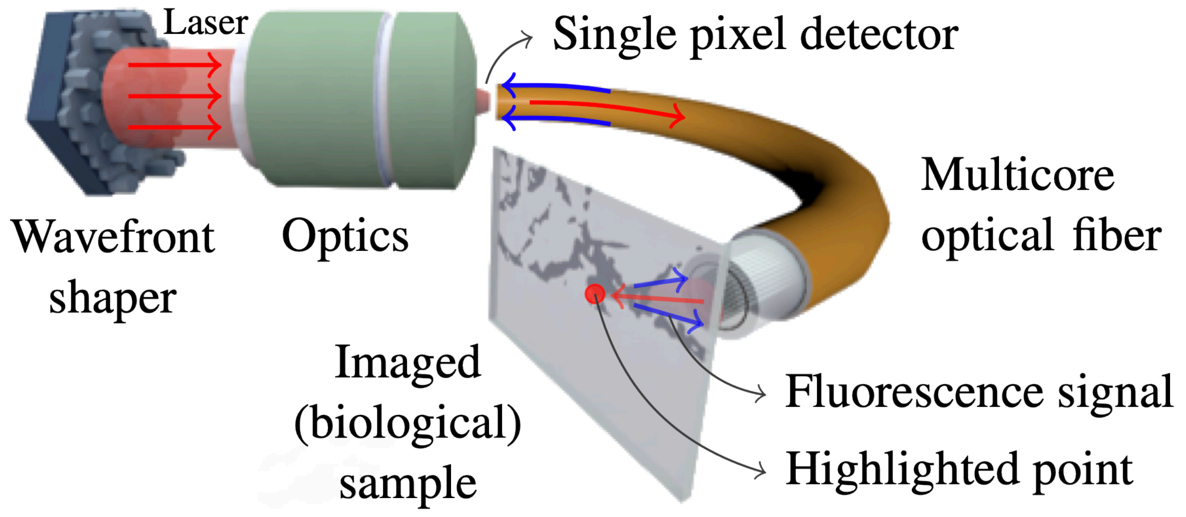

H. Rigneault†


S. Sivankutty‡

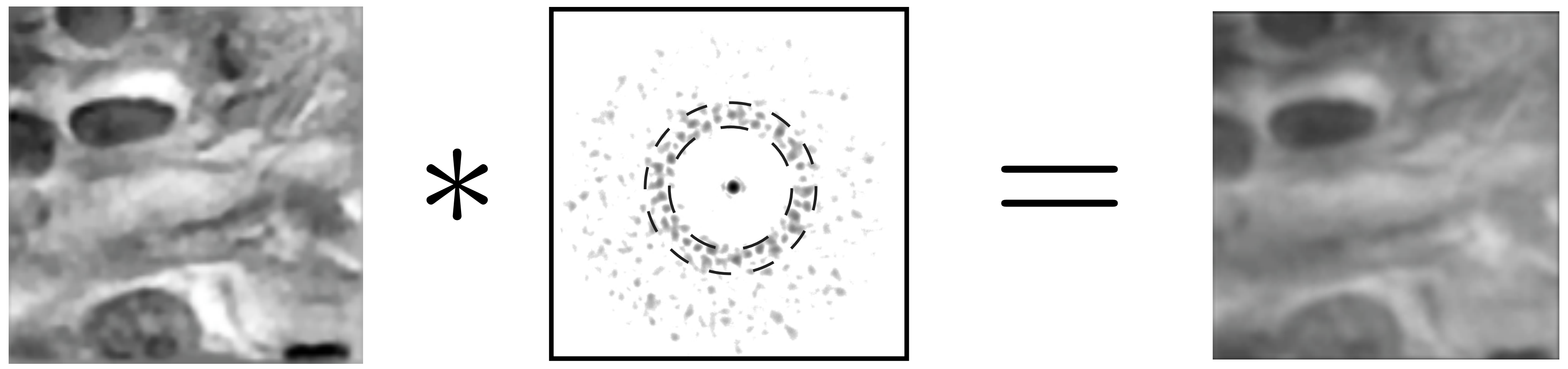
*: ISPGroup, INMA, UCLouvain, Belgium. †: Institut Fresnel, France. ‡: PhLAM, France.

Lensless endoscopy: focused mode

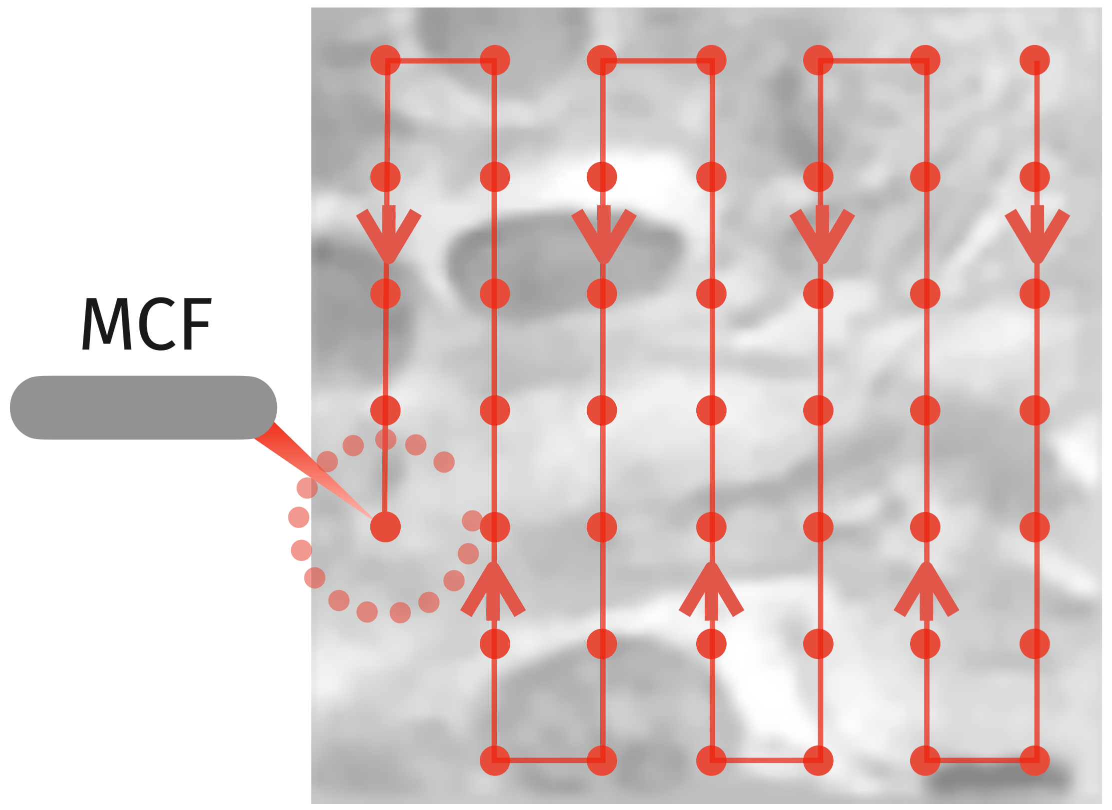




Sensing model



Raster Scanning

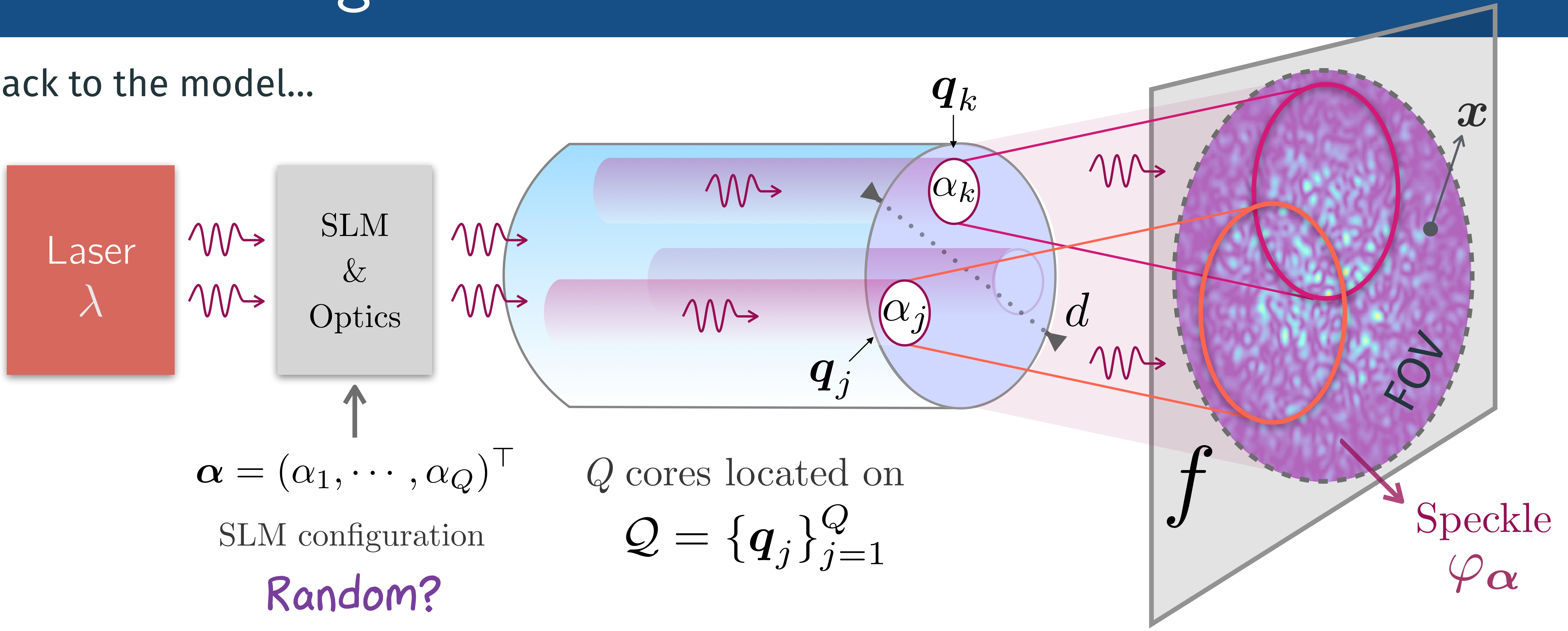


Biological sample

Andresen et al., 2016. Sivankutty et al., 2018.

A closer look to sensing model

Back to the model...



$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_Q)^\top$

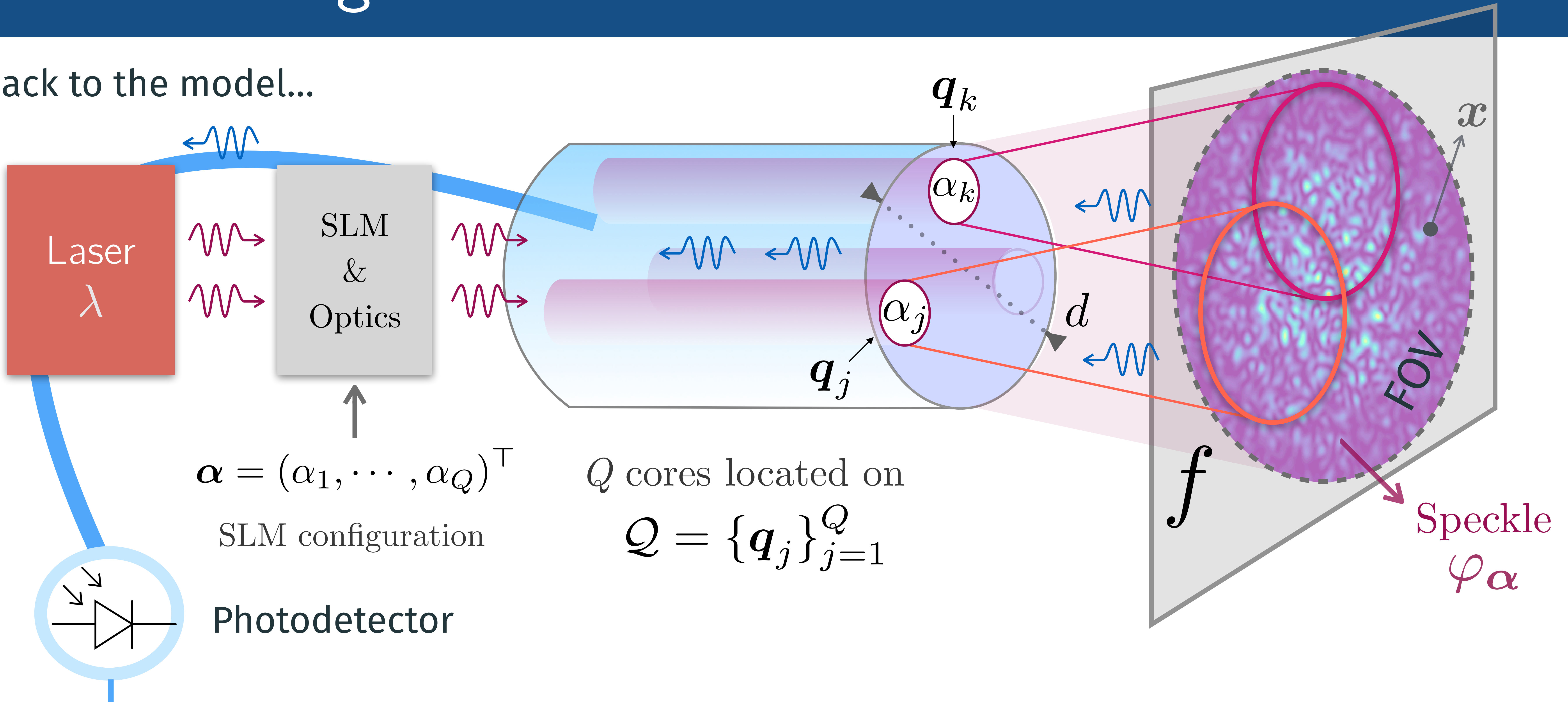
SLM configuration

Random?

Q cores located on

$\mathcal{Q} = \{\boldsymbol{q}_j\}_{j=1}^{Q}$

Back to the model...



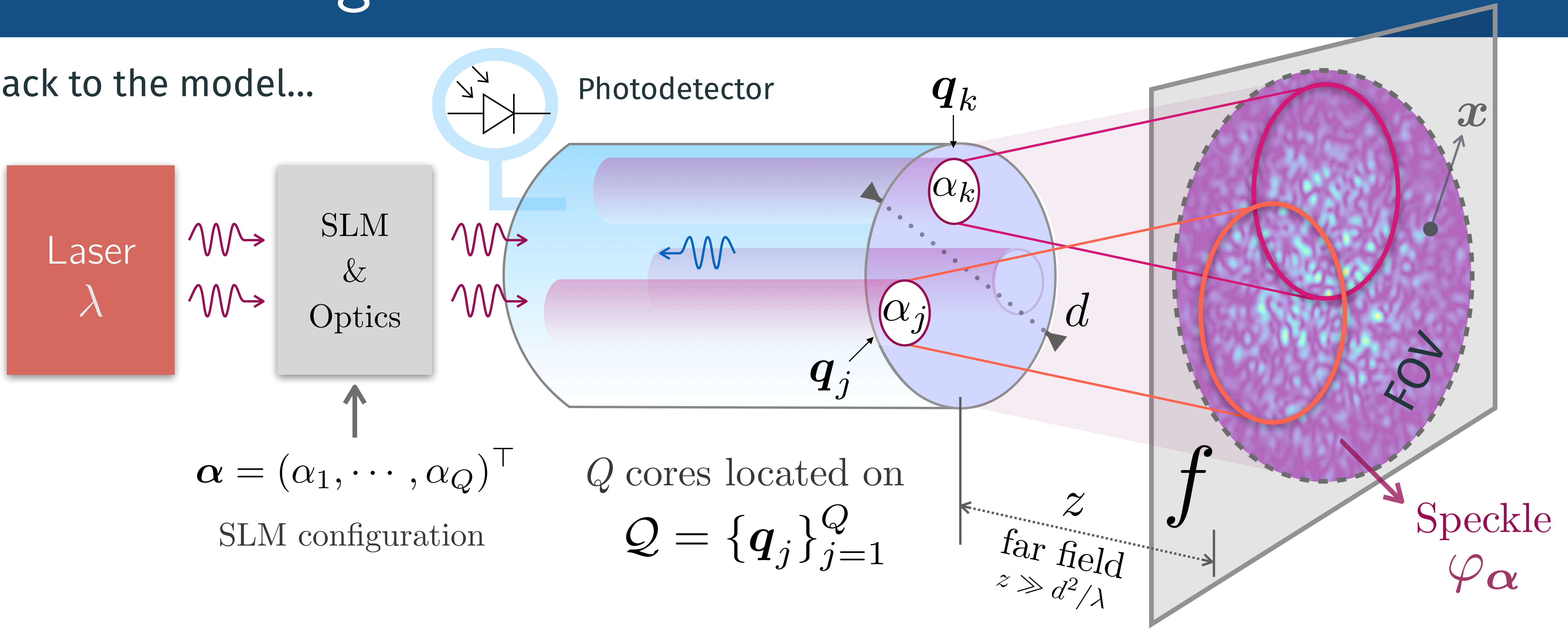
Speckle

$$y_{\boldsymbol{\alpha}} \propto \int_{\mathbb{R}^2} \varphi_{\boldsymbol{\alpha}}(\boldsymbol{x}) f(\boldsymbol{x})\, \mathrm{d}^2\boldsymbol{x} = \langle \varphi_{\boldsymbol{\alpha}}, f \rangle$$

Measurement model

A closer look to sensing model

Back to the model...



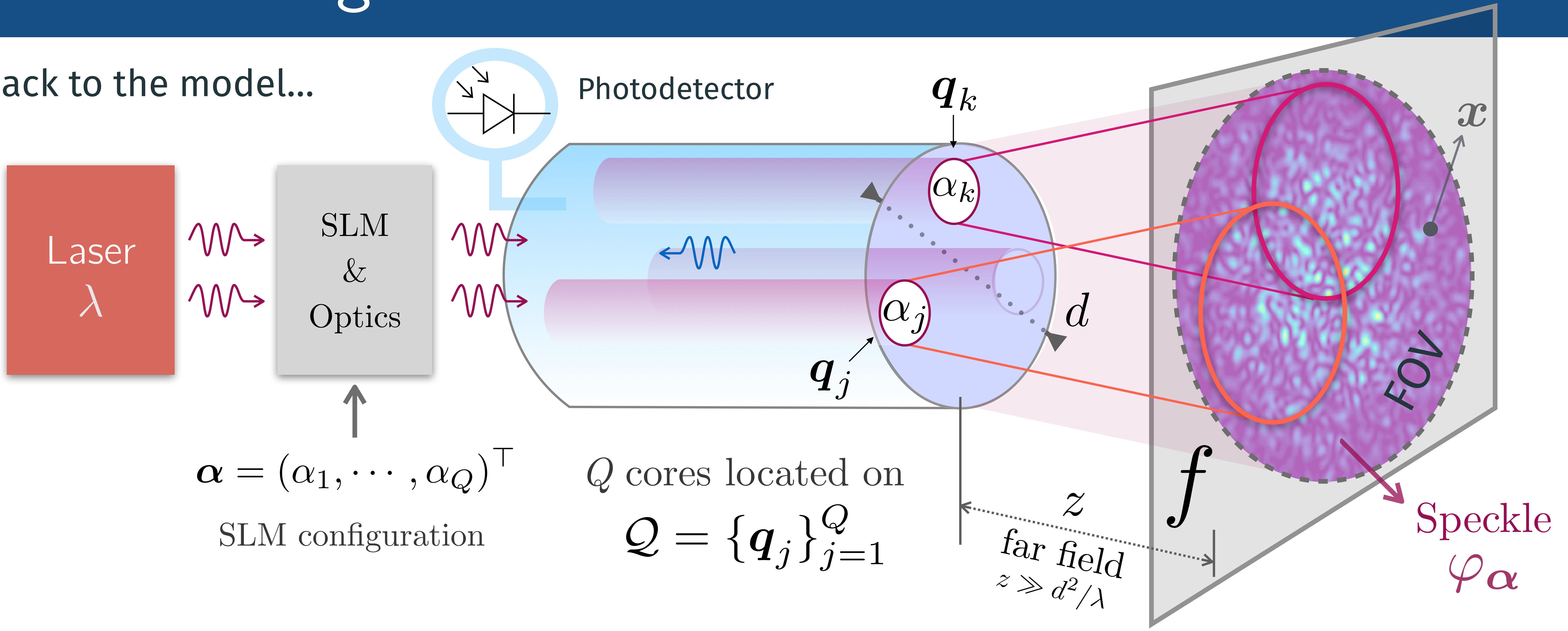
However, speckles are interferences: (Under far-field approximation)

$$\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \propto \underbrace{w(\boldsymbol{x})}_{\text{FOV window}} \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \underbrace{e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top \boldsymbol{x}}}_{\text{Core pair interference}}$$

Can we do compressive sensing?

A closer look to sensing model

Back to the model...



$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_Q)^\top$

SLM configuration

Q cores located on

$\mathcal{Q} = \{\boldsymbol{q}_j\}_{j=1}^{Q}$

However, speckles are interferences: (Under far-field approximation)

$$\langle f(\boldsymbol{x}), \varphi_{\boldsymbol{\alpha}}(\boldsymbol{x})\rangle \propto \langle w(\boldsymbol{x}) f(\boldsymbol{x}), \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top \boldsymbol{x}}\rangle$$

Can we do compressive sensing?

Therefore

$$\langle f, \varphi_{\boldsymbol{\alpha}} \rangle = \boxed{\textstyle\sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top \boldsymbol{x}} w(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right]}$$

$$\boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}[wf]\, \boldsymbol{\alpha}$$ → ROP!!

with the (Hermitian) *interferometric matrix* $\boldsymbol{\mathcal{I}}[wf] \in \mathbb{C}^{Q \times Q}$ s.t.

$$(\boldsymbol{\mathcal{I}}[wf])_{j,k} := \int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top \boldsymbol{x}}\, w(\boldsymbol{x}) f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

(noiseless) Interferometric sensing model

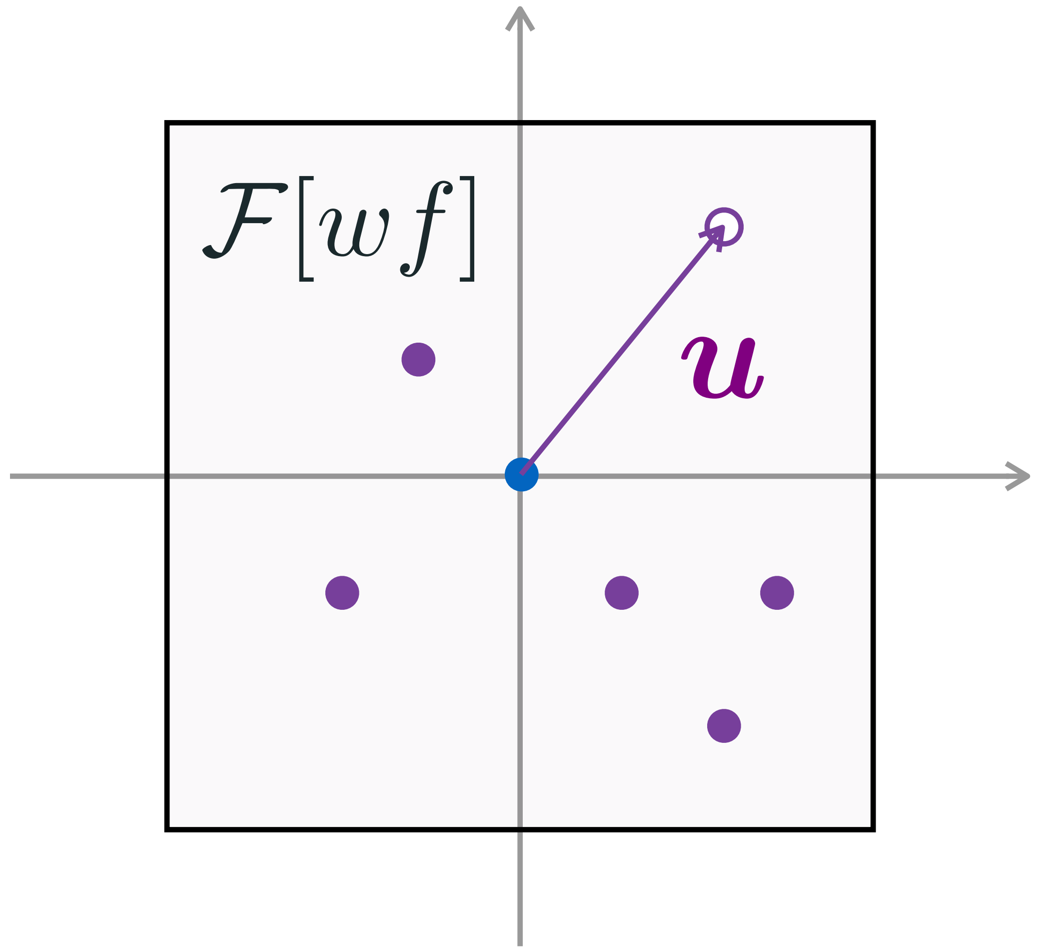
Therefore

$$\langle f, \varphi_{\boldsymbol\alpha}\rangle = \boxed{\textstyle\sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top \boldsymbol{x}} w(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\right]}$$

$$\dashrightarrow \boldsymbol\alpha^* \boldsymbol{\mathcal{I}}[wf]\, \boldsymbol\alpha \quad \to \text{ROP!!}$$

with the (Hermitian) *interferometric matrix* $\boldsymbol{\mathcal{I}}[wf] \in \mathbb{C}^{Q\times Q}$ s.t.

$$(\boldsymbol{\mathcal{I}}[wf])_{j,k} := \int_{\mathbb{R}^2} e^{\underbrace{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top}_{\boldsymbol{u}} \boldsymbol{x}}\, w(\boldsymbol{x}) f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \mathcal{F}[wf](\mathcal{V})$$

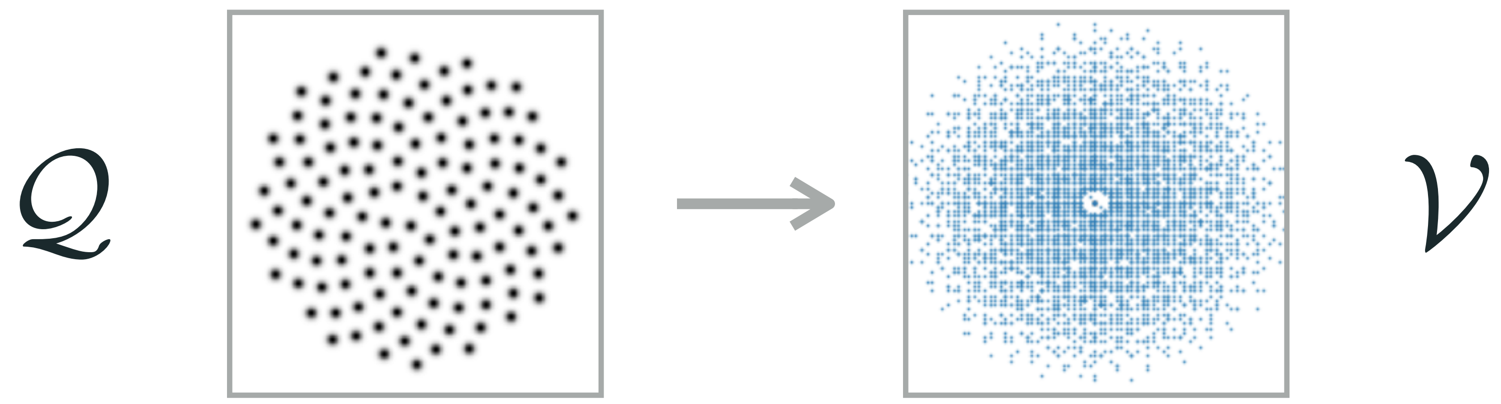


$$\boldsymbol{u} \in \mathcal{V} := \frac{1}{\lambda z}(\mathcal{Q} - \mathcal{Q})$$

Observation 1: denser Fourier sampling if

$$|\mathcal{V}| \simeq Q^2$$

- ✦ Lattices are bad core arrangements
- ✦ Fermat's spiral is not bad



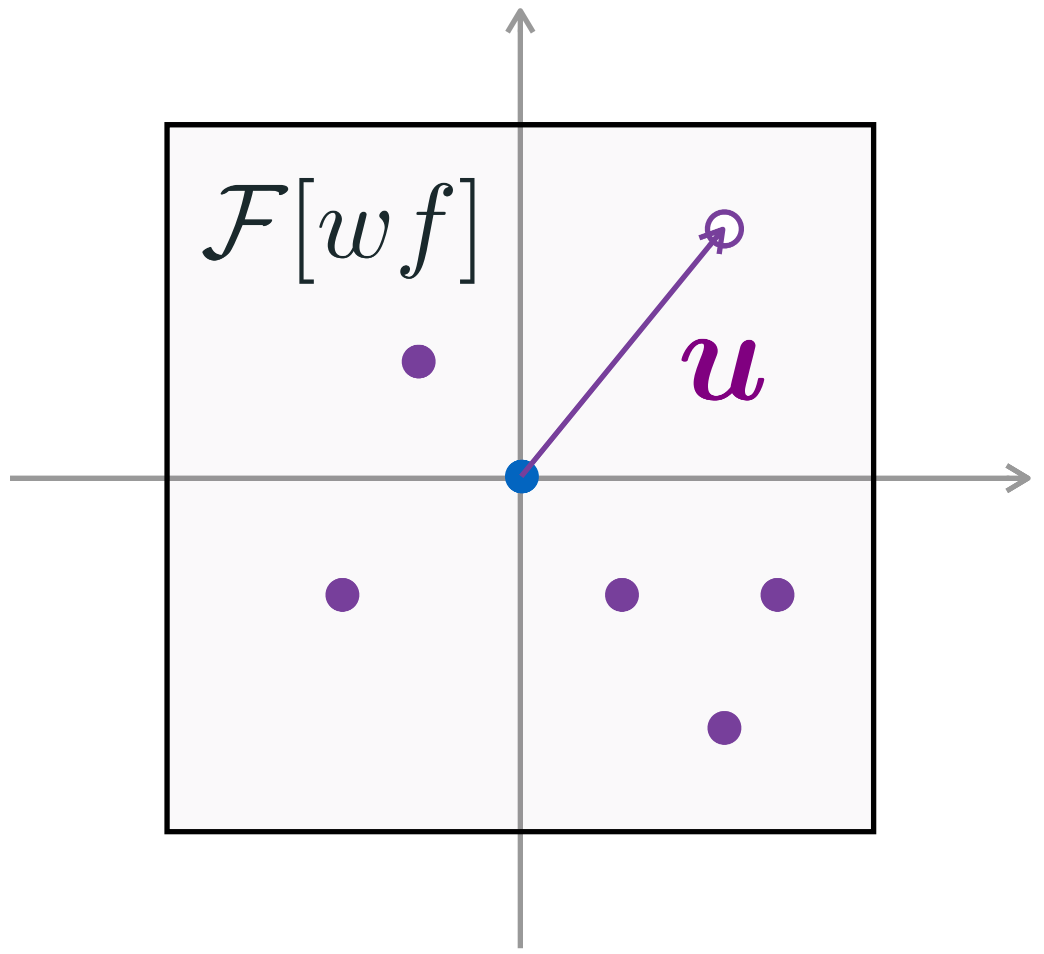
Therefore

$$\langle f, \varphi_{\boldsymbol{\alpha}} \rangle = \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)^\top \boldsymbol{x}} w(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right]$$

$$\boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}[wf]\, \boldsymbol{\alpha}$$ → ROP!!

with the (Hermitian) *interferometric matrix* $\boldsymbol{\mathcal{I}}[wf] \in \mathbb{C}^{Q \times Q}$ s.t.

$$(\boldsymbol{\mathcal{I}}[wf])_{j,k} := \int_{\mathbb{R}^2} e^{\underbrace{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{q}_j - \boldsymbol{q}_k)}_{\boldsymbol{u}}{}^\top \boldsymbol{x}} w(\boldsymbol{x}) f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \mathcal{F}[wf](\mathcal{V})$$



$$\boldsymbol{u} \in \mathcal{V} := \frac{1}{\lambda z}(\mathcal{Q} - \mathcal{Q})$$

Observation 2:

Low-complexity on f
→
Low-complexity on $\boldsymbol{\mathcal{I}}$

e.g., sparsity → low-rank

Interferometric sensing model

Composition of two sensing methods

$$\boldsymbol{y} = (y_{\boldsymbol{\alpha}_1}, \cdots, y_{\boldsymbol{\alpha}_m})^\top = \underbrace{\boldsymbol{\Phi}}_{\textcircled{2}\ m \times Q^2}(\overbrace{\boldsymbol{\mathcal{I}}}^{Q \times Q\ \textcircled{1}}[wf]) + \text{noise},$$

with $\boldsymbol{\Phi}(\boldsymbol{M}) := \{\langle \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j^*, \boldsymbol{M}\rangle_{\mathrm{F}}\}_{j=1}^m$.

Sample complexities of interest:

2 Does $\boldsymbol{\Phi}$ capture enough from $\boldsymbol{\mathcal{I}}$? $\leftrightarrow$ m big enough?

1 Does $\boldsymbol{\mathcal{I}}$ capture enough from f? $\leftrightarrow$ Q big enough?
Core arrangement?

A few answers from a few simplifications ...

Theory + Simulations + Experimental results

Theoretical guarantees

Given

- a discretisation $\boldsymbol{f}$ of wf over N pixels
- a frequency coverage $\mathscr{V}$ respecting usual CS conditions (RIP)

(under specific simplifying assumptions)

If the $\{\boldsymbol{\alpha}_i\}$ are (sub)Gaussian, given a sparsity level K

and provided $M = O(K)$ and $Q^2 = O(K)$ (up to logs),

then, with high probability, given the observations $\boldsymbol{z} = \boldsymbol{\Phi}'[\boldsymbol{f}] + \underbrace{\text{noise}}_{\|\cdot\|_1 \leqslant \epsilon}$,

an ℓ_1-minimization program gives an estimate $\boldsymbol{f}'$ with

$$\|\boldsymbol{f} - \boldsymbol{f}'\|_2 \leqslant C \frac{\|\boldsymbol{f} - \boldsymbol{f}_K\|_1}{\sqrt{K}} + D \frac{\epsilon}{M}$$

for some $C, D > 0$.

Proof idea: $\boldsymbol{\Phi}' =$ centering of $\boldsymbol{\Phi}$; show that $\boldsymbol{\Phi}'$ respects a variants of the restricted isometry property.

1-D simulations: phase transition diagrams

Simplified setting:

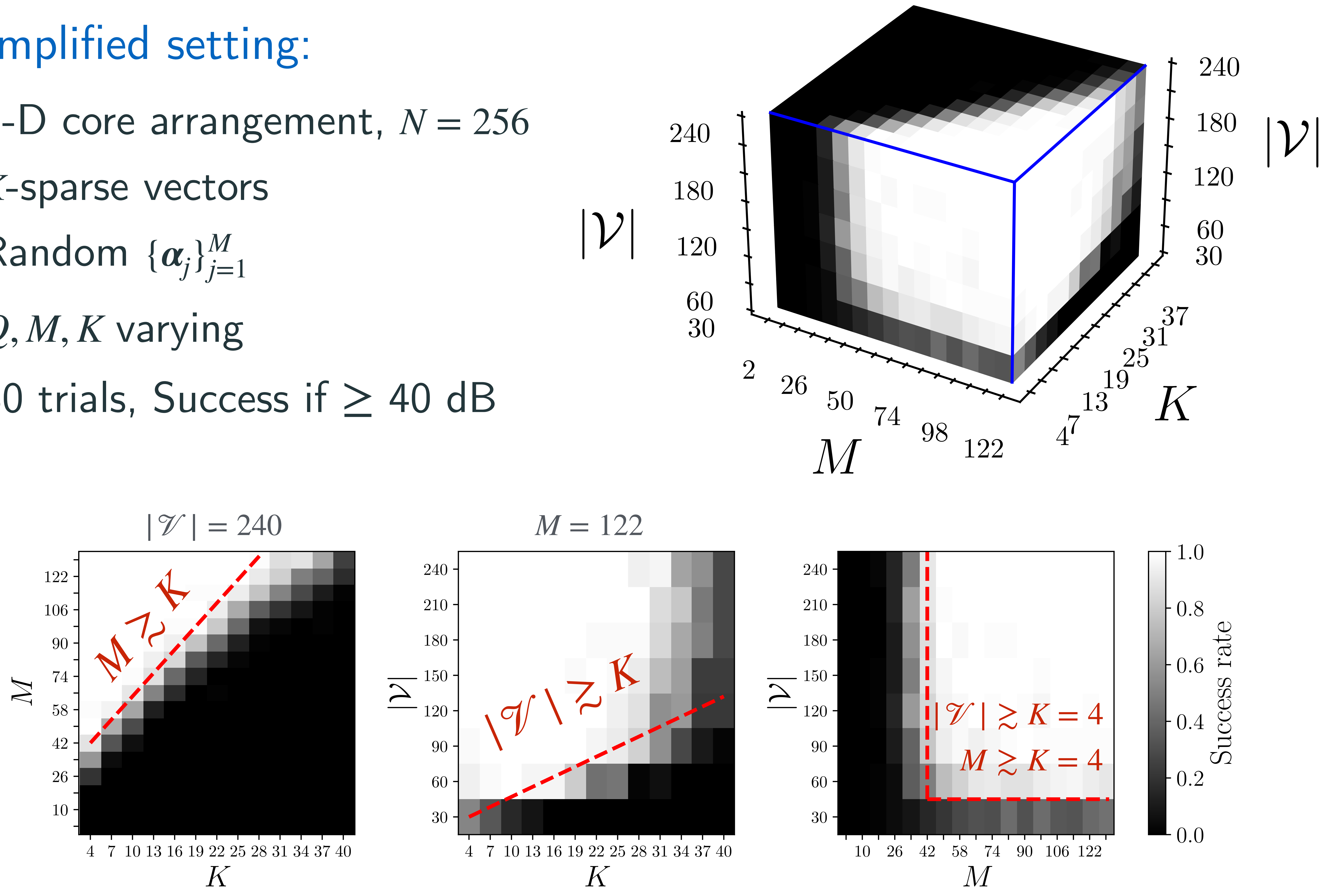
1-D core arrangement, $N = 256$

K-sparse vectors

Random $\{\boldsymbol{\alpha}_j\}_{j=1}^{M}$

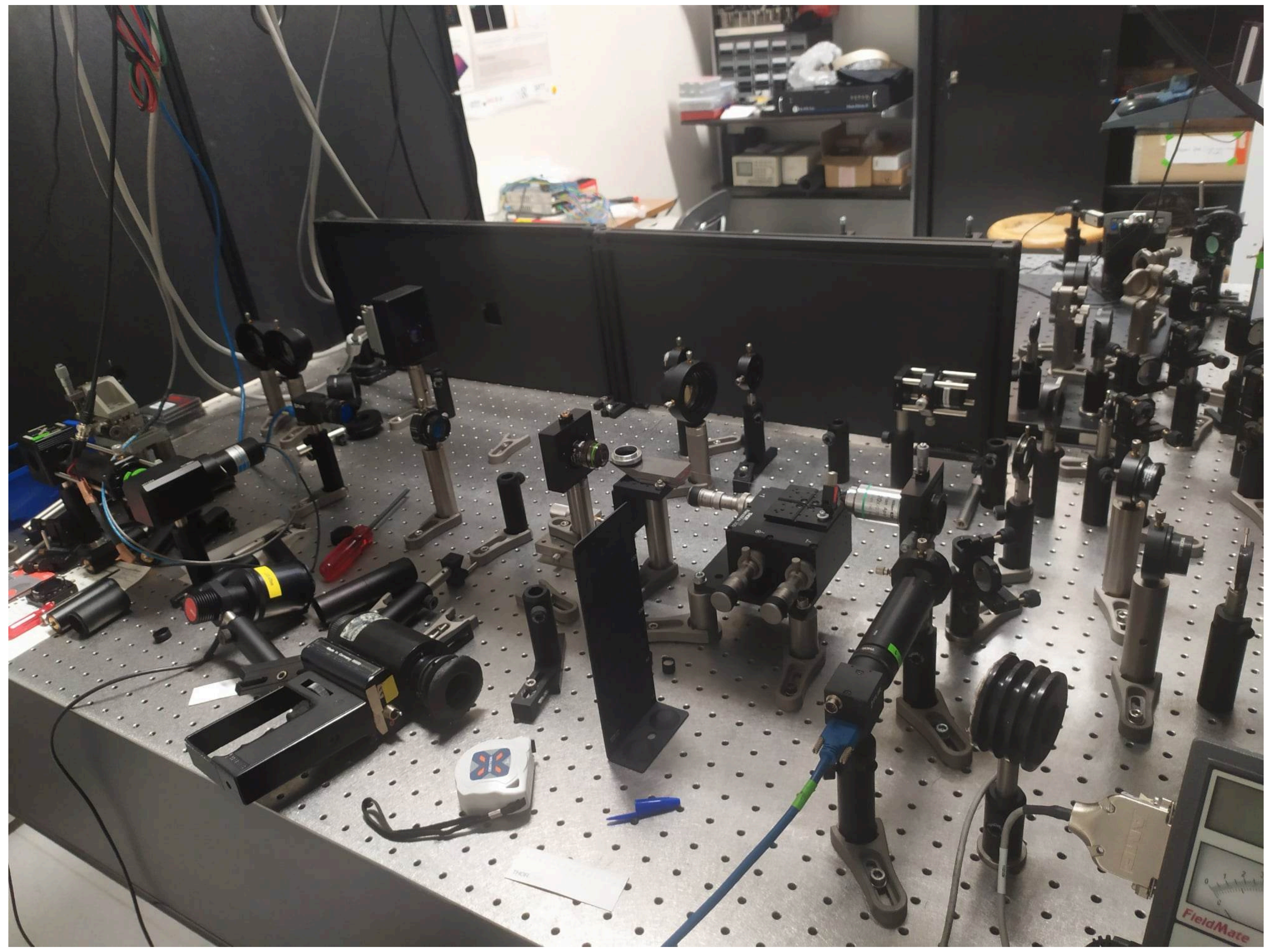
Q, M, K varying

80 trials, Success if $\geq$ 40 dB

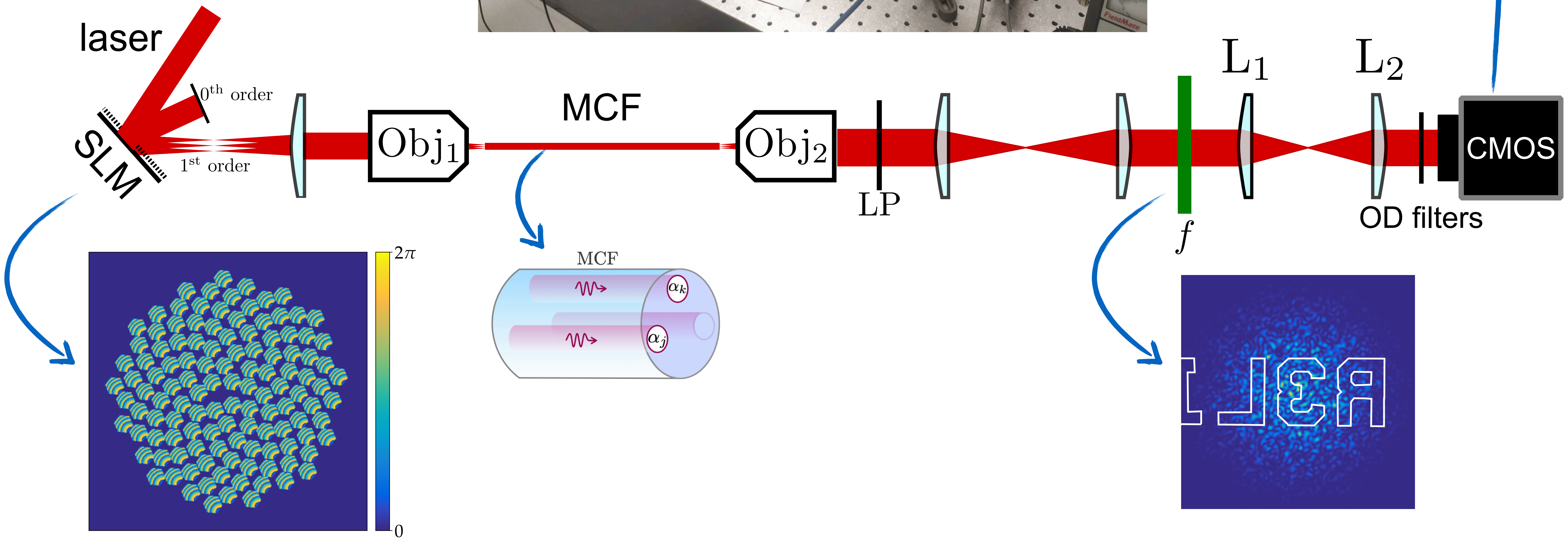


Experiments (in Institut Fresnel, France)

(Adapted from xkcd #1233)



Single pixel
=
Pixel summation



$+$ a lot of calibrations & validations

Experiments (in Institut Fresnel, France)